Solved Problems on the Particle Nature of Matter

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Given here are solutions to 5 problems on the particle nature of matter.

The solutions were used as a learning-tool for students in the introductory undergraduate course Physics 200 *Relativity and Quanta* given by Malcolm McMillan at UBC during the 1998 and 1999 Winter Sessions. The solutions were prepared in collaboration with Charles Asman and Adam Monaham who were graduate students in the Department of Physics at the time.

The problems are from Chapter 3 *The Particle Nature of Matter* of the course text *Modern Physics* by Raymond A. Serway, Clement J. Moses and Curt A. Moyer, Saunders College Publishing, 2nd ed., (1997).

Coulomb's Constant and the Elementary Charge

When solving numerical problems on the particle nature of matter it is useful to note that the product of Coulomb's constant

$$k = 8.9876 \times 10^9 \text{ m}^2/\text{ C}^2 \tag{1}$$

and the square of the elementary charge

$$e = 1.6022 \times 10^{-19} \text{ C} \tag{2}$$

is

$$ke^2 = 1.4400 \text{ eV nm} = 1.4400 \text{ keV pm} = 1.4400 \text{ MeV fm} \tag{3}$$

where

$$\text{eV} = 1.6022 \times 10^{-19} \text{ J} \tag{4}$$

Breakdown of the Rutherford Scattering Formula: Radius of a Nucleus

Problem 3.9, page 39

It is observed that α particles with kinetic energies of 13.9 MeV or higher, incident on copper foils, do not obey Rutherford's $(\sin\phi/2)^{-4}$ scattering formula.

• Use this observation to estimate the radius of the nucleus of a copper atom. Assume that the nucleus remains fixed in a head-on collision with an α particle.

Solution

Rutherford's $(\sin\phi/2)^{-4}$ scattering formula governs the scattering of an α particle by an atomic nucleus when the interaction between the α particle and the nucleus is due entirely to the Coulomb force.

At the distance d of closest approach in a head-on collision, the kinetic energy of the α particle is zero and therefore, by conservation of energy, the incident kinetic energy K of the α particle is equal to the potential energy of the system:

$$K = \frac{k(Ze)(2e)}{d} \tag{5}$$

where Z is the atomic number of the nucleus.

Rutherford's formula breaks down when the α particle is close enough to the nucleus to be influenced by the strong nuclear force. The distance d calculated with the lowest kinetic energy K at which the Rutherford's formula breaks down is an estimate of the radius of the nucleus.

For copper, $Z = 29$ and $K = 13.9\,\text{MeV}$, so $d = 6.00\,\text{fm}$. The value of the radius calculated using $1.3A^{1/3}\,\text{fm}$ with $A = 64$ is 5.2 fm.

Bohr's Model of the Atom: Radii and Speeds

Problem 3.14, page 39

- Use text Eq. (3.35):

$$r_n = \frac{\hbar^2 n^2}{m_e k e^2 Z} = \frac{a_0 n^2}{Z} = \frac{52.9 n^2}{Z}\ \text{pm} \tag{6}$$

to calculate the radius of the first, second and third Bohr orbits of hydrogen; $a_0 = \hbar^2/(m_e k e^2) = 52.9$ pm is the Bohr radius of hydrogen. That is, $a_0 = r_1$ when $Z = 1$.

- Determine the electron's speed in the same three orbits. Is a relativistic correction necessary? Explain.

Solution

Eq. (6) gives the radius r_n of the nth Bohr orbit of a single electron orbiting a fixed nucleus of charge $+Ze$. For hydrogen ($Z = 1$), Eq. (6) yields

$$\begin{aligned} r_1 &= 52.9\ \text{pm} && (7)\\ r_2 &= 212\ \text{pm} && (8)\\ r_3 &= 476\ \text{pm} && (9) \end{aligned}$$

It follows using text Eq. (3.24):

$$m_e v r = n\hbar \tag{10}$$

that the speed v_n of an electron in the nth Bohr orbit is

$$v_n = \frac{\hbar}{n m_e a_0} = \frac{7.30 \times 10^{-3} c}{n} \tag{11}$$

so

$$\begin{aligned} v_1 &= 7.30 \times 10^{-3} c && (12)\\ v_2 &= 3.65 \times 10^{-3} c && (13)\\ v_3 &= 2.43 \times 10^{-3} c && (14) \end{aligned}$$

These speeds are all very much less than c so relativistic corrections are small. Energy corrections are of the order of μeV.

Bohr's Model of the Atom: Photon Emission

Problem 3.24, page 140

A hydrogen atom originally at rest in the $n = 3$ state decays to the ground state with the emission of a photon.

- Calculate the wavelength of the emitted photon.
- Estimate the recoil momentum of the atom. Where does this energy come from?

Solution

Text Eq. (3.33):

$$\frac{1}{\lambda} = R\left(\frac{1}{n_f^2} - \frac{1}{n_i^2}\right) \tag{15}$$

where

$$R = \frac{ke^2}{2a_0hc} = 1.0973 \times 10^7\,\mathrm{m}^{-1} = 1/91.13\,\mathrm{nm} \tag{16}$$

gives the wavelength λ of the photon emitted when a hydrogen atom initially at rest in the n_i state decays to the n_f state.

It follows that in the decay from $n = 3$ state to the ground state ($n = 1$), the wavelength λ of the emitted photon is

$$\lambda = \frac{9}{8R} = 103\,\mathrm{nm}. \tag{17}$$

The energy E and magnitude p of the momentum of the emitted photon are

$$E = \frac{hc}{\lambda} = 12.0\,\mathrm{eV} \tag{18}$$

$$p = \frac{E}{c} = 12.0\,\mathrm{eV}/c. \tag{19}$$

Momentum is conserved in the process and therefore p is also the magnitude of the momentum of the recoiling hydrogen atom. The kinetic energy K of the recoiling hydrogen atom is

$$K = \frac{p^2}{2m_H} = 77.2\,\mathrm{neV}. \tag{20}$$

Eq. (17) assumes that all the transition energy is carried off by the photon and that $K = 0$. Comparison of Eqs. (18) and (20) shows that this is a reasonable assumption.

Bohr's Model of the Atom: Muonic Atom

Problem 3.33, page 141

A muon is a particle with a charge equal to that of an electron and a mass equal to 207 times the mass of the electron.

Muonic lead is formed when ^{208}Pb captures a muon to replace an electron. Assume that the muon moves in such a small orbit that it sees a nuclear charge Z=82.

- According to Bohr's theory, what are the radius and energy of the ground state of muonic lead? Use the concept of reduced mass in solving the problem.

Solution

Eq. (6) gives the radius of the nth Bohr orbit and text Eq. (3.36):

$$E_n = -\frac{ke^2}{2a_0}\left(\frac{Z^2}{n^2}\right) = -\left(\frac{13.58Z^2}{n^2}\right)\,\mathrm{eV} \tag{21}$$

gives the energy of the nth Bohr level for a single electron orbiting a fixed nucleus of charge $+Ze$.

The model may be extended further when the electron is replaced by a particle of mass m and charge e. In this case, Eqs. (6) and (21) are replaced by

$$r_n = \left(\frac{m_e}{\mu}\right)\left(\frac{52.9n^2}{Z}\right) \text{ pm} \tag{22}$$

$$E_n = -\left(\frac{\mu}{m_e}\right)\left(\frac{13.58Z^2}{n^2}\right) \text{ eV} \tag{23}$$

where

$$\mu = \frac{mm_N}{m + m_N} \tag{24}$$

where m_N is the mass of the nucleus; μ is the reduced mass of the orbiting particle and the nucleus.

A negative muon has charge e and mass $m = 207m_e$. Muonic lead is formed when ^{208}Pb captures a negative muon to replace an electron. Assuming that the muon moves in such a small orbit that it only feels a positive charge corresponding to $Z = 82$ (i.e., that screening effects caused by the other electrons surrounding the nucleus are negligible), then Eqs. (22) and (23) with $n = 1$ give the radius r_1 and ground state energy E_1 of the ground state of muonic lead according to Bohr theory:

$$r_1 = 3.1 \text{ fm} \tag{25}$$

$$E_1 = -19 \text{ MeV}. \tag{26}$$

Nuclear physics studies indicate that the nuclear radius $r_{nucleus}$ is given approximately by

$$r_{nucleus} = 1.3\ A^{\frac{1}{3}} \text{ fm} = 7.7 \text{ fm when } A = 207. \tag{27}$$

The Bohr model prediction (Eq. (25)) is of the right order of magnitude but too small; the assumption that the muon feels a positive charge corresponding to $Z = 82$ is incorrect. The radius r of the ground state of muonic lead is equal to the nuclear radius R when the effective charge in the Bohr model corresponds to $Z = 33$. The calculations indicate that muonic atoms can be used to probe the atomic nucleus.

Bohr's Model of the Atom: Positronium

Problem 3.35, page 141

Positronium is a hydrogen-like atom consisting of a positron (a positively charged electron) and an electron revolving around each other.

• Using the Bohr model, find the allowed radii (relative to the center of mass of the two particles) and the allowed energies of the system. Use the concept of reduced mass in solving the problem.

Solution

Eqs. (22) and (23) with $m = m_N = m_e$ and $Z = 1$ give the Bohr model values for the allowed radii r_n and energies E_n of positronium:

$$r_n = 105.8n^2 \text{ pm} \tag{28}$$

$$E_n = -\frac{6.79}{n^2} \text{ eV} \tag{29}$$