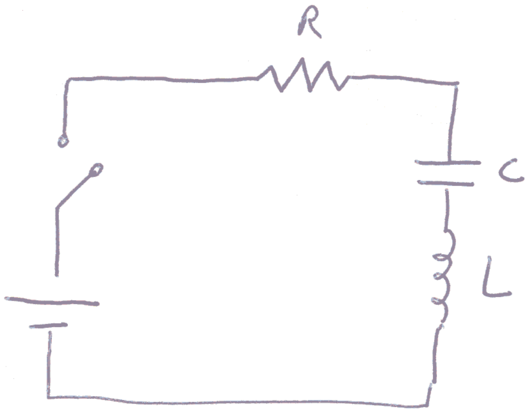
Transients in an RLC Series Circuit



What happens when the switch is closed?

KVL: $V = iR + \frac{q}{C} + L\frac{di}{dt}$

differentiate:

(1) $$0 = R\frac{di}{dt} + \frac{i}{C} + L\frac{d^2 i}{dt^2}$$

2nd order, linear d.e.

Look for solutions of the form:

$$i = e^{\lambda t}$$

so $$\frac{di}{dt} = \lambda e^{\lambda t}$$

$$\frac{d^2 i}{dt^2} = \lambda^2 e^{\lambda t}$$

plug into (1):

$$0 = \lambda^2 + \frac{R}{L}\lambda + \frac{1}{LC} = 0$$

$$\lambda = \frac{-\frac{R}{L} \pm \sqrt{\left(\frac{R}{L}\right)^2 - \frac{4}{LC}}}{2}$$

define $\alpha = \frac{R}{2L}$ and $\omega_0^2 = \frac{1}{LC}$

$$\lambda_+ = -\alpha + \sqrt{\alpha^2 - \omega_0^2} \qquad \lambda_- = -\alpha - \sqrt{\alpha^2 - \omega_0^2}$$

3 Cases

Case I: $\alpha^2 > \omega_0^2$ $\left[R^2 > \frac{4L}{C} \right]$ "Overdamped"

λ_+ and λ_- are both real #'s and are < 0

so our solution looks like

$$i(t) = Ae^{\lambda_1 t} + Be^{\lambda_2 t}$$

To determine A and B we need two initial conditions.

$i(0) = 0$ $\left[\text{otherwise } \left.\frac{di}{dt}\right|_{t=0} = \infty\right]$

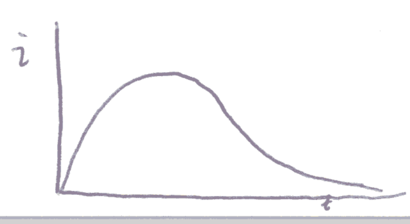
get $i'(0)$ from KVL:

$$V = iR + \frac{q}{C} + L\frac{di}{dt}$$

$$\rightarrow \left.\frac{di}{dt}\right|_{t=0} = \frac{V}{L}$$

so

$$0 = A + B$$
$$\frac{V}{L} = A\lambda_1 + B\lambda_2 \quad \Rightarrow \quad A = \frac{V}{L(\lambda_+ - \lambda_-)} = -B$$



Case II : $\alpha^2 = \omega_0^2$ "critically damped"

so $\lambda = -\alpha$ $\qquad R^2 = \frac{4L}{C}$

But now we have only 1 solution. Our second order d.e. must have 2 solutions.

Try: $i = f(t)\,e^{-\alpha t}$, plug into ①

find: $f''(t) = 0$

so $i = (A + Bt)\,e^{-\alpha t}$

case III $\alpha^2 < \omega_0^2$ "under damped"

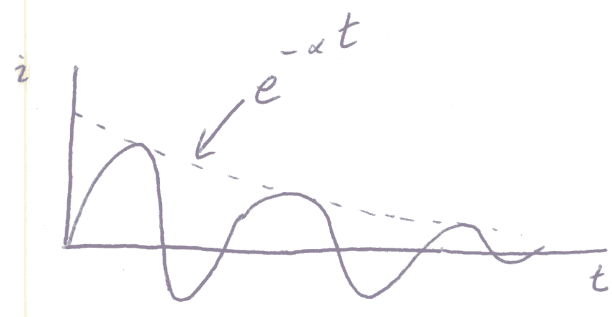
$$\lambda = -\alpha \pm j\sqrt{\omega_0^2 - \alpha^2}$$

so $i(t) = Ae^{-\alpha t}\sin \omega t + Be^{-\alpha t}\cos \omega t$ where $\omega = \sqrt{\omega_0^2 - \alpha^2}$

using the initial conditions we find $B = 0$

and $A = \frac{V}{\omega L}$

so $\boxed{i = \frac{V}{\omega L}\sin(\omega t)\,e^{-\alpha t}}$



In the frequency domain, the bandwidth $\omega_2 - \omega_1 = \frac{R}{L} = 2\alpha$

ie the ring-down time constant $= \frac{2L}{R} \propto \frac{1}{BW}$