Phys 209 Jan 18,

Errors

When you make a single measurement, you get a value and an uncertainty: $x \pm \delta x$

δx should be your best estimate of the standard deviation of the probability distribution for x (see below)

If you measure x many times: $x_1, x_2, x_3 \ldots$

then the best answer for x is $\bar{x} = \sum_{i=1}^{N} x_i$

But what is the uncertainty?

The standard deviation:

$$\sigma = \sqrt{\frac{\sum_{i=1}^{N} (\bar{x} - x_i)^2}{N-1}}$$

should match δx above. Because we've made more measurements now, the uncertainty of the true value in x should be smaller.

It is: $\sigma_m = \dfrac{\sigma}{\sqrt{N}}$

So repeating measurements gives a more accurate determination of x and also information about the probability distribution $P_x(x)$.

It's easy to see that σ_m, the standard error of the mean should be $\frac{\sigma}{\sqrt{N}}$.

Imagine $N = 2$

$$\bar{x} = (x_1 + x_2)/2$$

and the uncertainty in each of x_1 and x_2 is δx

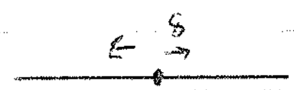
so the uncertainty in $\bar{x}$ is $\frac{\sqrt{(\delta x)^2 + (\delta x)^2}}{2} = \frac{\delta x}{\sqrt{2}}$

The same argument shows it is $\frac{\delta x}{\sqrt{N}}$ for N measurements.

Quadrature

Why do we add errors in quadrature?

Imagine a random walk: Consider an object starting at the origin that takes steps either left or right, of size δ.



Label each step:

$$\Delta x_1 = \pm \delta$$
$$\Delta x_2 = \pm \delta$$
$$\vdots$$

x_i = position after i steps

$$x_N = \sum_{i=1}^{N} \Delta x_i$$

After N steps, where do we expect to be?

$$\langle x_N \rangle = \sum_{i=1}^{N} \langle \Delta x_i \rangle$$

$$= 0$$ if $+\delta$ and $-\delta$ are equally likely.

But on any given run, we are unlikely to end up exactly at 0.

How far from the starting point do we expect to be?

Consider: $\langle x_N^2 \rangle = \left\langle \left(\sum_{i=1}^{N} \Delta x_i \right)^2 \right\rangle$

$$\langle x_1^2 \rangle = \langle \Delta x_1^2 \rangle = \delta^2$$

$$\langle x_2^2 \rangle = \langle (\Delta x_1 + \Delta x_2)^2 \rangle$$
$$= \langle (\Delta x_1^2 + 2\Delta x_1 \Delta x_2 + \Delta x_2^2) \rangle$$

if the steps are independent (ie Δx_2 doesn't depend on Δx_1), then $\langle \Delta x_1 \Delta x_2 \rangle = 0$

so $\langle x_2^2 \rangle = 2\delta^2$

and $\langle x_N^2 \rangle = N\delta^2$

so that $\sqrt{\langle x_N^2 \rangle} = \sqrt{N}\,\delta$

and $\dfrac{\sqrt{\langle x_N^2 \rangle}}{N} = \dfrac{\delta}{\sqrt{N}}$

Errors are similar. You can think of each step as corresponding to the error in a measurement. If we do N measurements, the error in the sum does increase $\sim \sqrt{N}\,\delta$, but when the average is taken, the error decreases: $\delta/\sqrt{N}$.

Quadrature part II

Imagine measuring quantities x and y where we want $z = x + y$

Assume the errors in x and y are Gaussian distributed and independent.

So $$P_x(x) = \frac{1}{\sqrt{2\pi}\,\sigma_x} e^{-(x-x_0)^2/2\sigma_x^2}$$

and $$P_y(y) = \frac{1}{\sqrt{2\pi}\,\sigma_y} e^{-(y-y_0)^2/2\sigma_y^2}$$

x_0 and y_0 are the true values and σ_x and σ_y are the widths of the distributions.

Make many measurements of x and y:

now: $z_i = x_i + y_i$

What does $P_z(z)$ look like?

Claim: $$P_z(z) = \frac{1}{\sqrt{2\pi}\,\sigma_z} e^{-(z-z_0)^2/2\sigma_z^2}$$

with $\sigma_z = \sqrt{\sigma_x^2 + \sigma_y^2}$

$z_0 = x_0 + y_0$

Proof:

Since the errors are independent:

$$P_z(z) = \int_{x=-\infty}^{\infty} P_x(x)\, P_y(z-x)\, dx.$$

There are many ways to arrive at any particular value of z. We integrate over all possible x, y pairs that give $z = x + y$.

$$P_z(z) = \int_{-\infty}^{\infty} \frac{1}{2\pi \sigma_x \sigma_y} e^{-(x-x_0)^2/2\sigma_x^2} e^{-(y-y_0)^2/2\sigma_y^2}$$

Strategy: expand the arguments of the exponentials, then complete the square

So we have something like:

$$\frac{1}{2\pi\sigma_x\sigma_y} \int_{-\infty}^{\infty} e^{-(x-A)^2/2B^2} f(z)\, dx$$

then: $$\int_{-\infty}^{\infty} e^{-(x-A)^2/2B^2} = \sqrt{2\pi}\, B$$

after a bit of algebra you can prove the claim above.
This is exactly what was demonstrated in homework 3.