Solved Problems on Matter Waves

Charles Asman, Adam Monahan and Malcolm McMillan Department of Physics and Astronomy University of British Columbia, Vancouver, British Columbia, Canada

Fall 1999; revised 2011 by Malcolm McMillan

Given here are solutions to 6 problems on matter waves.

The solutions were used as a learning-tool for students in the introductory undergraduate course Physics 200 *Relativity and Quanta* given by Malcolm McMillan at UBC during the 1998 and 1999 Winter Sessions. The solutions were prepared in collaboration with Charles Asman and Adam Monaham who were graduate students in the Department of Physics at the time.

The problems are from Chapter 4 *Matter Waves* of the course text *Modern Physics* by Raymond A. Serway, Clement J. Moses and Curt A. Moyer, Saunders College Publishing, 2nd ed., (1997).

Planck's Constant and the Speed of Light.

When solving numerical problems in Quantum Mechanics is useful to note that the product of Planck's constant

$$h = 6.6261 \times 10^{-34} \,\mathrm{J\,s} \tag{1}$$

and the speed of light

$$c = 2.9979 \times 10^8 \,\mathrm{m \, s^{-1}} \tag{2}$$

is

$$hc = 1239.8 \text{ eV nm} = 1239.8 \text{ keV pm} = 1239.8 \text{ MeV fm}$$
 (3)

where

$$eV = 1.6022 \times 10^{-19} \,\text{J} \tag{4}$$

Also,

$$\hbar c = 197.32 \text{ eV} \text{ nm} = 197.32 \text{ keV} \text{ pm} = 197.32 \text{ MeV} \text{ fm}$$
 (5)

where $\hbar = h/2\pi$.

De Broglie Matter Waves

Problem 4.5, page 181

To observe small objects, one measures the diffraction of particles whose de Broglie wavelength is approximately equal to the object's size.

• Determine the kinetic energy (in electron volts) required for electrons to resolve a large organic molecule of size 10 nm, an atomic features of size 100 pm and a nucleus of size 10 fm.

• Repeat these calculations using alpha particles in place of electrons.

Solution

A small object can be observed by a wave if the wave can be effectively scattered by the object. This happens if the object's size and the wavelength of the wave are about the same. That is, to observe an object of length d, a wave of wavelength $\lambda \simeq d$ is needed. Because of their wavelike properties, electrons can be used to observe small objects. This is the basis of electron microscopy.

An electron (rest mass $m_e = 511 \text{ keV}/c^2$) with momentum p has energy

$$E = \sqrt{p^2 c^2 + m_e^2 c^4} \tag{6}$$

and kinetic energy

$$K = \sqrt{p^2 c^2 + m_e^2 c^4} - m_e c^2 \tag{7}$$

and de Broglie wavelength

$$\lambda = \frac{h}{p} \tag{8}$$

 \mathbf{SO}

$$K = \sqrt{\left(\frac{hc}{\lambda}\right)^2 + m_e^2 c^4} - m_e c^2,\tag{9}$$

the nonrelativistic limit of which is

$$K = \frac{h^2}{2m_e\lambda^2}.$$
(10)

Eq. (10) is a good approximation to Eq. (9) when $p \ll m_e c$ or $\lambda \gg h/m_e c = 2.43$ pm.

Setting $d = \lambda$, it follows from Eqs. (9) and (10) that K = 15 meV when d = 10 nm; K = 150 eV when d = 100 pm; and K = 123 MeV when d = 10 fm.

Eq. (10) is a good approximation to Eq. (9) for an alpha particle (rest mass $m_{\alpha} = 7295 \ m_e$) when $\lambda \gg 0.332$ fm.

Setting $d = \lambda$, it follows from Eq. (10) that for an alpha particle $K=2.06 \ \mu \text{eV}$ when d=10 nm; K=20.6 meV when d=100 pm; and K=2.06 MeV when d=10 fm.

Low-Energy Electron Diffraction

Problem 4.14, page 182

• Show that the formula for low-energy electron diffraction when electrons are incident perpendicular to a crystal surface is

$$\sin\phi_n = \frac{nh}{d(2m_e K)^{1/2}}$$
(11)

where n is the order of the maximum, d is the atomic spacing, m_e is the electron mass, K is the electron's kinetic energy and ϕ is the angle between the incident and diffracted beams.

• Calculate the atomic spacing in a crystal that has consecutive diffraction maxima at 24.1° and 54.9° for 100 eV electrons.

Solution

Eq. (11) follows from text Eq. (4.9):

$$d\sin\phi_n = n\lambda\tag{12}$$

and Eq. (10).

It follows from Eq. (11) that

$$d = \frac{h}{(2mK)^{\frac{1}{2}}(\sin\phi_{n+1} - \sin\phi_n)}$$
(13)

from which d = 0.299 nm for a crystal which exhibits consecutive diffraction maxima at 24.1° and 54.9° for 100 eV electrons.

Problem 4.17, page 182

The dispersion relation for a free relativistic electron wave is

$$\omega(k) = c\sqrt{k^2 + k_e^2} \tag{14}$$

where $k_e = m_e c/\hbar$.

• Obtain expressions for the phase speed v_p and group speed v_g for this wave and show that their product is constant, independent of k.

• What can you conclude about v_q ?

Solution

Eq. (14) follows from Eq. (6) on writing $E = \hbar \omega$ and $p = \hbar k$. The phase speed v_p of a wave with frequency f and wavelength λ is defined as

$$v_p = f\lambda \tag{15}$$

It follows from Eq. (8) and $f = 2\pi\omega$ that the phase speed v_p of a free electron wave is

$$v_p = \frac{\omega(k)}{k} = c \left(1 + \frac{k_e^2}{k^2}\right)^{1/2} \ge c$$
 (16)

The group speed v_g of a wave is defined as

$$v_g = \frac{d\omega(k)}{dk} \tag{17}$$

It follows that the group speed v_g of a free electron wave is

$$v_g = c \left(1 + \frac{k_e^2}{k^2} \right)^{-1/2} \le c$$
 (18)

It follows that

$$v_p v_q = c^2. (19)$$

The group speed v_g of a free electron wave is equal to the electron particle speed v defined by

$$p = \gamma m_e v \qquad E = \gamma m_e c^2 \tag{20}$$

That is,

$$v = \frac{pc^2}{E} = c \left(1 + \frac{k_e^2}{k^2}\right)^{-1/2} = v_g \tag{21}$$

Heisenberg's Uncertainty Relation for Position and Momentum: Application to a Photon

Problem 4.24, page 182

We wish to measure simultaneously the wavelength and position of a photon.

Assume the wavelength measurement gives $\lambda = 600$ nm to an accuracy to one part in a million, that is, $\Delta \lambda / \lambda = 10^{-6}$.

• Determine the minimum uncertainty in the position of the photon.

Solution

We use the Heisenberg Uncertainly Relation (text Eq. (4.31))

$$\Delta x \Delta p \ge \hbar/2 \tag{22}$$

to define the minimum uncertainty Δx_{min} in the position of a photon with momentum uncertaintly Δp :

$$\Delta x_{min} = \frac{\hbar}{2\Delta p} \tag{23}$$

We use the de Broglie relation Eq. (8) to write

$$\Delta p \simeq \left| \frac{dp}{d\lambda} \right| \Delta \lambda = \frac{h\Delta\lambda}{\lambda^2} \tag{24}$$

 \mathbf{SO}

$$\Delta x_{min} \simeq \frac{\lambda^2}{4\pi\Delta\lambda}.$$
(25)

Thus, $\Delta x_{min} \simeq 4.8$ cm when $\lambda = 600$ nm and $\Delta \lambda / \lambda = 10^{-6}$.

Heisenberg's Uncertainty Relation for Energy and Time: Lifetime of the Delta Particle

Problem 4.30, page 183

Measurements to determine the mass of the subatomic delta particle give a mass peak at $1236 \,\mathrm{MeV}/c^2$ with $110 \,\mathrm{MeV}/c^2$ full width at half maximum.

• Estimate the lifetime of the delta particle.

Solution

The lifetime τ of the delta particle is too short to be measured directly. It is defined instead using the Heisenberg Uncertainly Relation (text Eq. (4.34))

$$\Delta E \Delta t \ge \hbar/2 \tag{26}$$

which relates the uncertainty ΔE in the energy of a system with the smallest time interval Δt required to measure a change in the system.

 τ is defined as

$$\tau = \frac{\hbar}{\Gamma} \tag{27}$$

where Γ is the full-width at half-maximum of the delta mass distribution. This gives $\tau = 6.0 \times 10^{-24}$ s.

Principle of Superposition: Two-Slit Diffraction of Electrons

Problem 4.33, page 183

A two-slit electron diffraction experiment is done with slits of unequal widths.

When only slit 1 is open, the number of electrons reaching the screen per second is 25 times the number of electrons reaching the screen per second when only slit 2 is open.

When both slits are open, an interference pattern results in which the destructive interference is not complete.

• Determine the ratio of the probability of an electron arriving at an interference maximum to the probability of an electron arriving at an adjacent minimum.

Solution

We denote the wavefunction describing the electron as Ψ . The probability P of finding the electron at a given location is proportional to $|\Psi|^2$.

We denote the wavefunction describing the electron when only slit 1 is open as Ψ_1 and the probability of finding the electron in this case as P_1 .

We denote the wavefunction describing the electron when only slit 2 is open as Ψ_2 and the probability of finding the electron in this case as P_2 .

Then,

$$\frac{P_1}{P_2} = \frac{|\Psi_1|^2}{|\Psi_2|^2} = 25 \tag{28}$$

 \mathbf{SO}

$$\frac{|\Psi_1|}{|\Psi_2|} = 5. \tag{29}$$

When both slits are open, the wavefunction of the electron is $\Psi = \Psi_1 + \Psi_2$. At an interference maximum Ψ_1 and Ψ_2 are in phase so

$$P_{max} = (|\Psi_1| + |\Psi_2|)^2 \tag{30}$$

At an interference minima Ψ_1 and Ψ_2 are out of phase so

$$P_{min} = (|\Psi_1| - |\Psi_2|)^2 \tag{31}$$

It follows that

$$\frac{P_{max}}{P_{min}} = \frac{\left(|\Psi_1| / |\Psi_2| + 1\right)^2}{\left(|\Psi_1| / |\Psi_2| - 1\right)^2} = 2.25 \tag{32}$$