1 Statistics and Repeated Measurements

In some situations you will be able to repeat a measurement many times in order to assess the uncertainty caused by random sources of variation. If you take sufficient measurements, you can draw a histogram that displays how frequently the measurement lands in different ranges of values. Three things can be seen in such a histogram. First, the data is often clustered in some way and mathematically we estimate the value they are clustered around by calculating the **mean**. If x_i are the measured values and if you took N different measurements ($i = 1, ...N$), the mean is calculated using

$$\overline{x} = \frac{1}{N}\sum_{i=1}^{N} x_i. \tag{1}$$

Note that the mean $\overline{x}$ is still an estimate based on a finite number of measurements and if you kept taking more measurements the mean would change somewhat.

The second thing to notice in the histogram is that the values are distributed on either side of the mean value, with some characteristic width. The width is usually expressed in terms of the **standard deviation** σ

$$\sigma = \sqrt{\frac{1}{N-1}\sum_{i=1}^{N}(x_i - \overline{x})^2}. \tag{2}$$

About 68% of the measurements will lie within the range from $\overline{x} - \sigma$ to $\overline{x} + \sigma$. So if you took just a single measurement it would have a 68% chance of landing within $\pm\sigma$ of the mean value. However, if you have gone to all of the trouble of taking N different measurements, the uncertainty in the mean value $\overline{x}$ is quite a bit better than the standard deviation. The uncertainty in the mean is called the **standard uncertainty** σ_m, given by

$$\sigma_m = \frac{\sigma}{\sqrt{N}}. \tag{3}$$

If you do one hundred times as many measurements, the uncertainty only decreases by a factor of ten - hard work!

The third thing you can see in a histogram, if you take enough measurements, is that the distribution of the measured values has a characteristic shape. These can vary a lot depending on the particular experiment, but a common shape is a Gaussian, the so-called bell curve.

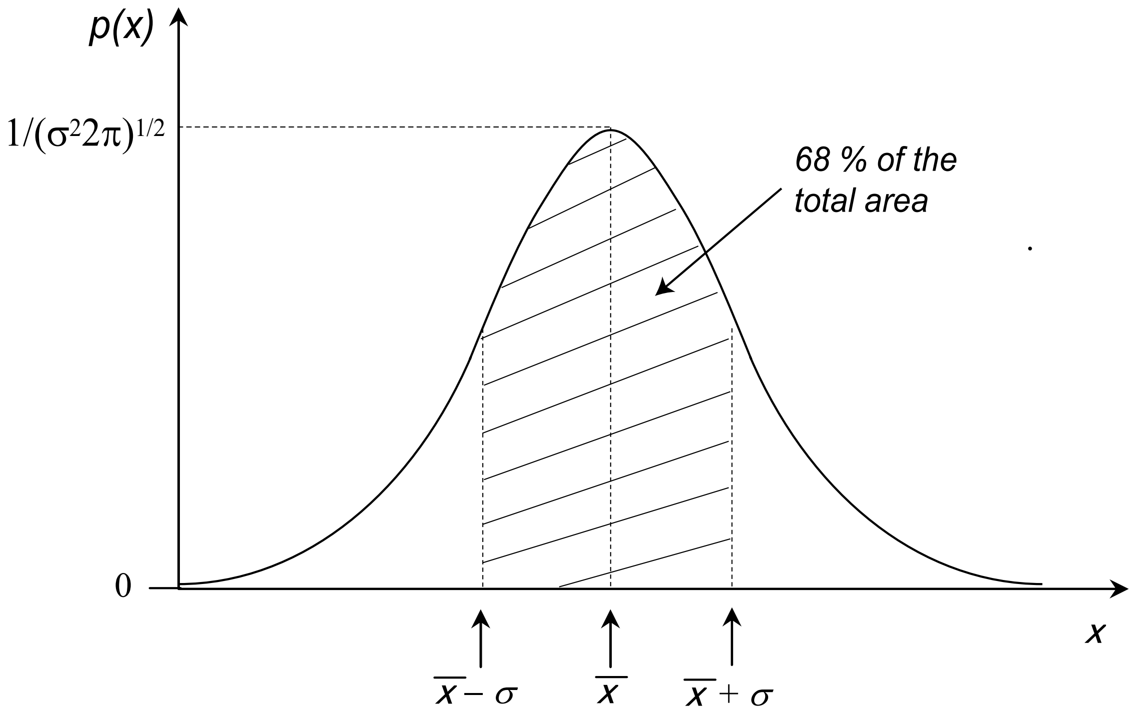

Figure 1: The probability density p(x) for a Gaussian distribution. The area under this function over some range of x gives the probability that x lies in that range. For instance, the total probability of x being within one standard deviation of the mean is 68 %

2 Probability Distributions

Next we take a leap from histograms of data to probability distributions. The figure above shows the **probability density** for a Gaussian distribution. The mathematical expression for the Gaussian is

$$p(x) = \frac{1}{\sigma\sqrt{2\pi}} exp\left[-\frac{(x-\overline{x})^2}{2\sigma^2}\right] \tag{4}$$

where $\overline{x}$ is the mean of the distribution and σ is the standard deviation. It is important to note that $p(x)$ is **not** the probability that the measurement takes on a particular value of x. The probability for any exact value is actually **zero**. $p(x)$ is is not a probability, it is a probability density. This means that for some range of values of x, the area under $p(x)$ over that range is the probability that x lies in that range. Expressed as an integral, this means that $\int_{x_1}^{x_2} p(x)dx$ is the probability that x is between x_1 and x_2. Note that since the integral is a probability, then $p(x)$ must have units that are the inverse of the units of x. For example, if x is some measurement of a time in seconds (s), such as the period of a pendulum, then $p(x)$ has units of s^{-1}. Also, the total area under the curve must be one, which is why the normalization factor $1/(\sigma\sqrt{2\pi})$ is in front of the expression above for the Gaussian distribution.

3 Uncertainty in Single Measurements

There are often situations where you take only a single measurement and then attach an uncertainty that is based on how well you can read the measured value. If it is a digital instrument, you have uncertainty due to the rounding of the number. For instance, a digital reading on a scale might display 24.5 kg, which means that the mass could be anywhere between 24.45 and 24.55 kg. Since you have no idea where the mass lies in this range, you attach a probability that is equal for any value in this range.

Figure 2 shows a correctly normalized probability density for a digital measurement. We should treat this type of measurement in a way that is consistent with the case above, where we were able to use statistics for multiple trials. To do this, we should use the standard deviation of this distribution, rather than saying that the uncertainty is the maximum spread due to the rounding off of the measured value. We can get the standard deviation from an integral. The rectangular distribution in Fig. 2 can be divided into small slices dx as shown in the figure. Each of these slices has a square deviation from the mean value given by $(x-\overline{x})^2$ and the probability associated with the slice is the area $dx/(2a)$. So, the standard uncertainty of the whole distribution is

$$\sigma = \int_{\overline{x}-a}^{\overline{x}+a} \frac{1}{2a}(x-\overline{x})^2 dx = \frac{a}{\sqrt{3}}. \tag{5}$$

Thus, for digital measurements with a maximum rounding uncertainty of $\pm a$, the standard uncertainty that you would write down is $\pm a/\sqrt{3}$.

For a single measurement on an analog scale, we use a triangular distribution as is shown in Fig. 3. In this case the normalization for the total area to be unity requires that the peak value of the distribution is $1/a$, where a is the maximum uncertainty in the reading. Following the line of argument for the rectangular distribution, an integral can show that the standard error for the triangular probability density is $\pm a/\sqrt{6}$.

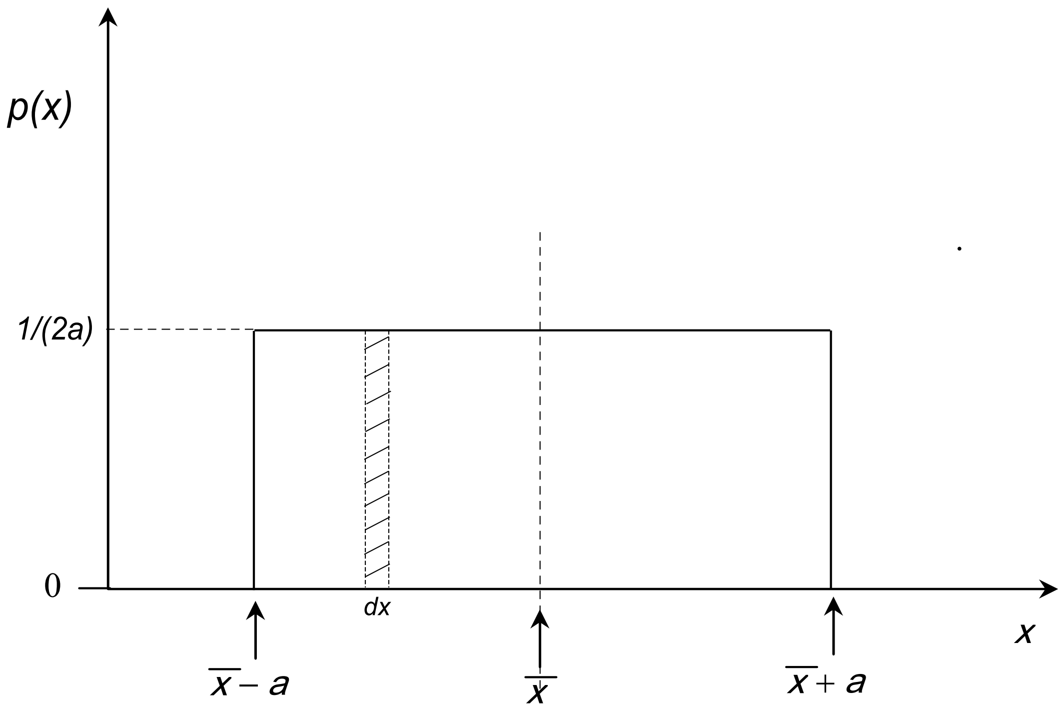

Figure 2: The probability density p(x) for a digital measurement. The reading on the meter is $\overline{x}$, which is also the mean value of the distribution. The maximum uncertainty due to rounding is $\pm a$. The value of $p(x)$ within this uncertainty range is uniform and takes on the value $1/(2a)$ so that the integral of the whole distribution is unity.

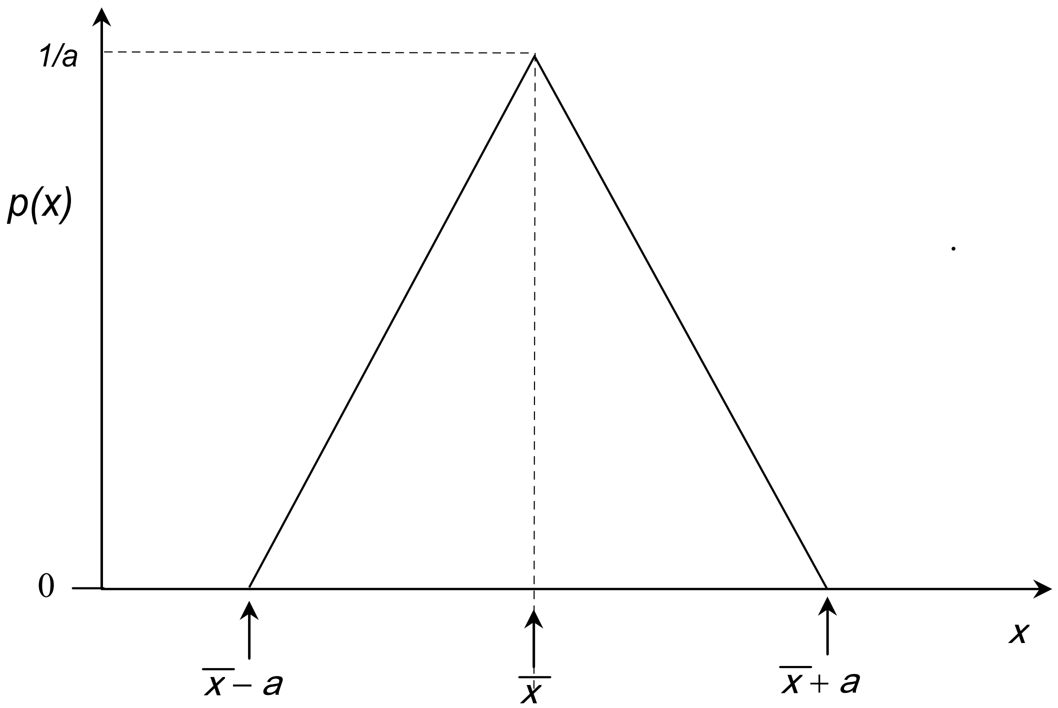

Figure 3: The probability density p(x) for an analog measurement. The reading on the scale is $\overline{x}$, which is also the mean value of the distribution. The maximum uncertainty estimated from the scale is $\pm a$. The value of $p(x)$ within this uncertainty range is peaked at value $1/a$, normalized so that the integral of the whole distribution is unity.

4 Propagating Uncertainty Through a Calculation

You will often have a measurement with some uncertainty and then need calculate something else with it. This requires a scheme for carrying the uncertainty along through the calculation. We'll denote the calculation as being some function $y = f(x)$, where $x \pm \delta x$ is a measured value that has an uncertainty δx. You need to calculate not just y, but also the uncertainty in y which we'll call δy. For example, if you simply divide by a number, say 10,

$$y \pm \delta y = \frac{x}{10} \pm \frac{\delta x}{10}. \tag{6}$$

You simply divide both by the same number. The same rule works for multiplication by a number. Notice that the **relative uncertainty** is not changed by this operation. The relative uncertainty in the measurement is $\frac{\delta x}{x}$, which is a dimensionless measure of how uncertain the measurement is. Think of it as a measure of quality. ($100\% \times \frac{\delta x}{x}$) is the **percent uncertainty**. In the example above, $\frac{\delta y}{y} = \frac{\delta x}{x}$, so the 'quality' of the number hasn't changed when dividing by a number that has no uncertainty of its own.

Another example you have seen is the uncertainty in the area of a circle, calculated from an uncertain radius. A circle of radius r has an area $A(r) = \pi r^2$. If the radius is slightly larger, $r + \delta r$, then $A(r + \delta) \approx A(r) + 2\pi r \times \delta r$. So, if you have an uncertain radius $r \pm \delta r$, then the uncertainty in the area is $\delta A \approx 2\pi r \times \delta r$. You can arrive at this geometrically, or by using a Taylor expansion. However, it's worth noticing another way to get this; the coefficient is the derivative of the area with respect to radius - dA/dR!

This is a general result. When propagating the uncertainty of $x \pm \delta x$ through the function $y = f(x)$, one can use

$$y \pm \delta y \approx f(x) \pm \frac{d}{dx} f(x) \times \delta x. \tag{7}$$

Try this with the simple example of division shown above.

5 Combining Multiple Sources of Uncertainty

If a measurement that you perform has more than one source of uncertainty, you add the uncertainties in **quadrature**. If you measure something and the result is a, but you have an uncertainty from the scale of $\pm\delta a_1$ and an uncertainty from some other cause that is $\pm\delta a_2$, then the measured value and its total uncertainty is

$$a \pm \sqrt{(\delta a_1)^2 + (\delta a_2)^2} \tag{8}$$

When adding measured quantities together, you add the uncertainties together as well, but the addition is in **quadrature**. If you add two quantities $a \pm \delta a$ and $b \pm \delta b$, the result is

$$a + b \pm \sqrt{(\delta a)^2 + (\delta b)^2} \tag{9}$$

The same expression is used if you subtract two quantities from one another.

$$a - b \pm \sqrt{(\delta a)^2 + (\delta b)^2} \tag{10}$$

When multiplying or dividing, you add the relative uncertainties in quadrature. That is, if $z = a \times b$, or if $z = a/b$, then the relative uncertainty in z is

$$\frac{\delta z}{z} = \sqrt{\left(\frac{\delta a}{a}\right)^2 + \left(\frac{\delta b}{b}\right)^2} \tag{11}$$