Solved Problems in the Quantum Theory of Light

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Given here are solutions to 7 problems in the Quantum Theory of Light.

The solutions were used as a learning-tool for students in the introductory undergraduate course Physics 200 *Relativity and Quanta* given by Malcolm McMillan at UBC during the 1998 and 1999 Winter Sessions. The solutions were prepared in collaboration with Charles Asman and Adam Monaham who were graduate students in the Department of Physics at the time.

The problems are from Chapter 2 *The Quantum Theory of Light* of the course text *Modern Physics* by Raymond A. Serway, Clement J. Moses and Curt A. Moyer, Saunders College Publishing, 2nd ed., (1997).

Planck's Constant and the Speed of Light.

When solving numerical problems in the Quantum Theory of Light it is useful to note that the product of Planck's constant

$$h = 6.6261 \times 10^{-34}\ \mathrm{J\,s} \tag{1}$$

and the speed of light

$$c = 2.9979 \times 10^{8}\ \mathrm{m\,s^{-1}} \tag{2}$$

is

$$hc = 1239.8\,\mathrm{eV\ nm} = 1239.8\,\mathrm{keV\ pm} = 1239.8\,\mathrm{MeV\ fm} \tag{3}$$

where

$$\mathrm{eV} = 1.6022 \times 10^{-19}\ \mathrm{J} \tag{4}$$

Wien's Displacement Law

Problem 2.2, page 92

The temperature of your skin is approximately 35^oC.

- What is the wavelength at which the peak occurs in the radiation emitted from your skin?

Solution

Wien's displacement law, given by text Eq. (2.6):

$$\lambda_{max} T = 2898\ \mu\mathrm{m} \cdot \mathrm{K} \tag{5}$$

gives the wavelength λ_{max} of maximum power emission of a black body at temperature T. Thus, $\lambda_{max} = 9.40\ \mu$m when $T = 35^o$C.

Photons from a Radio Transmitter

Problem 2.9, page 93

An FM radio transmitter has a power output of 100 kW and operates at a frequency of 94 MHz.

• How many photons per second does the transmitter emit?

Solution

A photon of frequency 94 MHz has energy

$$E = hf = 6.23 \times 10^{-26}\ \mathrm{J} \tag{6}$$

The radio transmitter emits energy at a rate of 100 kJ/s so

$$\frac{\#\ \text{photons}}{\text{unit time}} = \frac{\text{energy/unit time}}{\text{energy/photon}} = 1.61 \times 10^{30}\ \mathrm{s}^{-1}. \tag{7}$$

Photoelectric Effect

Problem 2.19, page 93

A light source of wavelength λ illuminates a metal and ejects photoelectrons with a maximum kinetic energy of 1.00 eV. A second light source with half the wavelength of the first ejects photoelectrons with a maximum kinetic energy of 4.00 eV.

• Determine the work function of the metal.

Solution

Text Eq. (2.24):

$$K_{max} = \frac{hc}{\lambda} - \phi \tag{8}$$

relates the maximum kinetic energy K_{max} of a photoelectron with the wavelength λ of the light producing the photoelectron and the work function ϕ of the metal.

It was with this equation that Einstein introduced the notion of light quanta in 1905 and for which he received the Nobel Prize for Physics in 1922.

Let λ_1 and λ_2 be the wavelengths of the light emitted by the first and second sources, respectively, and let K_1 and K_2 be the maximum kinetic energies of the corresponding photoelectrons.

It follows from Eq. (8) that

$$K_1 = \frac{hc}{\lambda_1} - \phi \tag{9}$$

$$K_2 = \frac{hc}{\lambda_2} - \phi \tag{10}$$

where $\lambda_2 = 0.5\,\lambda_1$, so

$$K_2 - 2K_1 = \phi \tag{11}$$

and therefore $\phi = 2\,\mathrm{eV}$ when $K_1 = 1\,\mathrm{eV}$ and $K_2 = 4\,\mathrm{eV}$.

Compton Effect

Problem 2.24, page 94

X-rays with an energy of 300 keV undergo Compton scattering with a target.

• If the scattered X-rays are detected at 30° relative to the incident X-rays, determine the Compton shift at this angle, the energy of the scattered X-ray, and the energy of the recoiling electron.

Solution

Text Eq. (2.28) gives the increase in a photon's wavelength (the Compton shift) when it is scattered through an angle θ by an electron:

$$\lambda' - \lambda = \lambda_e(1 - \cos\theta) \tag{12}$$

where

$$\lambda_e = \frac{h}{m_e c} = 2.43\,\text{pm} \tag{13}$$

is the Compton wavelength of the electron and $m_e = 511\,\text{keV}/c^2$ is the mass of the electron. It follows that the Compton shift is 0.325 pm when $\theta = 30°$.

The energy E' of the scattered photon is

$$E' = \frac{hc}{\lambda'} \tag{14}$$

and

$$\lambda = \frac{hc}{E} \tag{15}$$

with E=300 keV is the wavelength of the incoming photon. It follows that $E' = 278\,\text{keV}$. By conservation of energy, the energy lost by the photon in the collision is converted into kinetic energy K of the recoiling electron so $K = 22\,\text{keV}$.

Derivation of the Compton Formula

Problem 2.32, page 95

• Derive the Compton formula Eq. (12).

Solution

A photon with momentum $p = h/\lambda$ collides with a stationary electron. The photon scatters at an angle θ from the incident direction of the photon with momentum $p' = hc/\lambda'$. The electron scatters at an angle ϕ from the incident direction of the photon with momentum p_e and energy $E_e = \sqrt{p_e^2c^2 + m^2c^4}$.

In the collision, energy is conserved:

$$pc + m_e c^2 = p'c + E_e \tag{16}$$

and so is momentum in the direction of the incident photon:

$$p = p'\cos\theta + p_e\cos\phi \tag{17}$$

and perpendicular to the direction of the incident photon:

$$p'\sin\theta = p_e\sin\phi \tag{18}$$

The above equations may be simplified to

$$\frac{1}{\lambda} + \frac{1}{\lambda_c} = \frac{1}{\lambda'} + \sqrt{\frac{1}{{\lambda_e}^2} + \frac{1}{\lambda_c^2}} \tag{19}$$

$$\frac{1}{\lambda} = \frac{\cos\theta}{\lambda'} + \frac{\cos\phi}{\lambda_e} \tag{20}$$

$$\frac{\sin\theta}{\lambda'} = \frac{\sin\phi}{\lambda_e} \tag{21}$$

where we have written $p_e = h/\lambda_e$.

We eliminate ϕ and λ_e from these equations. Eliminating ϕ from last two equations yields

$$\frac{1}{\lambda_e^2} = \frac{1}{\lambda^2} + \frac{1}{\lambda'^2} - \frac{2\cos\theta}{\lambda\lambda'} \tag{22}$$

which, with Eq. (19), yields Eq. (12).

Comments on the Compton Formula.

The derivation of the Compton formula in 1922 was the first application of the idea that the momentum p of a photon is related to its wavelength λ by $p = h/\lambda$.

The reconciliation of this "wave-particle duality" was resolved with the subsequent invention of Quantum Mechanics and the development Quantum Electrodynamics.

Quantum Electrodynamics (QED) is the relativistic quantum theory of Maxwell's Equations of Electrodynamics and systems of electrons, positrons and photons. QED is the most successful relativistic quantum theory ever developed. There are no known discrepancies between QED and experiment.

Comparing the Compton Effect and the Photoelectric Effect

Problem 2.37, page 95

In a Compton collision with an electron, a photon of violet light ($\lambda = 400$ nm) is backward scattered through an angle 180°.

- How much energy is transferred to the electron in this collision?
- Compare the result with the energy the electron would acquire in a photoelectric process with the same photon.
- Could violet light eject electrons from a metal by Compton collision? Explain.

Solution

Eq. (12) with Eq. (13) gives the shift in the wavelength of a photon when it is scattered through an angle θ by an electron. The wavelength shift (the Compton shift) is 4.86 pm when $\theta = 180°$.

The energy lost by the photon during the collision is

$$\frac{hc}{\lambda} - \frac{hc}{\lambda'} = hc\frac{\lambda' - \lambda}{\lambda\lambda'} \simeq hc\frac{\lambda' - \lambda}{\lambda^2} = 37.7\ \mu\text{eV}. \tag{23}$$

By conservation of energy, this energy is converted into kinetic energy K_c of the recoiling electron so $K_c =$ 37.7 μeV.

In a photoelectric process, the entire energy of the incident photon is converted into kinetic energy K_p of the electron:

$$K_p = \frac{hc}{\lambda} = 3.10\ \text{eV}. \tag{24}$$

The work function for a metal is typically on the order of a few eV. A Compton collision involving violet light could not eject an electron from a metal surface since the kinetic energy K_c transferred to the electron is much too small.

Bragg Equation

Problem 2.38, page 95

X-rays of wavelength $\lambda = 62.6\,\text{pm}$ from a molybdenum target are incident on a crystal with atomic plane spacing $d = 400\,\text{pm}$.

- Determine the three smallest angles at which intensity maxima occur in the diffracted beam.

Solution

Constructive interference occurs as given by text Eq. (2.26) (the Bragg Equation):

$$n\lambda = 2d\sin\theta \qquad (n = 1, 2, 3, \ldots) \tag{25}$$

where n is the order of the intensity maximum and θ is the angle for an intensity maximum in the diffracted beam as measured from the top atomic plane.

The three smallest angles at which intensity maxima occur are

$$\theta_1 = \sin^{-1}\left(\frac{\lambda}{2d}\right) = 4.49^\circ \tag{26}$$

$$\theta_2 = \sin^{-1}\left(\frac{\lambda}{d}\right) = 9.00^\circ \tag{27}$$

$$\theta_3 = \sin^{-1}\left(\frac{3\lambda}{2d}\right) = 13.60^\circ \tag{28}$$