# QUANTUM LEAPS AND BOUNDS 

## Scattering Theory

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## Preface

The six volumes of notes Quantum Leaps and Bounds (QLB) form the basis of the introductory graduate quantum mechanics course I have given in the Department of Physics at the University of British Columbia at various times since 1973.

The six volumes of $Q L B$ are

- Introductory Topics: a collection of miscellaneous topics in introductory quantum mechanics
- Scattering Theory: an introduction to the basic ideas of quantum scattering theory by considering the scattering of a relativistic spinless particle from a fixed target
- Quantum Mechanics in Fock Space: an introduction to the second-quantization description of nonrelativistic many-body systems
- Relativistic Quantum Mechanics: an introduction to incorporating special relativity in quantum mechanics
- Some Lorentz Invariant Systems: some examples of systems incorporating special relativity in quantum mechanics
- Relativistic Quantum Field Theory: an elementary introduction to the relativistic quantum field theory of spinless bosons, spin $\frac{1}{2}$ fermions and antifermions and to quantum electrodynamics, the relativistic quantum field theory of electrons, positrons and photons
$Q L B$ assumes no familiarity with relativistic quantum mechanics. It does assume that students have taken undergraduate courses in nonrelativistic quantum mechanics which include discussion of the nonrelativistic Schrodinger equation and the solutions of some standard problems (e.g., the one-dimensional harmonic oscillator and the hydrogen atom) and perturbation theory and other approximation
methods.
$Q L B$ assumes also that students will take other graduate courses in condensed matter physics, nuclear and particle physics and relativistic quantum field theory. Accordingly, our purpose in $Q L B$ is to introduce some basic ideas and formalism and thereby give students sufficient background to read the many excellent texts on these subjects.

I am happy to have this opportunity to thank my friends and colleagues I.K. Affleck, R. Barrie, B. Bergersen, M. Bloom, J. Feldman, D.H. Hearn, W.W. Hsieh, R.I.G. Hughes, F.A. Kaempffer, A.H. Monahan, P.A. Kalyniak, R.H. Landau, E.L. Lomon, W. Opechowski, M.H.L. Pryce, A. Raskin, P. Rastall, L. Rosen, G.W. Semenoff, L. Sobrino, F. Tabakin, A.W. Thomas, W.G. Unruh, E.W. Vogt, G.M. Volkoff and N. Weiss for sharing their knowledge of quantum mechanics with me.

I also thank my wife, Henrietta, for suggesting the title for these volumes of notes. Quite correctly, she found my working title Elements of Intermediate Quantum Mechanics a bore.

## SCATTERING THEORY

## Chapter 1

INTRODUCTORY REMARKS

Much of our understanding of the structure of matter including the existence of the atomic nucleus and the existence of quarks comes from analyses of scattering experiments.

Our purpose in this part of $Q L B$ is to introduce the basic ideas of quantum scattering theory for the simplest case, the scattering of a spinless particle by a fixed target, and thereby give the reader sufficient background to read research papers and the many excellent texts on the subject.

Goldberger and Watson (1964), Landau (1996), Morse and Feshbach (1953), Newton (1966), Taylor (1972) and Weinberg (1995) are recommended for further reading.

Monahan (1995) is recommended for the theory of scattering of two relativistic particles with spin, scattering in relativistic three-particle systems and scattering in a relativistic two-body system where a third particle can be created.

In the six chapters which follow, the quantum mechanics of a single spinless particle is reviewed in Chapter 2, scattering states and Møller operators are defined and discussed in Chapter 3, the scattering operator is defined and discussed in Chapter 4, the $T$ operator is defined and discussed in Chapter 5, standard methods for solving the scattering problem are given in Chapters 5 and 6 and extensions to complex momentum and energy and complex angular momentum are given in Chapter 7. A construction of a familty of interaction potentials which are equivalent for scattering is given in Appendix A and some properties of Riccati functions which are important for solving the scattering problem are given in Appendix B.

### 2.1 Introductory remarks

In this chapter we consider a relativistic spinless particle of rest mass $m$ interacting with a fixed target. The material presented in this chapter is a review and extension of QLB: Some Lorentz Invariant Systems Chapter 2 where the quantum mechanics of a free relativistic spinless particle is discussed in detail.

The machinery of quantum mechanics is set up and properties of the interaction potential between the particle and target are discussed in Section 2.2; coordinate/momentum and angular momentum bases for the Hilbert space are discussed in Sections 2.3 and 2.4 and Green's operators are defined and discussed in Section 2.5 .

### 2.2 Fundamentals

The physical system is a spinless particle of rest mass $m$ interacting with a target fixed at the origin of the coordinate system. The fundamental dynamical variables of the system are the Cartesian coordinates and momenta

$$
\begin{equation*}
X^{1}, X^{2}, X^{3}, P^{1}, P^{2}, P^{3} \tag{2.1}
\end{equation*}
$$

which satisfy the fundamental quantum conditions

$$
\begin{gather*}
{\left[X^{j}, X^{k}\right]=0}  \tag{2.2}\\
{\left[P^{j}, P^{k}\right]=0}  \tag{2.3}\\
{\left[X^{j}, P^{k}\right]=i \hbar \delta_{j k}} \tag{2.4}
\end{gather*}
$$

where $j, k=1,2,3$.

The state of the particle $\mid \psi(t)>$ at time $t$ is

$$
\begin{equation*}
|\psi(t)>=U(t)| \psi> \tag{2.5}
\end{equation*}
$$

where $\mid \psi>$ is the state at time zero and $U(t)$ is the evolution operator, that is,

$$
\begin{equation*}
U(t)=e^{-i H t / \hbar} \tag{2.6}
\end{equation*}
$$

where

$$
\begin{equation*}
H=H_{0}+V \tag{2.7}
\end{equation*}
$$

where

$$
\begin{equation*}
H_{0}=\sqrt{P^{2} c^{2}+m^{2} c^{4}} \tag{2.8}
\end{equation*}
$$

where $P^{2}=\vec{P} \cdot \vec{P}$ and

$$
\begin{equation*}
V=V\left(X^{1}, X^{2}, X^{3}, P^{1}, P^{2}, P^{3}\right) \equiv V(\vec{X}, \vec{P}) \tag{2.9}
\end{equation*}
$$

The potential $V$ given by (2.9) specifies the interaction of the particle with the fixed target. $V$ is a local potential if

$$
\begin{equation*}
V=V(\vec{X}) \tag{2.10}
\end{equation*}
$$

$V$ is a central potential if

$$
\begin{equation*}
V=V(R) \tag{2.11}
\end{equation*}
$$

where $R=\sqrt{\vec{X} \cdot \vec{X}}$.

## Comments

## 1. Relativistic kinematics

(2.8) indicates that the speed of the particle is restricted only the the principle of special relativity.

Relativistic kinematics will be used throughout this material unless stated otherwise.

## 2. Nonrelativistic kinematics

When nonrelativistic kinematics are used (2.8) is replaced by

$$
\begin{equation*}
H_{0}=\frac{P^{2}}{2 m} \tag{2.12}
\end{equation*}
$$

## 3. Restrictions on the interaction potential

We assume that the physical system is invariant under rotations, space inversion and time reversal. As discussed in QLB: Relativistic Quantum Mechan$i c s$, it follows that $V$ must satisfy

$$
\begin{equation*}
\mathrm{T} V \mathrm{~T}^{\dagger}=V \tag{2.15}
\end{equation*}
$$

where P is the space-inversion operator, T is the time-reversal operator and $R^{j}(\theta)$ is the rotation operator for a rotation by $\theta$ about the $j$-axis. That is,

$$
\begin{equation*}
R^{j}(\theta)=e^{-i L^{j} \theta / \hbar} \tag{2.16}
\end{equation*}
$$

$$
\begin{align*}
& R^{j}(\theta) V R^{\dagger j}(\theta)=V  \tag{2.13}\\
& \mathrm{P} V \mathrm{P}^{\dagger}=V \tag{2.14}
\end{align*}
$$

where

$$
\begin{equation*}
\vec{L}=\vec{X} \times \vec{P} \tag{2.17}
\end{equation*}
$$

is the angular momentum operator. It follows from (2.13) that

$$
\begin{equation*}
\left[V, L^{j}\right]=0 \tag{2.18}
\end{equation*}
$$

that is, angular momentum is conserved and it follows from (2.14) that parity is conserved.

It follows from (2.13) to (2.15) that

$$
\begin{equation*}
V(\vec{X}, \vec{P})=V\left(\vec{X}_{R^{j}}, \vec{P}_{R^{j}}\right)=V(-\vec{X},-\vec{P})=V(\vec{X},-\vec{P}) \tag{2.19}
\end{equation*}
$$

where

$$
\begin{align*}
\vec{X}_{R^{j}} & =R^{j}(\theta) \vec{X} R^{j \dagger}(\theta)  \tag{2.20}\\
\vec{P}_{R^{j}} & =R^{j}(\theta) \vec{P} R^{j \dagger}(\theta) \tag{2.21}
\end{align*}
$$

(2.19) holds if $V$ is a central potential (2.11).

### 2.3 Coordinate/momentum bases

The operators $X^{1}, X^{2}, X^{3}$ and $P^{1}, P^{2}, P^{3}$ each form a complete set of compatible observables. We denote their simultaneous eigenkets by

$$
\begin{align*}
& |\vec{x}\rangle=\mid x^{1}, x^{2}, x^{3}>  \tag{2.22}\\
& |\vec{p}\rangle=\mid p^{1}, p^{2}, p^{3}> \tag{2.23}
\end{align*}
$$

respectively. These eigenkets may be used as bases for the Hilbert space. That is,

$$
\begin{gather*}
X^{j}=\int d^{3} x\left|\vec{x}>x^{j}<\vec{x}\right|  \tag{2.24}\\
P^{j}=\int d^{3} p\left|\vec{p}>p^{j}<\vec{p}\right|  \tag{2.25}\\
1=\int d^{3} x|\vec{x}><\vec{x}|=\int d^{3} p|\vec{p}><\vec{p}| \tag{2.26}
\end{gather*}
$$

$$
\begin{align*}
& <\vec{x}|\vec{y}\rangle=\delta(\vec{x}-\vec{y})  \tag{2.27}\\
& <\vec{p} \mid \vec{q}>=\delta(\vec{p}-\vec{q}) \tag{2.28}
\end{align*}
$$

where $\delta(\vec{x}-\vec{y})$ and $\delta(\vec{p}-\vec{q})$ are 3-dimensional Dirac delta functions. ${ }^{1}$
$1 \quad \delta(\vec{x}-\vec{y})=\delta\left(x^{1}-y^{1}\right) \delta\left(x^{2}-y^{2}\right) \delta\left(x^{3}-y^{3}\right)$

$$
\begin{equation*}
|\psi(\vec{x}, t)|^{2} d x^{1} d x^{2} d x^{3} \tag{2.32}
\end{equation*}
$$

is the probability that the particle is in the volume $d x^{1} d x^{2} d x^{3}$ about the point $\left(x^{1}, x^{2}, x^{3}\right)$ at time $t$.

## 4. Momentum-space wave function

The momentum-space wave function for the particle is

$$
\begin{equation*}
\psi(\vec{p}, t)=<\vec{p} \mid \psi(t)> \tag{2.33}
\end{equation*}
$$

and

$$
\begin{equation*}
|\psi(\vec{p}, t)|^{2} d p^{1} d p^{2} d p^{3} \tag{2.34}
\end{equation*}
$$

is the probability that the particle has momentum in the volume $d p^{1} d p^{2} d p^{3}$ about the point $\left(p^{1}, p^{2}, p^{3}\right)$ at time $t$.

## 5. Relationship between wave functions

## Comments

## 1. Transformation function

It follows from (2.2) to (2.4) that

$$
\begin{equation*}
<\vec{x} \left\lvert\, \vec{p}>=\left(\frac{1}{2 \pi \hbar}\right)^{\frac{3}{2}} e^{i \vec{k} \cdot \vec{x}}\right. \tag{2.29}
\end{equation*}
$$

where

$$
\begin{equation*}
\vec{p}=\hbar \vec{k} \tag{2.30}
\end{equation*}
$$

## 2. Notation

$\vec{p}$ will usually be replaced by $\hbar \vec{k}$ as per (2.30).
And we write $f(\vec{k})$ in place of $f(\vec{p})$ if it is convenient.
We choose not to use units where $\hbar=c=1$ for pedagogic reasons.

## 3. Coordinate-space wave function

The coordinate-space wave function for the particle is

$$
\begin{equation*}
\psi(\vec{x}, t)=<\vec{x} \mid \psi(t)> \tag{2.31}
\end{equation*}
$$

and

It follows from (2.29) that

$$
\begin{align*}
& \psi(\vec{x}, t)=\left(\frac{\hbar}{2 \pi}\right)^{\frac{3}{2}} \int d^{3} k e^{i \vec{k} \cdot \vec{x}} \psi(\vec{k}, t)  \tag{2.35}\\
& \psi(\vec{k}, t)=\left(\frac{1}{2 \pi \hbar}\right)^{\frac{3}{2}} \int d^{3} x e^{-i \vec{k} \cdot \vec{x}} \psi(\vec{x}, t) \tag{2.36}
\end{align*}
$$

## 6. Spectral decomposition of the free-particle Hamiltonian

The free-particle Hamiltonian (2.8) is a function of momentum so

$$
\begin{equation*}
H_{0}=\int d^{3} p\left|\vec{p}>\epsilon_{p}<\vec{p}\right| \tag{2.37}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon_{p}=\sqrt{p^{2} c^{2}+m^{2} c^{4}} \tag{2.38}
\end{equation*}
$$

where $p^{2}=\vec{p} \cdot \vec{p}$.

## 7. Lorentz factor

The Lorentz factor $\gamma$ is

$$
\begin{equation*}
\gamma=\frac{\epsilon_{p}}{m c^{2}}=\frac{\omega}{c \mu_{C}}=\sqrt{1+\frac{k^{2}}{\mu_{C}^{2}}}=\frac{1}{\sqrt{1-v^{2} / c^{2}}} \tag{2.39}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon_{p}=\hbar \omega \tag{2.40}
\end{equation*}
$$

$$
\begin{equation*}
\mu_{C}=\frac{m c}{\hbar} \tag{2.41}
\end{equation*}
$$

and $v^{2}=\vec{v} \cdot \vec{v}$ where

$$
\begin{equation*}
\vec{v}=\frac{c^{2} \vec{p}}{\epsilon_{p}} \tag{2.42}
\end{equation*}
$$

$\mu_{C}$ is the inverse of the Compton wavelength of the particle and $\vec{v}$ is the velocity of the particle when its momentum is $\vec{p}$.
8. Spectral decomposition of the free-particle evolution operator

The free-particle evolution operator $U_{0}(t)$ is defined as

$$
\begin{equation*}
U_{0}(t)=e^{-i H_{0} t / \hbar} \tag{2.43}
\end{equation*}
$$

It follows from (2.37) that

$$
\begin{equation*}
U_{0}(t)=\int d^{3} p\left|\vec{p}>e^{-i \epsilon_{p} t / \hbar}<\vec{p}\right| \tag{2.44}
\end{equation*}
$$

## 9. Form of the Hamiltonian

It follows using (2.37) that the Hamiltonian (2.7) may be expressed as

$$
\begin{align*}
H & =\int d^{3} p\left|\vec{p}>\epsilon_{p}<\vec{p}\right|+\int d^{3} x d^{3} y|\vec{x}><\vec{x}| V|\vec{y}><\vec{y}| \\
& =\int d^{3} p\left|\vec{p}>\epsilon_{p}<\vec{p}\right|+\int d^{3} p d^{3} q|\vec{p}><\vec{p}| V|\vec{q}><\vec{q}| \tag{2.45}
\end{align*}
$$

$<\vec{x}|V| \vec{y}>$ and $\langle\vec{p}| V|\vec{q}\rangle$ are, respectively, the coordinate-space and momentum-space representatives of the interaction potential.

If the potential is local, that is, if (2.10) holds, then

$$
\begin{equation*}
<\vec{x}|V| \vec{y}\rangle=V(\vec{x}) \delta(\vec{x}-\vec{y}) \tag{2.46}
\end{equation*}
$$

## 10. Nonrelativistic free-particle Hamiltonian

When nonrelativistic kinematics are used (2.37) is replaced by

$$
\begin{equation*}
H_{0}=\int d^{3} p\left|\vec{p}>\frac{p^{2}}{2 m}<\vec{p}\right| \tag{2.47}
\end{equation*}
$$

It follows from (2.29), (2.35) and (2.47) that

$$
\begin{align*}
<\vec{x}\left|H_{0}\right| \psi(t)> & =\int d^{3} p<\vec{x}\left|\vec{p}>\frac{p^{2}}{2 m}<\vec{p}\right| \psi(t)> \\
& =-\frac{\hbar^{2}}{2 m} \nabla^{2} \psi(\vec{x}, t) \tag{2.48}
\end{align*}
$$

## 11. Nonrelativistic Schrodinger equation

When nonrelativistic kinematics are used in the coordinate-space Schrodinger equation

$$
\begin{equation*}
\left.<\vec{x}|H| \psi(t)>=i \hbar \frac{\partial}{\partial t}<\vec{x} \right\rvert\, \psi(t)> \tag{2.49}
\end{equation*}
$$

it follows from (2.48) that, with a local potential (2.10), $\psi(\vec{x}, t)$ satisfies the nonrelativistic Schrodinger equation

$$
\begin{equation*}
-\frac{\hbar^{2}}{2 m} \nabla^{2} \psi(\vec{x}, t)+V(\vec{x}) \psi(\vec{x}, t)=i \hbar \frac{\partial \psi(\vec{x}, t)}{\partial t} \tag{2.50}
\end{equation*}
$$

### 2.4 Angular momentum bases

The operators $R, L^{2}, L_{z}$ and $P, L^{2}, L_{z}$ where

$$
\begin{gather*}
R=\sqrt{\vec{X} \cdot \vec{X}}  \tag{2.51}\\
P=\sqrt{\vec{P} \cdot \vec{P}}  \tag{2.52}\\
L^{2}=\vec{L} \cdot \vec{L}  \tag{2.53}\\
L_{z}=L^{3} \tag{2.54}
\end{gather*}
$$

each form a complete set of compatible observables. We denote their simultaneous eigenkets by

$$
\begin{align*}
& \mid r l m_{l}>  \tag{2.55}\\
& \left|p l m_{l}\right\rangle \tag{2.56}
\end{align*}
$$

respectively. These eigenkets may be used as bases for the Hilbert space. That
is,

$$
\begin{align*}
& R=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d r\left|r l m_{l}>r<r l m_{l}\right|  \tag{2.57}\\
& P=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>p<p l m_{l}\right| \tag{2.58}
\end{align*}
$$

$$
\begin{align*}
L^{2} & =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d r\left|r l m_{l}>l(l+1) \hbar^{2}<r l m_{l}\right| \\
& =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>l(l+1) \hbar^{2}<p l m_{l}\right| \tag{2.59}
\end{align*}
$$

$$
\begin{align*}
L_{z} & =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d r\left|r l m_{l}>m_{l} \hbar<r l m_{l}\right| \\
& =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>m_{l} \hbar<p l m_{l}\right| \tag{2.60}
\end{align*}
$$

$$
\begin{align*}
1 & =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d r\left|r l m_{l}><r l m_{l}\right| \\
& =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}><p l m_{l}\right| \tag{2.61}
\end{align*}
$$

$$
\begin{align*}
& <r l m_{l} \mid r^{\prime} l^{\prime} m_{l}^{\prime}>=\delta\left(r-r^{\prime}\right) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}  \tag{2.62}\\
& <p l m_{l} \mid p^{\prime} l^{\prime} m_{l}^{\prime}>=\delta\left(p-p^{\prime}\right) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}} \tag{2.63}
\end{align*}
$$

## Comments

## 1. Transformation functions

The transformation functions connecting the $|\vec{x}\rangle,|\vec{p}\rangle,\left|r l m_{l}\right\rangle$ and | $p l m_{l}>$ eigenkets are

$$
\begin{gather*}
<\vec{x} \left\lvert\, r^{\prime} l m_{l}>=\frac{\delta\left(r-r^{\prime}\right)}{r} Y_{l m_{l}}\left(\theta_{x}, \varphi_{x}\right)\right.  \tag{2.64}\\
<\vec{x} \left\lvert\, p l m_{l}>=i^{l} \sqrt{\frac{2}{\pi \hbar}} \frac{\widehat{j}(k r)}{r} Y_{l m_{l}}\left(\theta_{x}, \varphi_{x}\right)\right. \tag{2.65}
\end{gather*}
$$

$$
\begin{gather*}
\langle\vec{p}| p^{\prime} l m_{l}>=\frac{\delta\left(p-p^{\prime}\right)}{p} Y_{l m_{l}}\left(\theta_{p}, \varphi_{p}\right)  \tag{2.66}\\
<\vec{p} \left\lvert\, r l m_{l}>=(-i)^{l} \sqrt{\frac{2}{\pi \hbar^{3}}} \frac{\widehat{j}_{l}(k r)}{k} Y_{l m_{l}}\left(\theta_{p}, \varphi_{p}\right)\right. \tag{2.67}
\end{gather*}
$$

$$
\begin{equation*}
<r l m_{l} \left\lvert\, p l^{\prime} m_{l}^{\prime}>=i^{l} \sqrt{\frac{2}{\pi \hbar}} \widehat{j}_{l}(k r) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}\right. \tag{2.68}
\end{equation*}
$$

where $r=\sqrt{\vec{x} \cdot \vec{x}}$ in (2.64) and (2.65), $p=\sqrt{\vec{p} \cdot \vec{p}}$ in (2.66), $\widehat{j}_{l}(z)$ is a Riccati-Bessel function ${ }^{1}$ and $Y_{l m_{l}}(\theta, \varphi)$ is a spherical harmonic.

## 2. Partial-wave expansion of a plane wave

It follows using (2.29), (2.61), (2.64) and (2.67) that ${ }^{2}$

$$
\begin{gather*}
e^{i \vec{k} \cdot \vec{x}}=4 \pi \sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} i^{l} \frac{\widehat{j}_{l}(k r)}{k r} Y_{l m_{l}}^{*}\left(\theta_{k}, \varphi_{k}\right) Y_{l m_{l}}\left(\theta_{x}, \varphi_{x}\right) \\
=\sum_{l=0}^{\infty}(2 l+1) i^{l} \frac{\widehat{j}_{l}(k r)}{k r} P_{l}\left(\cos \theta_{k-x}\right) \tag{2.69}
\end{gather*}
$$

where $\theta_{k-x}$ is the angle between $\vec{k}$ and $\vec{x}$ and $P_{l}(z)$ is a Legendre polynomial.
(2.69) is the standard partial-wave expansion of a plane wave.

[^0]3. Coordinate-space/angular momentum wave function

The coordinate-space/angular momentum wave function for the particle is

$$
\begin{equation*}
\psi_{l m_{l}}(r, t)=<r \operatorname{lm}_{l} \mid \psi(t)> \tag{2.70}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\psi_{l m_{l}}(r, t)\right|^{2} d r \tag{2.71}
\end{equation*}
$$

is the probability that the particle is a distance from the origin between $r$ and $r+d r$ with angular momentum $l \hbar$ and z-component of angular momentum $m_{l} \hbar$ at time $t$.

It follows from (2.64) that the coordinate-space wave function (2.31) may be written as

$$
\begin{equation*}
\psi(\vec{x}, t)=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \frac{\psi_{l m_{l}}(r, t)}{r} Y_{l m_{l}}\left(\theta_{x}, \varphi_{x}\right) \tag{2.72}
\end{equation*}
$$

## 4. Momentum-space/angular momentum wave function

The momentum-space/angular momentum wave function for the particle is

$$
\begin{equation*}
\psi_{l m_{l}}(p, t)=<p l m_{l} \mid \psi(t)> \tag{2.73}
\end{equation*}
$$

and

$$
\begin{equation*}
\left|\psi_{l m_{l}}(p, t)\right|^{2} d p \tag{2.74}
\end{equation*}
$$

is the probability that the particle has magnitude of linear momentum between $p$ and $p+d p$ with angular momentum $l \hbar$ and $\mathbf{z}$-component of angular momentum $m_{l} \hbar$ at time $t$.

It follows from (2.66) that the momentum space wave function (2.33) may be written as

$$
\begin{equation*}
\psi(\vec{p}, t)=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \frac{\psi_{l m_{l}}(p, t)}{p} Y_{l m_{l}}\left(\theta_{p}, \varphi_{p}\right) \tag{2.75}
\end{equation*}
$$

## 5. Relationship of wave functions

It follows from (2.68) that the coordinate-space/angular momentum and momentum-space/angular momentum wave functions are related according to

$$
\begin{gather*}
\psi_{l m_{l}}(r, t)=i^{l} \sqrt{\frac{2 \hbar}{\pi}} \int_{0}^{\infty} d \widehat{j}_{l}(k r) \psi_{l m_{l}}(k, t)  \tag{2.76}\\
\psi_{l m_{l}}(k, t)=(-i)^{l} \sqrt{\frac{2}{\pi \hbar}} \int_{0}^{\infty} d r \widehat{j}_{l}(k r) \psi_{l m_{l}}(r, t) \tag{2.77}
\end{gather*}
$$

which equations are consistent with (B.27).

## 6. Spectral decomposition of the free-particle Hamiltonian

The free-particle Hamiltonian (2.8) is a function of $P$ defined by (2.52). It follows that

$$
\begin{equation*}
H_{0}=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>\epsilon_{p}<p l m_{l}\right| \tag{2.78}
\end{equation*}
$$

7. Spectral decomposition of the free-particle evolution operator

It follows from (2.78) that the free-particle evolution operator (2.43) may be expressed as

$$
\begin{equation*}
U_{0}(t)=\sum_{l=0}^{\infty} \sum_{m_{i}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>e^{-i \epsilon_{p} t / \hbar}<p l m_{l}\right| \tag{2.79}
\end{equation*}
$$

## 8. Form of the interaction potential

It follows using (2.18) that

$$
\begin{align*}
& <r l m_{l}|V| r^{\prime} l^{\prime} m_{l}^{\prime}>=v_{l}\left(r, r^{\prime}\right) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}  \tag{2.80}\\
& <p l m_{l}|V| p^{\prime} l^{\prime} m_{l}^{\prime}>=v_{l}\left(k, k^{\prime}\right) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}} \tag{2.81}
\end{align*}
$$

$v_{l}\left(r, r^{\prime}\right)$ and $v_{l}\left(k, k^{\prime}\right)$ are, respectively, the coordinate-space/angular momentum and momentum-space/angular momentum representatives of the interaction potential.

It follows from (2.68) that

$$
\begin{align*}
& v_{l}\left(k, k^{\prime}\right)=\frac{1}{\hbar} \int_{0}^{\infty} d r d r^{\prime} \widehat{j}_{l}(k r) v_{l}\left(r, r^{\prime}\right) \widehat{j}_{l}\left(k^{\prime} r^{\prime}\right)  \tag{2.82}\\
& v_{l}\left(r, r^{\prime}\right)=\hbar \int_{0}^{\infty} d k d k^{\prime} \widehat{j}_{l}(k r) v_{l}\left(k, k^{\prime}\right) \widehat{j}_{l}\left(k^{\prime} r^{\prime}\right) \tag{2.83}
\end{align*}
$$

## 9. Central potential

For the central potential (2.11)

$$
\begin{equation*}
v_{l}\left(r, r^{\prime}\right)=V(r) \delta\left(r-r^{\prime}\right) \tag{2.84}
\end{equation*}
$$

$$
\begin{equation*}
v_{l}\left(k, k^{\prime}\right)=\frac{1}{\hbar} \int_{0}^{\infty} d r \widehat{j}_{l}(k r) V(r) \widehat{j}_{l}\left(k^{\prime} r\right) \tag{2.85}
\end{equation*}
$$

## 10. Yukawa potential

$V(r)$ is a Yukawa potential if

$$
\begin{equation*}
V(r)=-g e^{-\mu r} / r \tag{2.86}
\end{equation*}
$$

The constants $g$ and $1 / \mu$ are the strength and range of the potential.

It follows from (2.85) and (B.28) that

$$
\begin{equation*}
v_{l}\left(k, k^{\prime}\right)=-\frac{g}{\pi \hbar} Q_{l}\left(\frac{k^{2}+k^{\prime 2}+\mu^{2}}{2 k k^{\prime}}\right) \tag{2.87}
\end{equation*}
$$

where $Q_{l}(z)$ is a Legendre function of the second kind.

## 11. Coulomb potential

The Coulomb potential

$$
\begin{equation*}
V(r)=Z_{1} Z_{2} e^{2} / r \tag{2.88}
\end{equation*}
$$

where $e$ is the charge of a proton is the special case of (2.86) when

$$
\begin{gather*}
\mu=0  \tag{2.89}\\
g=-Z_{1} Z_{2} e^{2} \tag{2.90}
\end{gather*}
$$

While the Coulomb potential decreases too slowly at large distances to satisfy the requirements of the scattering theory developed in Chapter 3, we will nevertheless recover the Rutherford cross-section formula in Chapter 6 via the special cases (2.89) and (2.90) of the Yukawa potential.

## 12. Superposition of Yukawa potentials

$V(r)$ is a superposition of Yukawa potentials if

$$
\begin{equation*}
V(r)=\int_{\mu}^{\infty} \frac{\rho(\sigma) e^{-\sigma r} d \sigma}{r} \tag{2.91}
\end{equation*}
$$

where $\rho(\sigma)$ is any arbitrary real function for which the integral converges.

## 13. Square-well potential

$V(r)$ is a square-well potential if

$$
\begin{align*}
V(r)=-V_{0} & & \text { when } & r \leq a  \tag{2.92}\\
& =0 & & \text { when }
\end{align*} \quad r>a
$$

The constants $V_{0}$ and $a$ are the strength and range of the potential.

## 14. Hard-sphere potential

$V(r)$ is a hard-sphere potential if

$$
\begin{array}{rlrl}
V(r) & =\infty & & \text { when } \\
& =0 & &  \tag{2.93}\\
& \text { when } & r>a
\end{array}
$$

The constant $a$ is the range of the potential.

## 15. Separable potential

The potential is a rank- 1 separable attractive potential if

$$
\begin{align*}
& v_{l}\left(k, k^{\prime}\right)=-v_{l}(k) v_{l}^{*}\left(k^{\prime}\right)  \tag{2.94}\\
& v_{l}\left(r, r^{\prime}\right)=-v_{l}(r) v_{l}^{*}\left(r^{\prime}\right) \tag{2.95}
\end{align*}
$$

It follows from (2.82) and (2.83) that

$$
\begin{align*}
& v_{l}(k)=\sqrt{\frac{1}{\hbar}} \int_{0}^{\infty} d r \widehat{j}_{l}(k r) v_{l}(r)  \tag{2.96}\\
& v_{l}(r)=\sqrt{\hbar} \int_{0}^{\infty} d k \widehat{j}_{l}(k r) v_{l}(k) \tag{2.97}
\end{align*}
$$

## 16. Boundary Condition Model

In The Boundary Condition Model the potential is assumed to vanish for distances greater than a boundary radius (which may be depend on the angular momentum) while at the boundary radius the logarithmic derivative of the coordinate-space/angular momentum wave function satisfies a boundary condition.

The Boundary Condition Model for nonrelativistic kinematics is equivalent to the rank-2 separable potential
$v_{l}\left(r, r^{\prime}\right)=\frac{\hbar^{2}}{2 m}\left[\frac{f_{l}}{r_{0_{l}}} \delta\left(r-r_{0_{l}}^{+}\right) \delta\left(r^{\prime}-r_{0_{l}}^{+}\right)+\delta\left(r-r_{0_{l}}^{-}\right) \delta^{1}\left(r^{\prime}-r_{0_{l}}^{-}\right)\right]$
where $f_{l} / r_{0_{l}}$ is the logarithmic derivative, $r_{0_{l}}$ is the boundary radius and $r_{0_{l}}^{ \pm}=r_{0_{l}} \pm 0 .{ }^{1}$

It follows from (2.82) that the momentum-space/angular momentum representative of (2.98) is

$$
\begin{equation*}
v_{l}\left(k, k^{\prime}\right)=\frac{\hbar}{2 m r_{0_{l}}}\left[f_{l} \widehat{j}_{l}(\alpha) \widehat{j}_{l}(\beta)-\beta \widehat{j}_{l}(\alpha) \hat{j}_{l}^{\prime}(\beta)\right] \tag{2.99}
\end{equation*}
$$

where prime on the right side means differentiation with respect to the argument and $\alpha=k r_{0_{l}}$ and $\beta=k^{\prime} r_{0_{l}}$

For further discussion of the Boundary Condition Model see Breit and Bouricius (1949) and Feshbach and Lomon (1956).

[^1]For further discussion of the separable potential (2.98) see McMillan (1961) and Lomon and McMillan (1963).

## 17. Expressions for the Hamiltonian

It follows from (2.78), (2.80) and (2.81) that the Hamiltonian (2.7) may be expressed as

$$
\begin{gather*}
H=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>\epsilon_{p}<p l m_{l}\right| \\
+\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d r d r^{\prime}\left|r l m_{l}>v_{l}\left(r, r^{\prime}\right)<r^{\prime} l m_{l}\right| \tag{2.100}
\end{gather*}
$$

$$
\begin{gather*}
H=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>\epsilon_{p}<p l m_{l}\right| \\
+\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p d p^{\prime}\left|p l m_{l}>v_{l}\left(k, k^{\prime}\right)<p^{\prime} l m_{l}\right| \tag{2.101}
\end{gather*}
$$

which forms show explicitly that angular momentum is conserved.

## 18. Nonrelativistic free-particle Hamiltonian

For nonrelativistic kinematics (2.78) is replaced by

$$
\begin{equation*}
H_{0}=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>\frac{p^{2}}{2 m}<p l m_{l}\right| \tag{2.102}
\end{equation*}
$$

It follows using (2.68), (2.76) and (2.102) that

$$
\begin{gather*}
<r l m_{l}\left|H_{0}\right| \psi(t)>=\int_{0}^{\infty} d p<r l m_{l}\left|p l m_{l}>\frac{p^{2}}{2 m}<p l m_{l}\right| \psi(t)>  \tag{2.103}\\
=\frac{\hbar^{2}}{2 m}\left[-\frac{\partial^{2}}{\partial r^{2}}+\frac{l(l+1)}{r^{2}}\right] \psi_{l m_{l}}(r, t)
\end{gather*}
$$

## 19. Nonrelativistic Schrodinger equation

When nonrelativistic kinematics are used in the coordinate-space Schrodinger equation

$$
\begin{equation*}
\left.<r l m_{l}|H| \psi(t)>=i \hbar \frac{\partial}{\partial t}<r l m_{l} \right\rvert\, \psi(t)> \tag{2.104}
\end{equation*}
$$

it follows from (2.103) that, with a central potential (2.11), $\psi_{l m_{l}}(r, t)$ satisfies
the nonrelativistic Schrodinger equation

$$
\begin{equation*}
\left[\frac{\hbar^{2}}{2 m}\left(-\frac{\partial^{2}}{\partial r^{2}}+\frac{l(l+1)}{r^{2}}\right)+V(r)\right] \psi_{l m_{l}}(r, t)=i \hbar \frac{\partial \psi_{l m_{l}}(r, t)}{\partial t} \tag{2.105}
\end{equation*}
$$

### 2.5 Green's operators

We define Green's operators

$$
\begin{align*}
G_{0}(z) & =\frac{1}{z-H_{0}}  \tag{2.106}\\
G(z) & =\frac{1}{z-H} \tag{2.107}
\end{align*}
$$

where $H_{0}$ is the free-particle Hamiltonian (2.8), $H$ is the Hamiltonian (2.7) including interaction, $V$ is the interaction potential (2.9) and $z$ is any complex number (which has the dimensions of energy) for which the inverses exist.

## Properties

## 1. Spectral decomposition

$G_{0}(z)$ is a known operator. It follows from (2.37) and (2.78) that

$$
\begin{gather*}
G_{0}(z)=\int d^{3} p\left|\vec{p}>\frac{1}{z-\epsilon_{p}}<\vec{p}\right| \\
=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>\frac{1}{z-\epsilon_{p}}<p l m_{l}\right| \tag{2.108}
\end{gather*}
$$

2. Lippmann-Schwinger equations

It follows from (2.106) and (2.107) that

$$
\begin{equation*}
G^{-1}(z)=G_{0}^{-1}(z)-V \tag{2.109}
\end{equation*}
$$

and therefore

$$
\begin{align*}
& G(z)=G_{0}(z)+G_{0}(z) V G(z)  \tag{2.110}\\
& G(z)=G_{0}(z)+G(z) V G_{0}(z) \tag{2.111}
\end{align*}
$$

(2.110) and (2.111) are Lippmann-Schwinger equations for $G(z)$.

## 3. Neumann series

It follows from (2.110) and (2.111) that

$$
\begin{equation*}
G(z)=\left[1-G_{0}(z) V\right]^{-1} G_{0}(z)=G_{0}(z)\left[1-V G_{0}(z)\right]^{-1} \tag{2.112}
\end{equation*}
$$

$$
\begin{equation*}
G(z)=G_{0}(z)+G_{0}(z) V G_{0}(z)+G_{0}(z) V G_{0}(z) V G_{0}(z)+\cdots \tag{2.113}
\end{equation*}
$$

(2.113) is the Neumann series for $G(z)$; it yields an approximation for $G(z)$ when truncated at a finite number of terms.

## 4. Analytic properties

It follows from (2.108) that the function $<\phi\left|G_{0}(z)\right| \chi>$ of the complex variable $z$ has branch points at $m c^{2}$ and $\infty$.

## 5. Principal-value Green's operator

In view of the identity

$$
\begin{equation*}
\frac{1}{x \pm i 0}=\frac{\mathcal{P}}{x} \mp i \pi \delta(x) \tag{2.114}
\end{equation*}
$$

where $\mathcal{P}$ denotes principal value, it follows that for real $\epsilon$

$$
\begin{equation*}
G_{0}(\epsilon \pm i 0)=\bar{G}_{0}(\epsilon) \mp i \pi \delta\left(\epsilon-H_{0}\right) \tag{2.115}
\end{equation*}
$$

where the principal-value Green's operator $\bar{G}_{0}(\epsilon)$ is defined as

$$
\begin{align*}
& \bar{G}_{0}(\epsilon)=\frac{\mathcal{P}}{\epsilon-H_{0}}=f d^{3} p\left|\vec{p}>\frac{1}{\epsilon-\epsilon_{p}}<\vec{p}\right| \\
& =\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} f_{0}^{\infty} d p\left|p l m_{l}>\frac{1}{\epsilon-\epsilon_{p}}<p l m_{l}\right| \tag{2.116}
\end{align*}
$$

where

$$
\begin{equation*}
f \tag{2.117}
\end{equation*}
$$

denotes that the principal value of the integral is taken.
$\bar{G}_{0}(\epsilon)$ is often called the standing-wave Green's operator.

## 6. Matrix elements

It follows from (2.29), (2.68) and (2.108) that

$$
\begin{gather*}
<\vec{x}\left|G_{0}(z)\right| \vec{y}>=\left(\frac{\hbar}{2 \pi}\right)^{3} \int \frac{d^{3} k e^{i \vec{k} \cdot(\vec{x}-\vec{y})}}{z-\epsilon_{p}}  \tag{2.118}\\
<r l m_{l}\left|G_{0}(z)\right| r^{\prime} l m_{l}>=\frac{2}{\pi} \int_{0}^{\infty} \frac{d k \widehat{j}_{l}(k r) \widehat{j}_{l}\left(k r^{\prime}\right)}{z-\epsilon_{p}} \tag{2.119}
\end{gather*}
$$

## 7. Nonrelativistic expressions

When $\epsilon_{p}$ is replaced by $m c^{2}+p^{2} / 2 m$ in (2.118) and (2.119) it follows using (B.29) that

$$
\begin{equation*}
<\vec{x}\left|G_{0}(z)\right| \vec{y}>=-\frac{m}{2 \pi \hbar^{2}} \frac{e^{i \sqrt{2 m\left(z-m c^{2}\right)}|\vec{x}-\vec{y}| / \hbar}}{|\vec{x}-\vec{y}|} \tag{2.120}
\end{equation*}
$$

$$
\begin{align*}
& <\vec{x}\left|G_{0}\left(\epsilon_{p} \pm i 0\right)\right| \vec{y}>=-\frac{m}{2 \pi \hbar^{2}} \frac{e^{ \pm i k|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|}  \tag{2.121}\\
& <\vec{x}\left|\bar{G}_{0}\left(\epsilon_{p}\right)\right| \vec{y}>=-\frac{m}{2 \pi \hbar^{2}} \frac{\cos k|\vec{x}-\vec{y}|}{|\vec{x}-\vec{y}|} \tag{2.122}
\end{align*}
$$

$$
\begin{align*}
& <r l m_{l}\left|G_{0}\left(\epsilon_{p} \pm i 0\right)\right| r^{\prime} l m_{l}>=-\frac{2 m}{\hbar^{2} k} \widehat{j}_{l}\left(k r_{<}\right) \widehat{h}_{l}^{ \pm}\left(k r_{>}\right)  \tag{2.123}\\
& \quad<r l m_{l}\left|\bar{G}_{0}\left(\epsilon_{p}\right)\right| r^{\prime} l m_{l}>=-\frac{2 m}{\hbar^{2} k} \widehat{j}_{l}\left(k r_{<}\right) \widehat{n}_{l}\left(k r_{>}\right) \tag{2.124}
\end{align*}
$$

### 3.1 Introductory remarks

We begin the description of scattering of the particle by the target in this chapter. Our main purpose here is to define and describe scattering states; further development of the theory and techniques for solving the scattering problem are given in Chapters 4, 5 and 6.

We define scattering states in Section 3.2, Møller operators in Section 3.3 and scattering eigenkets in Section 3.4. Coordinate-space wave functions are considered in Section 3.5 in order to give further insight into the scattering process. Preparation in a mixed state is considered in Section 3.6 and derivations of some results are given in Section 3.7.

### 3.2 In- and out-asymptotes

The essential feature of scattering is that the particle behaves as a free particle well before and well after collision with the target.

The average position $\vec{x}(t)=<\psi(t)|\vec{X}| \psi(t)>$ of the particle in a scattering state is a straight line well before and well after the collision; $\vec{x}(t)$ can be as shown in Figure 3.1.

Figure 3.1 A scattering state


In mathematical terms, a scattering state (2.5) satisfies

$$
\begin{gather*}
U(t)\left|\psi>\rightarrow U_{0}(t)\right| \psi_{\text {in }}>\quad \text { as } \quad t \rightarrow-\infty  \tag{3.1}\\
U(t)\left|\psi>\rightarrow U_{0}(t)\right| \psi_{\text {out }}>\quad \text { as } \quad t \rightarrow+\infty \tag{3.2}
\end{gather*}
$$

for some $\mid \psi_{\text {in }}>$ and $\mid \psi_{\text {out }}>$ where $U_{0}(t)$ is the free-particle evolution operator (2.43).

## Comments

## 1. Restrictions on the potential

The existence of the limits (3.1) and (3.2) restricts the form of the interaction potential. Clearly, $\langle\vec{x}| V \mid \vec{y}>$ must be sufficiently short-ranged, not too singular at the origin and reasonably smooth.

A central potential $V(r)$ must fall off faster that $r^{-3}$ at infinity and be less singular than $r^{-\frac{3}{2}}$ at the origin.

## 2. In- and out-asymptotes

$U_{0}(t) \mid \psi_{\text {in }}>$ is the in-asymptote of the scattering state $U(t) \mid \psi>$.
$U_{0}(t) \mid \psi_{\text {out }}>$ is the out-asymptote of the scattering state $U(t) \mid \psi>$.

## 3. Bound states

Not all states of the system are scattering states. If the potential is attractive and sufficiently strong, there may also be bound states

$$
\begin{equation*}
\left|1>,|2>, \cdots,| n_{b}>\right. \tag{3.3}
\end{equation*}
$$

which satisfy

$$
\begin{gather*}
H\left|b>=\epsilon_{b}\right| b>\quad b=1,2, \cdots, n_{b}  \tag{3.4}\\
<b \mid b^{\prime}>=\delta_{b b^{\prime}} \tag{3.5}
\end{gather*}
$$

No in- or out-asymptotes exist for the states

$$
\begin{equation*}
\left|\psi_{b}(t)>=U(t)\right| b>=e^{-i \epsilon_{b} t / \hbar} \mid b>\quad b=1,2, \cdots, n_{b} \tag{3.6}
\end{equation*}
$$

### 3.3 Møller operators

Møller operators $\Omega_{ \pm}$are defined by

$$
\begin{equation*}
\Omega_{ \pm}=\lim _{t \rightarrow \mp \infty} U^{\dagger}(t) U_{0}(t) \tag{3.7}
\end{equation*}
$$

## Properties

## 1. Relationship between states

It follows from (3.1), (3.2) and the unitarity of the evolution operator (2.6) that

$$
\begin{align*}
& \left|\psi>=\lim _{t \rightarrow-\infty} U^{\dagger}(t) U_{0}(t)\right| \psi_{\text {in }}>=\Omega_{+} \mid \psi_{\text {in }}>  \tag{3.8}\\
& \left|\psi>=\lim _{t \rightarrow+\infty} U^{\dagger}(t) U_{0}(t)\right| \psi_{\text {out }}>=\Omega_{-} \mid \psi_{\text {out }}> \tag{3.9}
\end{align*}
$$

That is, the Møller operators relate the actual state of the system with the free-particle in- and out-states.

## 2. Isometric operators

$$
\begin{gather*}
\Omega_{ \pm}^{\dagger} \Omega_{ \pm}=1  \tag{3.10}\\
\Omega_{ \pm} \Omega_{ \pm}^{\dagger}=1-B  \tag{3.11}\\
B \Omega_{ \pm}=\Omega_{ \pm}^{\dagger} B=0 \tag{3.12}
\end{gather*}
$$

where $B$ is the projection operator onto the bound states (3.3), that is,

$$
\begin{equation*}
B=\sum_{b=1}^{n_{b}}|b><b| \tag{3.13}
\end{equation*}
$$

$\Omega_{ \pm}$are one-sided unitary or isometric operators.
In view the fact that $|\psi>,| \psi_{\text {in }}>$ and $\mid \psi_{\text {out }}>$ all have unit norm, one might think that $\Omega_{ \pm}$must be unitary. The one-sidedness of (3.10) and (3.11), that is, the presence of $B$ in (3.11), results from the fact that not all states
of the system are scattering states.
(3.12) and the fact that $B$ is projection operator follow from (3.10) and (3.11).
(3.12) expresses the fact that scattering states are orthogonal to bound states.

## 3. Intertwining equation

We show in Section 3.7 that

$$
\begin{gather*}
U(t) \Omega_{ \pm}=\Omega_{ \pm} U_{0}(t)  \tag{3.14}\\
H \Omega_{ \pm}=\Omega_{ \pm} H_{0}  \tag{3.15}\\
\Omega_{ \pm}^{\dagger} H \Omega_{ \pm}=H_{0} \tag{3.16}
\end{gather*}
$$

(3.14) is the intertwining equation.
(3.16) is consistent with the fact that when $B \neq 0$ the Møller operators are not unitary and $H$ and $H_{0}$ have different spectra.

## 4. Space-time transformations

It follows from (2.13) to (2.15) that

$$
\begin{gather*}
R^{j}(\theta) \Omega_{ \pm} R^{\dagger j}(\theta)=\Omega_{ \pm}  \tag{3.17}\\
{\left[\Omega_{ \pm}, L^{j}\right]=0} \tag{3.18}
\end{gather*}
$$

$$
\begin{align*}
& \mathrm{P} \Omega_{ \pm} \mathrm{P}^{\dagger}=\Omega_{ \pm}  \tag{3.19}\\
& \mathrm{T} \Omega_{ \pm} \mathrm{T}^{\dagger}=\Omega_{\mp} \tag{3.20}
\end{align*}
$$

where $R^{j}(\theta)$ is the rotation operator (2.16), P is the space-inversion operator and T is the time-reversal operator.

## 5. Integral formula

We show in Section 3.7 that

$$
\begin{equation*}
\Omega_{ \pm}=1+\frac{i}{\hbar} \int_{0}^{\mp \infty} d t U^{\dagger}(t) V U_{0}(t) \tag{3.21}
\end{equation*}
$$

(3.21) is a key equation in developing Lippmann-Schwinger equations for solving the scattering problem.
6. Relationship to the Green's operator

We show in Section 3.7 that it follows from (3.21) that

$$
\begin{gather*}
\Omega_{ \pm}=1+\int d^{3} p G\left(\epsilon_{p} \pm i 0\right) V|\vec{p}><\vec{p}| \\
=1+\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p G\left(\epsilon_{p}+i 0\right) V\left|p l m_{l}><p l m_{l}\right| \tag{3.22}
\end{gather*}
$$

and therefore

$$
\begin{equation*}
<\vec{q}\left|\Omega_{ \pm}\right| \vec{p}>=\delta(\vec{p}-\vec{q})+<\vec{q}\left|G\left(\epsilon_{p} \pm i 0\right) V\right| \vec{p}> \tag{3.23}
\end{equation*}
$$

and

$$
\begin{gather*}
<q l m_{l}\left|\Omega_{ \pm}\right| p l^{\prime} m_{l}^{\prime}> \\
=\delta_{l l} \delta_{m_{l} m_{l}^{\prime}}\left[\delta(p-q)+<q l m_{l}\left|G\left(\epsilon_{p}+i 0\right) V\right| p l m_{l}>\right] \tag{3.24}
\end{gather*}
$$

We show in Section 5.5 how (3.23) and (3.24) lead to methods for solving the scattering problem.

### 3.4 Scattering eigenkets

We define scattering eigenkets $\mid \vec{p} \pm>$ and $\mid p l m_{l} \pm>$ by

$$
\begin{gather*}
\left|\vec{p} \pm>=\Omega_{ \pm}\right| \vec{p}>  \tag{3.25}\\
\left.\left|p l m_{l} \pm>=\Omega_{ \pm}\right| p l m_{l}\right\rangle \tag{3.26}
\end{gather*}
$$

## Properties

## 1. Eigenkets of the Hamiltonian

It follows using (3.15) that

$$
\begin{gather*}
H\left|\vec{p} \pm>=\epsilon_{p}\right| \vec{p} \pm>  \tag{3.27}\\
H\left|p l m_{l} \pm>=\epsilon_{p}\right| p l m_{l} \pm> \tag{3.28}
\end{gather*}
$$

That is, $\mid \vec{p} \pm>$ and $\mid p l m_{l} \pm>$ are eigenkets of $H$ belonging to spectral value $\epsilon_{p}$.

## 2. Orthonormality

We show in Section 3.7 that

$$
\begin{gather*}
<\vec{p} \pm \mid \vec{q} \pm>=\delta(\vec{p}-\vec{q})  \tag{3.29}\\
<\vec{p} \pm \mid b>=0 \tag{3.30}
\end{gather*}
$$

$$
\begin{gather*}
<p l m_{l} \pm \mid p^{\prime} l^{\prime} m_{l}^{\prime} \pm>=\delta\left(p-p^{\prime}\right) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}  \tag{3.31}\\
<p l m_{l} \pm \mid b>=0 \tag{3.32}
\end{gather*}
$$

## 3. Spanning the Hilbert space

It follows from (3.11), (3.25) and (3.26) that

$$
\begin{equation*}
1=\Omega_{ \pm} \Omega_{ \pm}^{\dagger}+B \tag{3.33}
\end{equation*}
$$

where $B$ is given by (3.13) and

$$
\begin{gather*}
\Omega_{ \pm} \Omega_{ \pm}^{\dagger}=\int d^{3} p|\vec{p} \pm><\vec{p} \pm| \\
=\sum_{l=0}^{\infty} \sum_{m_{i}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l} \pm><p l m_{l} \pm\right| \tag{3.34}
\end{gather*}
$$

(3.33) is a decomposition of the unit operator into orthogonal projections onto the scattering states and the bound states.

## 4. Spectral decomposition of the Hamiltonian

It follows from (3.27), (3.28) and (3.33) that

$$
\begin{equation*}
H=H_{s}+H_{b} \tag{3.35}
\end{equation*}
$$

where

$$
\begin{gather*}
H_{s}=\int d^{3} p\left|\vec{p} \pm>\epsilon_{p}<\vec{p} \pm\right| \\
=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l} \pm>\epsilon_{p}<p l m_{l} \pm\right| \tag{3.36}
\end{gather*}
$$

and

$$
\begin{equation*}
H_{b}=\sum_{b=1}^{n_{b}}\left|b>\epsilon_{b}<b\right| \tag{3.37}
\end{equation*}
$$

5. Spectral decomposition of the evolution operator

It follows from (3.29), (3.31) and (3.35) to (3.37) that

$$
\begin{equation*}
U(t)=U_{s}(t) U_{b}(t)=U_{b}(t) U_{s}(t) \tag{3.38}
\end{equation*}
$$

where

$$
\begin{gather*}
U_{s}(t)=\int d^{3} p\left|\vec{p} \pm>e^{-i \epsilon_{p} t / \hbar}<\vec{p} \pm\right| \\
=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l} \pm>e^{-i \epsilon_{p} t / \hbar}<p l m_{l} \pm\right| \tag{3.39}
\end{gather*}
$$

and

$$
\begin{equation*}
U_{b}(t)=\sum_{b=1}^{n_{b}}\left|b>e^{-i \epsilon_{b} t / \hbar}<b\right| \tag{3.40}
\end{equation*}
$$

## 6. Spectral decomposition of the Green's operator

It follows from (3.35) to (3.37) that

$$
\begin{equation*}
G(z)=G_{s}(z)+G_{b}(z) \tag{3.41}
\end{equation*}
$$

where

$$
\begin{gather*}
G_{s}(z)=\int d^{3} p\left|\vec{p} \pm>\frac{1}{z-\epsilon_{p}}<\vec{p} \pm\right| \\
=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l} \pm>\frac{1}{z-\epsilon_{p}}<p l m_{l} \pm\right| \tag{3.42}
\end{gather*}
$$

$$
\begin{equation*}
G_{b}(z)=\sum_{b=1}^{n_{b}}\left|b>\frac{1}{z-\epsilon_{b}}<b\right| \tag{3.43}
\end{equation*}
$$

## 7. Analytic properties

It follows from (3.41) to (3.43) that the function $<\phi|G(z)| \chi>$ of the complex variable $z$ has branch points at $m c^{2}$ and $\infty$ and poles at the boundstate energies $\epsilon_{1}, \cdots, \epsilon_{n_{b}}$ of $H$.

### 3.5 Coordinate-space wave functions

## In- and out-wave functions

The coordinate-space scattering wave function $\psi(\vec{x}, t)$ defined by (2.31) satisfies

$$
\begin{align*}
& \psi(\vec{x}, t) \rightarrow \psi_{\text {in }}(\vec{x}, t) \quad \text { as } \quad t \rightarrow-\infty  \tag{3.44}\\
& \psi(\vec{x}, t) \rightarrow \psi_{\text {out }}(\vec{x}, t) \quad \text { as } \quad t \rightarrow+\infty \tag{3.45}
\end{align*}
$$

where the in- and out-wave functions $\psi_{\mathrm{in}}(\vec{x}, t)$ and $\psi_{\text {out }}(\vec{x}, t)$ are defined by

$$
\begin{align*}
\psi_{\text {in }}(\vec{x}, t) & =<\vec{x}\left|U_{0}(t)\right| \psi_{\text {in }}>  \tag{3.46}\\
\psi_{\text {out }}(\vec{x}, t) & =<\vec{x}\left|U_{0}(t)\right| \psi_{\text {out }}> \tag{3.47}
\end{align*}
$$

It follows from (2.37) and (2.43) that

$$
\begin{align*}
\psi_{\text {in }}(\vec{x}, t) & =\left(\frac{\hbar}{2 \pi}\right)^{\frac{3}{2}} \int d^{3} k e^{i(\vec{k} \cdot \vec{x}-\omega t)} \psi_{\text {in }}(\vec{k})  \tag{3.48}\\
\psi_{\text {out }}(\vec{x}, t) & =\left(\frac{\hbar}{2 \pi}\right)^{\frac{3}{2}} \int d^{3} k e^{i(\vec{k} \cdot \vec{x}-\omega t)} \psi_{\text {out }}(\vec{k}) \tag{3.49}
\end{align*}
$$

where

$$
\begin{align*}
\psi_{\text {in }}(\vec{k}) & =<\vec{p} \mid \psi_{\text {in }}>  \tag{3.50}\\
\psi_{\text {out }}(\vec{k}) & =<\vec{p} \mid \psi_{\text {out }}> \tag{3.51}
\end{align*}
$$

## Comments

## 1. Wave-packet

It follows from (3.44) and (3.46) that as $t \rightarrow-\infty$ the scattering wave function $\psi(\vec{x}, t)$ is a wave-packet whose shape is determined by $\psi_{\text {in }}(\vec{k})$.
$\psi_{\text {in }}(\vec{k})$ is a known function determined by the preparation apparatus for the particle.
$\psi_{\text {out }}(\vec{k})$, on the other hand, is not known a priori; properties of $\psi_{\text {out }}(\vec{k})$ are determined by the detection apparatus for the particle.
2. Average initial momentum

The average momentum of the particle

$$
\begin{equation*}
<\psi(t)|\vec{P}| \psi(t)>\rightarrow \vec{p}_{\text {in }} \quad \text { as } \quad t \rightarrow-\infty \tag{3.52}
\end{equation*}
$$

where

$$
\begin{gather*}
\vec{p}_{\text {in }}=<U_{0}(t) \psi_{\text {in }}|\vec{P}| U_{0}(t) \psi_{\text {in }}> \\
=<\psi_{\text {in }}|\vec{P}| \psi_{\text {in }}>=\int d^{3} p \vec{p}\left|\psi_{\text {in }}(\vec{p})\right|^{2} \tag{3.53}
\end{gather*}
$$

$\vec{p}_{\mathrm{in}}$ is the average initial momentum of the particle.
In practice, $\vec{p}_{\text {in }}$ is pointed towards the target and $\psi_{\text {in }}(\vec{p})$ is sharply peaked about $\vec{p}_{\text {in }}$.
3. Average position well before the collision

The average position of the particle

$$
\begin{equation*}
<\psi(t)|\vec{X}| \psi(t)>\rightarrow \vec{x}_{\text {in }}(t) \quad \text { as } \quad t \rightarrow-\infty \tag{3.54}
\end{equation*}
$$

where

$$
\begin{gather*}
\vec{x}_{\text {in }}(t)=<U_{0}(t) \psi_{\text {in }}|\vec{X}| U_{0}(t) \psi_{\text {in }}> \\
=\vec{x}_{\text {in }}+\vec{v}_{\text {in }} t \tag{3.55}
\end{gather*}
$$

where

$$
\begin{equation*}
\vec{x}_{\text {in }}=<\psi_{\text {in }}|\vec{X}| \psi_{\text {in }}>=\int d^{3} x \vec{x}\left|\psi_{\text {in }}(\vec{x})\right|^{2} \tag{3.56}
\end{equation*}
$$

and

$$
\begin{equation*}
\vec{v}_{\mathrm{in}}=<\psi_{\mathrm{in}}\left|\frac{c^{2} \vec{P}}{H_{0}}\right| \psi_{\mathrm{in}}>=\int d^{3} p \frac{c^{2} \vec{p}}{\epsilon_{p}}\left|\psi_{\mathrm{in}}(\vec{p})\right|^{2} \tag{3.57}
\end{equation*}
$$

where

$$
\begin{equation*}
\psi_{\mathrm{in}}(\vec{x})=<\vec{x} \mid \psi_{\mathrm{in}}> \tag{3.58}
\end{equation*}
$$

$\vec{x}_{\mathrm{in}}(t)$ is the average position of the particle well before the collision.
The last line of (3.55) indicates that the particle behaves as a relativistic free particle well before the collision.

Similar equations hold for average position $\vec{x}_{\text {out }}(t)$ of the particle well after the collision.

## Scattering functions

We write the coordinate-space representative of the scattering eigenket $\mid \vec{p}+>$ defined by (3.25) as

$$
\begin{equation*}
<\vec{x} \left\lvert\, \vec{p}+>=\left(\frac{1}{2 \pi \hbar}\right)^{\frac{3}{2}} \psi_{\vec{p}}(\vec{x})\right. \tag{3.59}
\end{equation*}
$$

which equation defines the scattering function $\psi_{\vec{p}}(\vec{x})$.

We write the coordinate-angular momentum representative of the scattering eigenket $\mid p l m_{l}+>$ defined by (3.26) as

$$
\begin{equation*}
<r l m_{l} \left\lvert\, p l^{\prime} m_{l}^{\prime}+>=i^{l} \sqrt{\frac{2}{\pi \hbar}} \psi_{l}(r, k) \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}\right. \tag{3.60}
\end{equation*}
$$

which equation defines the partial-wave scattering function $\psi_{l}(r, k)$.

## Comments

## 1. Notation

We use a Greek letter to label the functions defined by (3.59) and (3.60) for consistency with other works on scattering theory; $\psi_{\vec{p}}(\vec{x})$ and $\psi_{l}(r, k)$ are not representatives of a state of the system.

## 2. Normalization

It follows from (2.26), (2.61), (3.29) and (3.59) that

$$
\begin{equation*}
\left(\frac{1}{2 \pi \hbar}\right)^{3} \int d^{3} x \psi_{\vec{p}}(\vec{x}) \psi_{\vec{q}}^{*}(\vec{x})=\delta(\vec{p}-\vec{q}) \tag{3.61}
\end{equation*}
$$

$$
\begin{equation*}
\int_{0}^{\infty} d r \psi_{l}(r, k) \psi_{l}^{*}\left(r, k^{\prime}\right)=\frac{\pi}{2} \delta\left(k-k^{\prime}\right) \delta_{l l^{\prime}} \tag{3.62}
\end{equation*}
$$

## 3. Partial-wave expansion

We show in Section 3.7 that

$$
\begin{align*}
\psi_{\vec{k}+}(\vec{x})= & 4 \pi \sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} i^{l} \frac{\psi_{l}(r, k)}{k r} Y_{l m_{l}}^{*}\left(\theta_{k}, \varphi_{k}\right) Y_{l m_{l}}\left(\theta_{x}, \varphi_{x}\right)  \tag{3.63}\\
& =\sum_{l=0}^{\infty}(2 l+1) i^{l} \frac{\psi_{l}(r, k)}{k r} P_{l}\left(\cos \theta_{k-x}\right)
\end{align*}
$$

(3.63) is the partial-wave expansion of $\psi_{\vec{k}}(\vec{x})$.
(3.63) becomes the partial-wave expansion (2.69) of a plane-wave when the potential vanishes.
4. Relationship to the scattering wave function

We show in Section 3.7 that

$$
\begin{equation*}
\psi(\vec{x}, t)=\left(\frac{\hbar}{2 \pi}\right)^{\frac{3}{2}} \int d^{3} k e^{-i \omega t} \psi_{\vec{k}}(\vec{x}) \psi_{\mathrm{in}}(\vec{k}) \tag{3.64}
\end{equation*}
$$

(3.64) shows how the scattering wave function $\psi(\vec{x}, t)$ is related to the scattering function $\psi_{\vec{k}}(\vec{x})$.

The coordinate behavior of $\psi(\vec{x}, t)$ is approximately given by $\psi_{\vec{p}}^{\text {in }}$ ( $\left.\vec{x}\right)$ if $\psi_{\text {in }}(\vec{k})$ is sharply peaked about $\vec{p}_{\text {in }}$.

Comparison of (3.64) with (3.48) shows that $\psi(\vec{x}, t)$ becomes the in-wave function $\psi_{\text {in }}(\vec{x}, t)$ when $\psi_{\vec{k}}(\vec{x})$ is replaced by $e^{i \vec{k} \cdot \vec{x}}$.

## 5. Time-independent methods for solving the scattering problem

While scattering is inherently a time-dependent process, it follows from (3.64) that time-independent methods can be used to solve the scattering problem.

Methods for determining $\psi_{\vec{k}}(\vec{x})$ and $\psi_{l}(r, k)$ are given in Chapter 6.
Other time-independent methods for solving the scattering problem are given in Chapters 4 and 5.
6. Time-independent Schrodinger equation

It follows from (3.27) that $\psi_{\vec{p}}(\vec{x})$ satisfies

$$
\begin{equation*}
\int d^{3} y<\vec{x}|H| \vec{y}>\psi_{\vec{p}+}(\vec{y})=\epsilon_{p} \psi_{\vec{p}+}(\vec{x}) \tag{3.65}
\end{equation*}
$$

It follows from (3.28) that $\psi_{l}(r, k)$ satisfies

$$
\begin{equation*}
\int_{0}^{\infty} d r^{\prime}<r l m_{l}|H| r^{\prime} l m_{l}>\psi_{l}\left(r^{\prime}, k\right)=\epsilon_{p} \psi_{l}(r, k) \tag{3.66}
\end{equation*}
$$

## 7. Nonrelativistic time-independent Schrodinger equation

When the free Hamiltonian $H_{0}$ is replaced by its nonrelativistic form (2.12) and the interaction potential $V$ is local (2.10) it follows from (3.65) that $\psi_{\vec{k}}(\vec{x})$ satisfies

$$
\begin{equation*}
\left[\nabla^{2}+k^{2}-U(\vec{x})\right] \psi_{\vec{k}+}(\vec{x})=0 \tag{3.67}
\end{equation*}
$$

where $U(\vec{x})$ is defined by

$$
\begin{equation*}
V(\vec{x})=\frac{\hbar^{2}}{2 m} U(\vec{x}) \tag{3.68}
\end{equation*}
$$

Similarly, when the interaction potential $V$ is a central potential (2.11) it follows from (3.66) that $\psi_{l}(r, k)$ satisfies

$$
\begin{equation*}
\left[\frac{d^{2}}{d r^{2}}-\frac{l(l+1)}{r^{2}}+k^{2}-U(r)\right] \psi_{l}(r, k)=0 \tag{3.69}
\end{equation*}
$$

The most general free-particle solution of (3.69) is a linear combination of the Riccati-Bessel function $\widehat{j}_{l}(k r)$ and the Riccati-Neumann function $\widehat{n}_{l}(k r) .{ }^{1}$

Further discussion of how (3.69) is used to solve the scattering problem for a nonrelativistic particle are given in Chapter 6.

### 3.6 Preparation in a mixed state

So far in this chapter we have assumed that the particle is prepared in a pure state. When the particle emerges from the collimator of an accelerator, however, the particle is prepared in a mixed state because the collimator has a non-zero aperture size. We discuss the properties of such a mixed state in this section. We

[^2]assume that the particle is prepared in a pure state in the development of scattering theory in later chapters merely for convenience in writing.

A mixed state of the particle is represented by the nonidempotent density operator

$$
\begin{equation*}
\Psi(t)=U(t) \Psi U^{\dagger}(t) \tag{3.70}
\end{equation*}
$$

The scattering state (3.70) satisfies

$$
\begin{align*}
& U(t) \Psi U^{\dagger}(t) \rightarrow U_{0}(t) \Psi_{\mathrm{in}} U_{0}^{\dagger}(t) \quad \text { as } \quad t \rightarrow-\infty  \tag{3.71}\\
& U(t) \Psi U^{\dagger}(t) \rightarrow U_{0}(t) \Psi_{\mathrm{out}} U_{0}^{\dagger}(t) \quad \text { as } \quad t \rightarrow+\infty \tag{3.72}
\end{align*}
$$

for some $\Psi_{\text {in }}$ and $\Psi_{\text {out }}$ where $U_{0}(t)$ is the free-particle evolution operator (2.43).

We assume that the collimator of the accelerator produces a particle with average initial momentum perpendicular to the plane of the aperture of the collimator. In order to accommodate the nonzero size of the aperture $\Psi_{\mathrm{in}}$ is specified as

$$
\begin{equation*}
\Psi_{\mathrm{in}}=\int d^{2} a\left|D(\vec{a}) \psi_{\mathrm{in}}>p(\vec{a})<D(\vec{a}) \psi_{\mathrm{in}}\right| \tag{3.73}
\end{equation*}
$$

where $p(\vec{a})$ is the probability per unit area for preparing the particle in a pure state with in-asymptote determined by $\mid D(\vec{a}) \psi_{\text {in }}>$ where

$$
\begin{equation*}
D(\vec{a})=e^{-i \vec{P} \cdot \vec{a} / \hbar} \tag{3.74}
\end{equation*}
$$

is the operator for displacement of the system by $\vec{a}$.

We assume that $p(\vec{a})$ differs from zero only if $\vec{a}$ is perpendicular to average initial momentum of the particle. Accordingly, the integral in (3.73) is a 2 -dimensional integral. The function $p(\vec{a})$ is a characteristic of the aperture of the collimator and is restricted only by the condition

$$
\begin{equation*}
\int d^{2} a p(\vec{a})=1 \tag{3.75}
\end{equation*}
$$

## Comments

## 1. Mixed states and the density operator formalism

Mixed states of a physical system and the density operator formalism of quantum mechanics are described in QLB; Introductory Topics, Chapter 5.

This formalism is used in Chapter 4 to express the differential cross section in terms of the scattering amplitude.

## 2. Average initial momentum

The average initial momentum $\vec{p}_{\text {in }}$ of the particle is

$$
\begin{gather*}
\vec{p}_{\text {in }}=\operatorname{Tr}\left(\vec{P} U_{0}(t) \Psi_{\text {in }} U_{0}^{\dagger}(t)\right) \\
=\int d^{2} a p(\vec{a})<U_{0}(t) D(\vec{a}) \psi_{\text {in }}|\vec{P}| U_{0}(t) D(\vec{a}) \psi_{\text {in }}>  \tag{3.76}\\
=<\psi_{\text {in }}|\vec{P}| \psi_{\text {in }}>=\int d^{3} p \vec{p}\left|\psi_{\text {in }}(\vec{p})\right|^{2}
\end{gather*}
$$

which, of course, is the same as (3.53) because the mixed state (3.73) differs from the corresponding pure state form only by a space displacement which operation leaves momentum unchanged.

## 3. Average position well before the collision

The average position $\vec{x}_{\text {in }}(t)$ of the particle well before the collision is

$$
\begin{gather*}
\vec{x}_{\mathrm{in}}(t)=\operatorname{Tr}\left(\vec{X} U_{0}(t) \Psi_{\mathrm{in}} U_{0}^{\dagger}(t)\right) \\
=\int d^{2} a p(\vec{a})<U_{0}(t) D(\vec{a}) \psi_{\mathrm{in}}|\vec{X}| U_{0}(t) D(\vec{a}) \psi_{\mathrm{in}}>  \tag{3.77}\\
=\overrightarrow{\widetilde{x}}_{\mathrm{in}}+\vec{v}_{\mathrm{in}} t
\end{gather*}
$$

where

$$
\begin{equation*}
\overrightarrow{\tilde{x}}_{\mathrm{in}}=\vec{x}_{\mathrm{in}}+\int d^{2} a \vec{a} p(\vec{a}) \tag{3.78}
\end{equation*}
$$

where $\vec{x}_{\text {in }}$ and $\vec{v}_{\text {in }}$ are given by (3.56) and (3.57), respectively.
The last line of (3.77) shows that the particle behaves as a relativistic free particle well before the collision.

The second term in (3.78) is a vector in the plane of the aperture of the collimator and reflects the lack of precision of the initial state of the particle by the collimator.

### 3.7 Some derivations

## Derivation of (3.14) to (3.16)

It follows from (2.6) and (3.7) that

$$
\begin{gather*}
U(t) \Omega_{ \pm}=e^{-i H t / \hbar} \lim _{\tau \rightarrow \mp \infty} e^{i H \tau / \hbar} e^{-i H_{0} \tau / \hbar} \\
=\lim _{\tau \rightarrow \mp \infty} e^{i H(\tau-t) / \hbar} e^{-i H_{0} \tau / \hbar}\left(e^{i H_{0} t / \hbar} e^{-i H_{0} t / \hbar}\right)  \tag{3.79}\\
=\lim _{\tau \rightarrow \mp \infty} e^{i H(\tau-t) / \hbar} e^{-i H_{0}(\tau-t) / \hbar} e^{-i H_{0} t / \hbar} \\
=\Omega_{ \pm} U_{0}(t)
\end{gather*}
$$

which is (3.14). (3.15) follows on differentiating (3.14) with respect to $t$ and setting $t=0$, and (3.16) follows using (3.10).

## Derivation of (3.21)

It follows from (3.7) that

$$
\begin{equation*}
\Omega_{ \pm}=\lim _{t \rightarrow \mp \infty} F(t) \tag{3.80}
\end{equation*}
$$

where

$$
\begin{equation*}
F(t)=U^{\dagger}(t) U_{0}(t) \tag{3.81}
\end{equation*}
$$

so (3.21) follows on noting that

$$
\begin{equation*}
F(t)=F(0)+\int_{0}^{t} d \tau \frac{d F(\tau)}{d \tau} \tag{3.82}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{d F(\tau)}{d \tau}=\frac{i}{\hbar} U^{\dagger}(\tau)\left(H-H_{0}\right) U_{0}(\tau)=\frac{i}{\hbar} U^{\dagger}(\tau) V U_{0}(\tau) \tag{3.83}
\end{equation*}
$$

## Derivation of (3.22)

It follows from (2.26), (2.44),

$$
\begin{equation*}
\int_{0}^{ \pm \infty} e^{i k x} d x=\frac{i}{x \pm i 0} \tag{3.84}
\end{equation*}
$$

and (2.107) that

$$
\begin{gather*}
\int_{0}^{\mp \infty} d t U^{\dagger}(t) V U_{0}(t)=\int_{0}^{\mp \infty} d t U^{\dagger}(t) V U_{0}(t) \int d^{3} p|\vec{p}><\vec{p}| \\
=\int_{0}^{\mp \infty} d t e^{i\left(H-\epsilon_{p}\right) t / \hbar} \int d^{3} p V|\vec{p}><\vec{p}|  \tag{3.85}\\
=-\frac{i}{\hbar} \int d^{3} p G\left(\epsilon_{p} \pm i 0\right) V|\vec{p}><\vec{p}|
\end{gather*}
$$

The first equality in (3.22) then follows from integral formula (3.21) for $\Omega_{ \pm}$.

The second equality in (3.22) is derived in a similar fashion.

## Derivation of (3.29) to (3.32)

(3.29) to (3.32) follow from (2.28), (2.63), (3.10), (3.12) (3.25) and (3.26).

## Derivation of (3.63)

It follows from (2.61), (2.64), (2.67), (3.26) and (3.60) that

$$
\begin{gather*}
<\vec{x}|\vec{p}+>=<\vec{x}| \Omega_{+} \mid \vec{p}>= \\
=\sum_{l m_{l} l^{\prime} m_{l}^{\prime}} \int_{0}^{\infty} d r^{\prime} d p^{\prime}<\vec{x}\left|r^{\prime} l m_{l}><r^{\prime} l m_{l}\right| \Omega_{+}\left|p^{\prime} l^{\prime} m_{l}^{\prime}><p^{\prime} l^{\prime} m_{l}^{\prime}\right| \vec{p}> \\
=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d r^{\prime} d p^{\prime}<\vec{x}\left|r^{\prime} l m_{l}><r^{\prime} l m_{l}\right| p^{\prime} l m_{l}+><p^{\prime} l m_{l} \mid \vec{p}>  \tag{3.86}\\
=\sqrt{\frac{2}{\pi \hbar^{3}}} \sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} i^{l} \frac{\psi_{l}(r, k)}{k r} Y_{l m_{l}}^{*}\left(\theta_{k}, \varphi_{k}\right) Y_{l m_{l}}\left(\theta_{x}, \varphi_{x}\right)
\end{gather*}
$$

which with (3.59) yields (3.63). The second equality in (3.63) follows using the addition theorem for spherical harmonics.

## Derivation of (3.64)

It follows using (3.8) that

$$
\begin{equation*}
\psi(\vec{x}, t)=<\vec{x}|U(t)| \psi>=<\vec{x}\left|U(t) \Omega_{+}\right| \psi_{\mathrm{in}}> \tag{3.87}
\end{equation*}
$$

the right side of which is

$$
\begin{align*}
& \int d^{3} p<\vec{x}\left|U(t) \Omega_{+}\right| \vec{p}><\vec{p} \mid \psi_{\mathrm{in}}> \\
= & \int d^{3} p<\vec{x}|U(t)| \vec{p}+><\vec{p} \mid \psi_{\mathrm{in}}>  \tag{3.88}\\
= & \int d^{3} p e^{-i \epsilon_{p} t / \hbar}<\vec{x}|\vec{p}+><\vec{p}| \psi_{\mathrm{in}}>
\end{align*}
$$

which using (3.50) and (3.59) is the right side of (3.64).

## Derivation of (3.77)

It follows using

$$
\begin{equation*}
U_{0}^{\dagger}(t) \vec{X} U_{0}(t)=\vec{X}+\vec{V} t \tag{3.89}
\end{equation*}
$$

and

$$
\begin{equation*}
D^{\dagger}(\vec{a}) \vec{X} D(\vec{a})=\vec{X}+\vec{a} \tag{3.90}
\end{equation*}
$$

that the expectation value in the integrand in the second line of (3.77) is

$$
\begin{equation*}
<U_{0}(t) D(\vec{a}) \psi_{\mathrm{in}}|\vec{X}| U_{0}(t) D(\vec{a}) \psi_{\mathrm{in}}>=\vec{x}_{\mathrm{in}}+\vec{a}+\vec{v}_{\mathrm{in}} t \tag{3.91}
\end{equation*}
$$

which leads immediately to the last line of (3.77).

### 4.1 Introductory remarks

In Chapter 3 we introduced Møller operators $\Omega_{ \pm}$which relate the actual state of the system with free-particle in- and out-states. In this chapter we introduce the scattering operator $S$ which relates the out-state with the in-state without direct reference to the actual state.

The scattering operator is defined in Section 4.2 and properties are given in Section 4.3. The scattering amplitude is introduced in Section 4.4 and the relationship between the scattering amplitude and measurable scattering cross sections is given in Section 4.5. Derivations of some results are given in Section 4.6.

### 4.2 Definition

It follows from (3.8) to (3.10) that

$$
\begin{equation*}
\left|\psi_{\text {out }}>=S\right| \psi_{\mathrm{in}}> \tag{4.1}
\end{equation*}
$$

where $\mid \psi_{\text {in }}>$ and $\mid \psi_{\text {out }}>$ are given by (3.1) and (3.2) and where the scattering operator $S$ is defined by

$$
\begin{equation*}
S=\Omega_{-}^{\dagger} \Omega_{+} \tag{4.2}
\end{equation*}
$$

where $\Omega_{ \pm}$are the Møller operators (3.7).

### 4.3 Properties

## 1. Relationship between states

(3.8) and (3.9) show that the Møller operators relate the actual state of the system with the free-particle in- and out-states.

The scattering operator (4.2), on the other hand, relates the out-state with the in-state without direct reference to the actual state.

## 2. Main goal of scattering theory

The main goal of scattering theory is to express the out-asymptote
$U_{0}(t) \mid \psi_{\text {out }}>$ in terms of the in-asymptote $U_{0}(t) \mid \psi_{\text {in }}>$ without further direct reference to the experimentally indeterminate details of the scattering state $U(t) \mid \psi>$.

The main goal of scattering theory therefore is to determine the scattering operator $S$.
3. Unitarity

We show in Section 4.6 that

$$
\begin{equation*}
S^{\dagger} S=S S^{\dagger}=1 \tag{4.3}
\end{equation*}
$$

Unlike the Møller operators, the scattering operator $S$ is unitary.

## 4. Conservation of energy

We show in Section 4.6 that

$$
\begin{equation*}
U_{0}(t) S U_{0}^{\dagger}(t)=S \tag{4.4}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
\left[S, H_{0}\right]=0 \tag{4.5}
\end{equation*}
$$

(4.5) states that energy is conserved in the scattering process.

The appearance of $H_{0}$ in (4.5) corresponds to the fact that the particle is asymptotically free.

It follows using (3.8), (3.9) and (3.16) that conservation of energy in the scattering process may be expressed

$$
\begin{gather*}
<\psi(t)|H| \psi(t)>=<\psi|H| \psi> \\
=<\psi_{\text {in }}\left|H_{0}\right| \psi_{\text {in }}>=<\psi_{\text {out }}\left|H_{0}\right| \psi_{\text {out }}> \tag{4.6}
\end{gather*}
$$

The last equality in (4.6) with (4.1) is consistent with (4.5).

## 5. Space-time transformations

It follows from (3.17) to (3.20) that

$$
\begin{gather*}
R^{j}(\theta) S R^{\dagger j}(\theta)=S  \tag{4.7}\\
{\left[S, L^{j}\right]=0} \tag{4.8}
\end{gather*}
$$

$$
\begin{gather*}
\mathrm{P} S \mathrm{P}^{\dagger}=S  \tag{4.9}\\
\mathrm{~T} S \mathrm{~T}^{\dagger}=S^{\dagger} \tag{4.10}
\end{gather*}
$$

where $R^{j}(\theta)$ is the rotation operator (2.16), P is the space-inversion operator and T is the time-reversal operator.
6. Spectral decomposition; phase shifts

It follows from (4.3), (4.5) and (4.8) that

$$
\begin{equation*}
S=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>e^{2 i \delta_{l}(p)}<p l m_{l}\right| \tag{4.11}
\end{equation*}
$$

where $\delta_{l}(p)$ is real. (The factor 2 in the exponential is inserted for later convenience.)
$\delta_{l}(p)$ is the $l$-th partial-wave phase shift. We show in Chapter 6 that $\delta_{l}(p)$ is the shift in the phase of the partial-wave scattering function due to scattering by the target.

It follows from (4.11) that the main goal of scattering theory is to determine the partial-wave phase shifts.

## 7. Reactance operator

The reactance operator $K$ is defined by

$$
\begin{equation*}
K=i \frac{1-S}{1+S} \tag{4.12}
\end{equation*}
$$

It follows from (4.12) that

$$
\begin{equation*}
S=\frac{1+i K}{1-i K} \tag{4.13}
\end{equation*}
$$

$K$ is hermitian; its spectral values are real. It follows from (4.11) that

$$
\begin{equation*}
K=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p\left|p l m_{l}>\tan \delta_{l}(p)<p l m_{l}\right| \tag{4.14}
\end{equation*}
$$

## 8. Dyson series

We show in Section 4.6 that

$$
\begin{equation*}
S=1+\sum_{n=1}^{\infty} \frac{1}{n!}\left(\frac{-i}{\hbar}\right)^{n} \int_{-\infty}^{\infty} d t_{1} \cdots d t_{n} P\left(V_{0}\left(t_{1}\right) \cdots V_{0}\left(t_{n}\right)\right) \tag{4.15}
\end{equation*}
$$

where

$$
\begin{equation*}
V_{0}(t)=e^{i H_{0} t / \hbar} V e^{-i H_{0} t / \hbar} \tag{4.16}
\end{equation*}
$$

and

$$
\begin{equation*}
P\left(V_{0}\left(t_{1}\right) \cdots V_{0}\left(t_{n}\right)\right)=V_{0}\left(t_{i}\right) \cdots V_{0}\left(t_{j}\right) \text { where } t_{i}>\cdots>t_{j} \tag{4.17}
\end{equation*}
$$

$V_{0}(t)$ is the potential in the interaction picture.

When operating on a product of time-labelled operators, the operator $P$ yields a time-ordered product of operators, the latest occurring first in the product.
(4.15) is the Dyson series for the scattering operator; it yields an approximation for $S$ when truncated at a finite number of terms.

It is clear from the derivation that (4.15) is not restricted to the scattering of a spinless particle by a fixed target. Valid quite generally it is one of the main tools used to calculate the scattering operator in relativistic quantum field theories.

## 9. Integral formula

We show in Section 4.6 that

$$
\begin{equation*}
S=1-\frac{i}{\hbar} \int_{0}^{\infty} d t U_{0}^{\dagger}(t)\{V, U(2 t)\} U_{0}^{\dagger}(t) \tag{4.18}
\end{equation*}
$$

where $\{A, B\}=A B+B A$.
(4.18) is a key equation for developing Lippmann-Schwinger equations for solving the scattering problem.

## 10. Relationship to the Green's operator

We show in Section 4.6 that it follows from (4.18) that

$$
\begin{equation*}
<\vec{q}|(S-1)| \vec{p}>=-\frac{1}{2}<\vec{q}|\{V, G(z)\}| \vec{p}> \tag{4.19}
\end{equation*}
$$

$$
\begin{gather*}
<q l m_{l}|(S-1)| p l^{\prime} m_{l}^{\prime}> \\
=-\frac{1}{2} \delta_{l l} \delta_{m_{l} m_{l}^{\prime}}<q l m_{l}|\{V, G(z)\}| p l m_{l}> \tag{4.20}
\end{gather*}
$$

where $z=\frac{1}{2}\left(\epsilon_{p}+\epsilon_{q}\right)+i 0$.
We show in Section 5.5 how (4.19) and (4.20) lead to methods for solving the scattering problem.

### 4.4 Scattering amplitude

It follows from (4.5) that the momentum representation of the scattering operator (4.2), that is, $\langle\vec{q}| S \mid \vec{p}>$ ("the $S$ matrix"), which corresponds to the scattering of a particle with initial momentum $\vec{p}$ to final momentum $\vec{q}$, has the form

$$
\begin{equation*}
<\vec{q}|S| \vec{p}>=\delta(\vec{p}-\vec{q})-2 \pi i \delta\left(\epsilon_{p}-\epsilon_{q}\right) t(\vec{p}, \vec{q}) \tag{4.21}
\end{equation*}
$$

which equation defines $t(\vec{p}, \vec{q})$.

Using

$$
\begin{equation*}
\frac{\delta\left(\epsilon_{p}-\epsilon_{q}\right)}{\gamma m}=\frac{\delta(p-q)}{p} \tag{4.22}
\end{equation*}
$$

and defining $f(\vec{p}, \vec{q})$ by

$$
\begin{equation*}
f(\vec{p}, \vec{q})=-(2 \pi)^{2} \hbar \gamma m t(\vec{p}, \vec{q}) \tag{4.23}
\end{equation*}
$$

it follows from (4.21) that

$$
\begin{equation*}
<\vec{q}|S| \vec{p}>=\delta(\vec{p}-\vec{q})+\frac{i}{2 \pi \hbar p} \delta(p-q) f(\vec{p}, \vec{q}) \tag{4.24}
\end{equation*}
$$

## Comments

## 1. On-shell $T$ matrix

As shown by (5.16), the momentum-space representative of the operator $T\left(\epsilon_{p}+i 0\right)$ defined by (5.1) is $t(\vec{p}, \vec{q})$.

Because of the factor $\delta\left(\epsilon_{p}-\epsilon_{q}\right)$ in (4.21), $t(\vec{p}, \vec{q})$ for given $\vec{p}$ only involves values of $\vec{q}$ on a spherical shell of radius $|\vec{p}|$.
$t(\vec{p}, \vec{q})$ is called the on-shell $T$ matrix.

## 2. Scattering amplitude

The function $f(\vec{p}, \vec{q})$, which has the dimensions of length, is the scattering amplitude.
$f(\vec{p}, \vec{q})$ corresponds to the scattering of a particle with initial momentum $\vec{p}$ to final momentum $\vec{q}$.

## 3. Restrictions by space-time invariances

It follows from (4.7) to (4.10) that

$$
\begin{equation*}
f(\vec{p}, \vec{q})=f\left(\vec{p}_{R^{j}}, \vec{q}_{R^{j}}\right)=f(-\vec{p},-\vec{q})=f(-\vec{q},-\vec{p}) \tag{4.25}
\end{equation*}
$$

It follows from (4.25) that $f(\vec{p}, \vec{q})$ is a function the variables $\vec{p} \cdot \vec{p}, \vec{q} \cdot \vec{q}, \vec{p} \cdot \vec{q}$. Because of the factor $\delta(p-q)$ in (4.24) we write

$$
\begin{equation*}
f(\vec{p}, \vec{q})=f(k, \theta) \tag{4.26}
\end{equation*}
$$

where $k=|\vec{p}| / \hbar$ and $\theta$ is the angle between $\vec{p}$ and $\vec{q}$.

## 4. Orientation of the coordinate system

When the 3-axis of the coordinate system is in the direction of $\vec{p}$ the angle $\theta$ defined in the previous item is a spherical polar coordinate of $\vec{q}$.

There is no dependence of $f(\vec{p}, \vec{q})$ on the azimuthal coordinate of $\vec{q}$ because of rotational invariance.
5. Partial-wave expansion of the scattering amplitude

We show in Section 4.6 that

$$
\begin{equation*}
f(k, \theta)=\sum_{l=0}^{\infty}(2 l+1) a_{l}(k) P_{l}(\cos \theta) \tag{4.27}
\end{equation*}
$$

where

$$
\begin{equation*}
a_{l}(k)=\frac{e^{2 i \delta_{l}(k)}-1}{2 i k} \tag{4.28}
\end{equation*}
$$

and where $P_{l}(z)$ is a Legendre polynomial.
(4.27) is the partial-wave expansion of the scattering amplitude.
$a_{l}(k)$ is the partial-wave scattering amplitude.

### 4.5 Cross sections

We show in this section how the scattering amplitude (4.26) is related to experimental observations.

## Differential cross section

In Section 4.6 we define the differential cross section

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}(\vec{p}, \vec{q}) \tag{4.29}
\end{equation*}
$$

for a particle with average initial momentum $\vec{p}$ in the 3-direction to scatter to final momentum $\vec{q}$ with spherical polar coordinates $\vec{q}=(p, \theta, \varphi)$.

We show in Section 4.6 that

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}(\vec{p}, \vec{q})=|f(k, \theta)|^{2} \tag{4.30}
\end{equation*}
$$

## Comments

## 1. Effective cross-sectional area of the target

The differential cross section (4.29) is the effective cross-sectional area of the target for the scattering of a particle with initial average momentum $\vec{p}$ to final momentum $\vec{q}$.

## 2. Experimental observations

We show in Section 4.6 how the differential cross section is related to the number of particles detected by the detector. That is, we show how (4.29) is measurable.

## 3. Dependence on the initial state

The differential cross section (4.30) depends on the average initial momentum of the particle; it is otherwise independent of the details of the initial state.

As shown in Section 4.6 this follows because the scattering amplitude is essentially constant over the region where momentum-space representative of the incoming wave function is appreciable.
4. Legendre series

It follows from (4.27) that

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}(k, \theta)=\sum_{l, l^{\prime}=0}^{\infty}(2 l+1)\left(2 l^{\prime}+1\right) a_{l}(k) a_{l^{\prime}}^{*}(k) P_{l}(\cos \theta) P_{l^{\prime}}(\cos \theta) \tag{4.31}
\end{equation*}
$$

## Total cross section

The total cross section $\sigma(p)$ for the scattering of a particle with initial average momentum $\vec{p}$ is defined as

$$
\begin{equation*}
\sigma(p)=\int d \Omega \frac{d \sigma}{d \Omega}(\vec{p}, \vec{q}) \tag{4.32}
\end{equation*}
$$

where $d \Omega=d(\cos \theta) d \varphi$.

## Comments

## 1. Effective cross-sectional area of the target

$\sigma(p)$ is the effective cross-sectional area of the target for the scattering of a particle with initial average momentum $\vec{p}$.

## 2. Optical theorem

We show in Section 4.6 that

$$
\begin{equation*}
\sigma(p)=\frac{4 \pi}{k} \operatorname{Im} f(\vec{p}, \vec{p}) \tag{4.33}
\end{equation*}
$$

(4.33) is the optical theorem.

## 3. Partial cross sections

It follows from (4.31) that

$$
\begin{equation*}
\sigma(k)=\sum_{l=0}^{\infty} \sigma_{l}(k) \tag{4.34}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma_{l}(k)=\frac{4 \pi}{k^{2}}(2 l+1) \sin ^{2} \delta_{l}(k) \tag{4.35}
\end{equation*}
$$

$\sigma_{l}(k)$ is total cross-sectional area of the target for a particle with momentum $\vec{p}$ and angular momentum $l \hbar$.

### 4.6 Some derivations

## Derivation of (4.3)

It follows from (4.2) and (3.10) to (3.12) that

$$
\begin{align*}
S^{\dagger} S & =\left(\Omega_{-}^{\dagger} \Omega_{+}\right)^{\dagger} \Omega_{-}^{\dagger} \Omega_{+}=\Omega_{+}^{\dagger} \Omega_{-} \Omega_{-}^{\dagger} \Omega_{+}  \tag{4.36}\\
& =\Omega_{+}^{\dagger}(1-B) \Omega_{+}=\Omega_{+}^{\dagger} \Omega_{+}=1 \\
S S^{\dagger} & =\Omega_{-}^{\dagger} \Omega_{+}\left(\Omega_{-}^{\dagger} \Omega_{+}\right)^{\dagger}=\Omega_{-}^{\dagger} \Omega_{+} \Omega_{+}^{\dagger} \Omega_{-}  \tag{4.37}\\
& =\Omega_{-}^{\dagger}(1-B) \Omega_{-}=\Omega_{-}^{\dagger} \Omega_{-}=1
\end{align*}
$$

## Derivation of (4.4) and (4.5)

It follows from (4.2) and (3.14) that

$$
\begin{align*}
& U_{0}(t) S U_{0}^{\dagger}(t)=U_{0}(t) \Omega_{-}^{\dagger} \Omega_{+} U_{0}^{\dagger}(t)=\left[\Omega_{-} U_{0}^{\dagger}(t)\right]^{\dagger} \Omega_{+} U_{0}^{\dagger}(t) \\
& =\left[U^{\dagger}(t) \Omega_{-}\right]^{\dagger} U^{\dagger}(t) \Omega_{+}=\Omega_{-}^{\dagger} U(t) U^{\dagger}(t) \Omega_{+}=\Omega_{-}^{\dagger} \Omega_{+}=S \tag{4.38}
\end{align*}
$$

(4.5) follows on differentiating (4.4) with respect to $t$ and setting $t=0$.

## Derivation of (4.15)

It follows from (4.2) that

$$
\begin{equation*}
S=\lim _{t \rightarrow+\infty} \lim _{t_{0} \rightarrow-\infty} F\left(t, t_{0}\right) \tag{4.39}
\end{equation*}
$$

where

$$
\begin{equation*}
F\left(t, t_{0}\right)=U_{0}^{\dagger}(t) U(t) U^{\dagger}\left(t_{0}\right) U_{0}\left(t_{0}\right) \tag{4.40}
\end{equation*}
$$

Now

$$
\begin{equation*}
F\left(t, t_{0}\right)=F\left(t_{0}, t_{0}\right)+\int_{t_{0}}^{t} d \tau \frac{\partial F\left(\tau, t_{0}\right)}{\partial \tau} \tag{4.41}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{\partial F\left(\tau, t_{0}\right)}{\partial \tau}=-\frac{i}{\hbar} V_{0}(\tau) F\left(\tau, t_{0}\right) \tag{4.42}
\end{equation*}
$$

so

$$
\begin{equation*}
F\left(t, t_{0}\right)=1-\frac{i}{\hbar} \int_{t_{0}}^{t} d \tau V_{0}(\tau) F\left(\tau, t_{0}\right) \tag{4.43}
\end{equation*}
$$

Thus, $S$ may be determined from the linear Volterra integral equation (4.43); (4.15) corresponds to the (iterative) Neumann solution to (4.43).

## Derivation of (4.18)

Taking the two limits simultaneously in (4.39) yields

$$
\begin{equation*}
S=\lim _{t \rightarrow+\infty} F(t) \tag{4.44}
\end{equation*}
$$

where

$$
\begin{equation*}
F(t)=U_{0}^{\dagger}(t) U(2 t) U_{0}^{\dagger}(t) \tag{4.45}
\end{equation*}
$$

(4.18) follows on noting that

$$
\begin{equation*}
F(t)=F(0)+\int_{0}^{t} d \tau \frac{d F(\tau)}{d \tau} \tag{4.46}
\end{equation*}
$$

where

$$
\begin{equation*}
\frac{d F(\tau)}{d \tau}=-\frac{i}{\hbar} U_{0}^{\dagger}(\tau)\{V, U(2 \tau)\} U_{0}^{\dagger}(\tau) \tag{4.47}
\end{equation*}
$$

## Derivation of (4.20)

It follows from (4.18), (2.61), (2.79), (2.107) and (3.84) that

$$
\begin{gather*}
<p l m_{l}|(S-1)| p^{\prime} l^{\prime} m_{l}^{\prime}> \\
=-\frac{i}{\hbar} \int_{0}^{\infty} d t<p l m_{l}\left|U_{0}^{\dagger}(t)\{V, U(2 t)\} U_{0}^{\dagger}(t)\right| p^{\prime} l^{\prime} m_{l}^{\prime}> \\
=-\frac{i}{\hbar} \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}<p l m_{l}\left|\left\{V, \int_{0}^{\infty} d t e^{i\left(\epsilon_{p}+\epsilon_{p^{\prime}}-2 H t\right) / \hbar}\right\}\right| p^{\prime} l^{\prime} m_{l}^{\prime}>  \tag{4.48}\\
=-\frac{1}{2} \delta_{l l^{\prime}} \delta_{m_{l} m_{l}^{\prime}}<p l m_{l}|\{V, G(z)\}| p^{\prime} l m_{l}>
\end{gather*}
$$

where $z=\frac{1}{2}\left(\epsilon_{p}+\epsilon_{p^{\prime}}\right)+i 0$.

## Derivation of (4.27)

It follows from (4.11) and (2.66) that

$$
\begin{gather*}
<\vec{q}|(S-1)| \vec{p}> \\
=\sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} \int_{0}^{\infty} d p^{\prime}<\vec{q}\left|p^{\prime} l m_{l}>\left(e^{2 i \delta_{l}\left(p^{\prime}\right)}-1\right)<p^{\prime} l m_{l}\right| \vec{p}>  \tag{4.49}\\
=\frac{\delta(p-q)}{p^{2}} \sum_{l=0}^{\infty} \sum_{m_{l}=-l}^{+l} Y_{l m_{l}}\left(\theta_{q}, \varphi_{q}\right) Y_{l m_{l}}^{*}\left(\theta_{p}, \varphi_{p}\right)\left(e^{2 i \delta_{l}(p)}-1\right)
\end{gather*}
$$

(4.27) follows from (4.24), (4.49) and the addition theorem for spherical harmonics.

## Derivation of (4.30)

We suppose that an accelerator and a detector are pointed toward the target which is at the origin of the coordinate system.. We suppose that a particle approaches the target with in-asymptote determined by $\mid \psi_{\text {in }}>$ (3.1) and with average initial momentum $\vec{p}_{\text {in }}$ (3.53) in the 3-direction.

The probability that after scattering the particle has momentum in the volume $d^{3} q$ about $\vec{q}=(q, \theta, \varphi)$ is

$$
\begin{gather*}
|<\vec{q}| U_{0}(t) \psi_{\text {out }}>\left|{ }^{2} d^{3} q=|<\vec{q}| S\right| \psi_{\text {in }}>\left.\right|^{2} d^{3} q \\
=|<\vec{q}| S\left|\psi_{\text {in }}>\right|^{2} q^{2} d q d \Omega \tag{4.50}
\end{gather*}
$$

where $d \Omega=d(\cos \theta) d \varphi$.

The probability $p\left(\psi_{\mathrm{in}}, \theta\right)$ that after scattering the particle emerges anywhere in the solid angle element $d \Omega$ about the direction of $\vec{q}$ is ${ }^{2}$

$$
\begin{equation*}
p\left(\psi_{\mathrm{in}}, \theta\right)=d \Omega \int_{0}^{\infty} q^{2} d q|<\vec{q}| S\left|\psi_{\mathrm{in}}>\right|^{2} \tag{4.51}
\end{equation*}
$$

So far we have assumed that the particle is prepared in a pure state. When the particle emerges from the collimator of the accelerator, however, it is actually prepared in a mixed state because the collimator has a non-zero aperture. For preparation by an accelerator the particle approaches the target with in-asymptote determined not by $\mid \psi_{\text {in }}>$ but by $\Psi_{\text {in }}$ as given by (3.73).

The probability that after scattering the particle emerges anywhere in the solid angle element $d \Omega$ about the direction of $\vec{q}$ when it has been prepared by the accelerator is

$$
\begin{equation*}
\int d^{2} a p(\vec{a}) p\left(D(\vec{a}) \psi_{\mathrm{in}}, \theta\right) \tag{4.52}
\end{equation*}
$$

[^3]When the experiment is repeated over and over again with a single particle or, more efficiently, when a beam of particles is used, the number of particles $n\left(\psi_{\mathrm{in}}, \theta\right)$ scattering into $d \Omega$ is

$$
\begin{equation*}
n\left(\psi_{\mathrm{in}}, \theta\right)=\int d^{2} a n(\vec{a}) p\left(D(\vec{a}) \psi_{\mathrm{in}}, \theta\right) \tag{4.53}
\end{equation*}
$$

where $n(\vec{a})$ is the number of particles per unit area with impact parameter $\vec{a}$. This number of particles is measurable.

We proceed to evaluate the right side of (4.53). We note first that in practice the beam spot is much larger than the target, so we can safely take $n(\vec{a})=$ constant $=n_{\text {in }}$ over the region where $p\left(D(\vec{a}) \psi_{\mathrm{in}}, \theta\right)$ is nonzero. Thus,

$$
\begin{equation*}
n\left(\psi_{\mathrm{in}}, \theta\right)=n_{\mathrm{in}} \sigma\left(\psi_{\mathrm{in}}, \theta\right) \tag{4.54}
\end{equation*}
$$

where

$$
\begin{equation*}
\sigma\left(\psi_{\mathrm{in}}, \theta\right)=\int d^{2} a p\left(D(\vec{a}) \psi_{\mathrm{in}}, \theta\right) \tag{4.55}
\end{equation*}
$$

$\sigma\left(\psi_{\mathrm{in}}, \theta\right)$ is the effective cross-sectional area of the target for scattering scattering of a particle with in-asymptote determined by $\mid \psi_{\text {in }}>$ into $d \Omega$. Now because of the factor $d \Omega$ in (4.51), we can write

$$
\begin{equation*}
\sigma\left(\psi_{\mathrm{in}}, \theta\right)=d \Omega \frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right) \tag{4.56}
\end{equation*}
$$

where, using (4.51),

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right)=\int d^{2} a \int_{0}^{\infty} q^{2} d q|<\vec{q}| S D(\vec{a})\left|\psi_{\mathrm{in}}>\right|^{2} \tag{4.57}
\end{equation*}
$$

$\frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right)$ is the differential cross section for an accelerator-prepared particle to scatter to spherical polar angles $(\theta, \varphi)$ when its in-asymptote is determined by $\mid \psi_{\text {in }}>$.

We evaluate the right side of (4.57). First,

$$
\begin{align*}
& <\vec{q}|S D(\vec{a})| \psi_{\mathrm{in}}>=\int d^{3} p e^{-i \vec{p} \cdot \vec{a} / \hbar}<\vec{q}|S| \vec{p}>\psi_{\mathrm{in}}(\vec{p}) \\
= & e^{-i \vec{q} \cdot \vec{a} / \hbar} \psi_{\mathrm{in}}(\vec{q})+\frac{i}{2 \pi \hbar q} \int d^{3} p e^{-i \vec{p} \cdot \vec{a} / \hbar} \delta(p-q) f(\vec{p}, \vec{q}) \psi_{\mathrm{in}}(\vec{p}) \tag{4.58}
\end{align*}
$$

We now assume that the detector is not placed in the beam, that is, we assume that $\vec{q} \neq \vec{p}_{\text {in }}$. For such $\vec{q}, \psi_{\text {in }}(\vec{q})=0$, so

$$
\begin{gather*}
\frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right)=\left(\frac{1}{2 \pi \hbar}\right)^{2} \int d^{2} a \int_{0}^{\infty} d q  \tag{4.59}\\
\int d^{3} p d^{3} p^{\prime} e^{i\left(\overrightarrow{p^{\prime}}-\vec{p}\right) \cdot \vec{a} / \hbar} \delta(p-q) \delta\left(p^{\prime}-q\right) f(\vec{p}, \vec{q}) f^{*}\left(\overrightarrow{p^{\prime}}, \vec{q}\right) \psi_{\text {in }}(\vec{p}) \psi_{\text {in }}^{*}\left(\overrightarrow{p^{\prime}}\right)
\end{gather*}
$$

The right side of (4.59) is a 9-dimensional integral. The double integral over $\vec{a}$ yields a two-dimensional delta function

$$
\begin{equation*}
\int d^{2} a e^{i\left(\overrightarrow{p^{\prime}}-\vec{p}\right) \cdot \vec{a} / \hbar}=(2 \pi \hbar)^{2} \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}^{\prime}\right) \tag{4.60}
\end{equation*}
$$

where $\vec{p}_{\perp}$ and $\overrightarrow{p^{\prime}} \perp$ are the components of $\vec{p}$ and $\overrightarrow{p^{\prime}}$ which are perpendicular to the average initial momentum $\vec{p}_{\text {in }}$ so (4.59) becomes

$$
\begin{gather*}
\frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right)=\int d^{3} p \int_{0}^{\infty} d q \delta(q-p)  \tag{4.61}\\
\int d^{3} p^{\prime} \delta\left(p-p^{\prime}\right) \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}\right) f(\vec{p}, \vec{q}) f^{*}\left(\overrightarrow{p^{\prime}}, \vec{q}\right) \psi_{\mathrm{in}}(\vec{p}) \psi_{\mathrm{in}}^{*}\left(\overrightarrow{p^{\prime}}\right)
\end{gather*}
$$

Now

$$
\begin{gather*}
\delta\left(p-p^{\prime}\right) \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}^{\prime}\right)=2 p \delta\left(p^{2}-p^{\prime 2}\right) \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}^{\prime}\right) \\
=2 p \delta\left(\left(\vec{p}_{/ /}\right)^{2}-\left(\vec{p}_{/ /}\right)^{2}\right) \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}^{\prime}\right) \\
=\frac{p}{p_{/ /}}\left[\delta\left(p / /-p_{/ /}^{\prime}\right)+\delta\left(p_{/ /}+p_{/ /}^{\prime}\right)\right] \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}^{\prime}\right)  \tag{4.62}\\
=\frac{p}{p_{/ /}}\left[\delta\left(\vec{p}-\overrightarrow{p^{\prime}}\right)+\delta\left(p_{/ /}+p_{/ /}^{\prime}\right) \delta\left(\vec{p}_{\perp}-\vec{p}_{\perp}^{\prime}\right)\right]
\end{gather*}
$$

where $p / /$ and $p_{/ /}^{\prime}$ are the components of $\vec{p}$ and $\overrightarrow{p^{\prime}}$ which are parallel to the average initial momentum $\vec{p}_{\text {in }}$.

There is no contribution to (4.61) from the second term in the last line of (4.62) because $\psi_{\text {in }}(\vec{p})$ and $\psi_{\mathrm{in}}^{*}\left(\overrightarrow{p^{\prime}}\right)$ vanish when $p / /$ or $p^{\prime} / /$ is negative. That is, the accelerator produces particles going away from it, not towards it. It follows then from (4.61) and (4.62) that

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right)=\int d^{3} p \int_{0}^{\infty} d q \delta(q-p) \frac{p}{p_{/ /}}|f(\vec{p}, \vec{q})|^{2}\left|\psi_{\mathrm{in}}(\vec{p})\right|^{2} \tag{4.63}
\end{equation*}
$$

We now assume that $\left|\psi_{\text {in }}(\vec{p})\right|^{2}$ is sufficiently peaked about $\vec{p}_{\text {in }}$ that $|f(\vec{p}, \vec{q})|^{2}$ is essentially constant over the region where $\psi_{\text {in }}(\vec{p})$ is appreciable. Thus we replace

$$
\begin{equation*}
\delta(q-p) \frac{p}{p / /}|f(\vec{p}, \vec{q})|^{2} \tag{4.64}
\end{equation*}
$$

by

$$
\begin{equation*}
\delta\left(q-p_{\text {in }}\right) \frac{p_{\text {in }}}{p_{\text {in }}}\left|f\left(\vec{p}_{\text {in }}, \vec{q}\right)\right|^{2} \tag{4.65}
\end{equation*}
$$

in (4.63). The 3-dimensional integral over $\vec{p}$ is unity and removes any further dependence of the right side of (4.63) on $\psi_{\text {in }}(\vec{p})$. Accordingly, we write

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}\left(\psi_{\mathrm{in}}, \theta\right)=\frac{d \sigma}{d \Omega}\left(p_{\mathrm{in}}, \theta\right)=\int_{0}^{\infty} d q \delta\left(q-p_{\mathrm{in}}\right)\left|f\left(\vec{p}_{\mathrm{in}}, \vec{q}\right)\right|^{2} \tag{4.66}
\end{equation*}
$$

the last equality in which is (4.30).

## Derivation of (4.33)

We write

$$
\begin{equation*}
S=1+R \tag{4.67}
\end{equation*}
$$

then (4.3) yields

$$
\begin{equation*}
R+R^{\dagger}=-R R^{\dagger} \tag{4.68}
\end{equation*}
$$

Using (4.24) it follows that

$$
\begin{gather*}
<\vec{q}\left|\left(R+R^{\dagger}\right)\right| \vec{p}>=<\vec{q}|R| \vec{p}>+<\vec{p}|R| \vec{q}>^{*} \\
=\frac{i}{2 \pi \hbar p} \delta(p-q)\left[f(\vec{p}, \vec{q})-f^{*}(\vec{q}, \vec{p})\right] \tag{4.69}
\end{gather*}
$$

and

$$
\begin{gather*}
<\vec{q}\left|R R^{\dagger}\right| \vec{p}>=\int d^{3} \vec{p}^{\prime}<\vec{q}|R| \overrightarrow{p^{\prime}}><\vec{p}|R| \overrightarrow{p^{\prime}}>^{*} \\
=\frac{\delta(p-q)}{(2 \pi \hbar p)^{2}} \int d^{3} \vec{p}^{-} \delta\left(p^{\prime}-p\right) f\left(\overrightarrow{p^{\prime}}, \vec{q}\right) f^{*}\left(\overrightarrow{p^{\prime}}, \vec{p}\right) \tag{4.70}
\end{gather*}
$$

so, from (4.68),

$$
\begin{equation*}
f(\vec{p}, \vec{q})-f^{*}(\vec{q}, \vec{p})=\frac{i}{2 \pi \hbar p} \int d^{3} \vec{p}^{\prime} \delta\left(p^{\prime}-p\right) f\left(\overrightarrow{p^{\prime}}, \vec{q}\right) f^{*}\left(\overrightarrow{p^{\prime}}, \vec{p}\right) \tag{4.71}
\end{equation*}
$$

which is (4.33) when $\vec{q}=\vec{p}$.

## Chapter 5

## $T$ OPERATOR

### 5.1 Introductory remarks

We consider methods for solving the scattering problem in this chapter and in Chapter 6 . Towards this end we define the $T$ operator which yields tractable expressions for matrix elements of the Møller operators and the scattering operator derived in Chapters 3 and 4.

The $T$ operator is defined in Section 5.2 and properties of the $T$ operator are given in Section 5.3. The standard method for determining phase shifts and bound-state energies is given in Section 5.4 and some derivations are given in Section 5.5.

### 5.2 Definition

The appearance of the Green's operator and the interaction potential in the combinations $G(z) V$ and $V G(z)$ in matrix elements (4.19) and (4.20) of the Møller operators and the scattering operator suggests that it is useful to define an operator which contains these combinations explicitly.

Accordingly, we define

$$
\begin{equation*}
T(z)=G_{0}^{-1}(z) G(z) V=V G(z) G_{0}^{-1}(z) \tag{5.1}
\end{equation*}
$$

where $z$ is any complex number for which $G(z)$ exists.
That one equality in (5.1) follows from the other follows from (2.110) and (2.111).

### 5.3 Properties

## 1. Products with the free Green's operator

It follows from (5.1) that

$$
\begin{align*}
& G_{0}(z) T(z)=G(z) V  \tag{5.2}\\
& T(z) G_{0}(z)=V G(z) \tag{5.3}
\end{align*}
$$

## 2. Conventional definition

It follows from (5.1), (5.5) and (5.6) that

$$
\begin{equation*}
T(z)=V+V G(z) V \tag{5.4}
\end{equation*}
$$

(5.4) is the conventional definition of $T(z)$.

## 3. Analytic properties

It follows from (5.1) and from (3.41) to (3.43) that $<\phi|T(z)| \chi>$ has branch points at $m c^{2}$ and $\infty$ and poles at the bound-state energies $\epsilon_{1}, \cdots, \epsilon_{n_{b}}$ of $H$.

## 4. Lippmann-Schwinger equations

It follows from (2.110), (2.111) and (5.1) that

$$
\begin{align*}
& T(z)=V+V G_{0}(z) T(z)  \tag{5.5}\\
& T(z)=V+T(z) G_{0}(z) V \tag{5.6}
\end{align*}
$$

(5.5) and (5.6) are Lippmann-Schwinger equations for $T(z)$.

## 5. Neumann series

It follows from (5.5) and (5.6) that

$$
\begin{equation*}
T(z)=\left[1-V G_{0}(z)\right]^{-1} V=V\left[1-G_{0}(z) V\right]^{-1} \tag{5.7}
\end{equation*}
$$

and therefore

$$
\begin{equation*}
T(z)=V+V G_{0}(z) V+V G_{0}(z) V G_{0}(z) V+\cdots \tag{5.8}
\end{equation*}
$$

(5.8) is the Neumann series for $T(z)$; it yields an approximation for $T(z)$ when truncated at a finite number of terms.

## 6. Green's operator as a propagator

A term in (5.8) corresponds to a sequence of interactions with free propagation of the particle between interactions. Accordingly, $G_{0}(z)$ is often called a propagator.

## 7. Principal-value $T$ operator

In analogy with (5.7) we define the principal-value $T$ operator $\bar{T}(\epsilon)$ for real $\epsilon$ by

$$
\begin{equation*}
\bar{T}(\epsilon)=\left[1-V \bar{G}_{0}(\epsilon)\right]^{-1} V \tag{5.9}
\end{equation*}
$$

where $\bar{G}_{0}(\epsilon)$ is the principal-value Green's operator (2.116).
$\bar{T}(\epsilon)$ is often called the $K$ operator. The relationship between spectral values (4.14) of the reactance operator $K$ and corresponding matrix elements of $\bar{T}\left(\epsilon_{p}\right)$ is given by (5.18).
8. Lippmann-Schwinger equation for the principal-value $T$ operator

It follows from (5.9) that

$$
\begin{equation*}
\bar{T}(\epsilon)=V+V \bar{G}_{0}(\epsilon) \bar{T}(\epsilon) \tag{5.10}
\end{equation*}
$$

9. Relationship between the $T$ operators

We show in Section 5.5 that

$$
\begin{equation*}
T(\epsilon+i 0)-\bar{T}(\epsilon)+i \pi \bar{T}(\epsilon) \delta\left(\epsilon-H_{0}\right) T(\epsilon+i 0)=0 \tag{5.11}
\end{equation*}
$$

(5.11) is often called Heitler's equation.

## 10. Lippmann-Schwinger equations for the partial-wave $T$ matrices

We define partial-wave $T$ matrix elements $t_{l}\left(p, p^{\prime} ; z\right)$ and $\bar{t}_{l}\left(p, p^{\prime} ; \epsilon\right)$ by

$$
\begin{equation*}
t_{l}\left(p, p^{\prime} ; z\right)=<p l m_{l}|T(z)| p^{\prime} l m_{l}> \tag{5.12}
\end{equation*}
$$

and

$$
\begin{equation*}
\left.\bar{t}_{l}\left(p, p^{\prime} ; \epsilon\right)=<p l m_{l}\left|\bar{T}_{l}(\epsilon)\right| p^{\prime} l m_{l}\right\rangle \tag{5.13}
\end{equation*}
$$

It follows from (5.5) and (5.10) and the spectral decompositions (2.108) and (2.116) that

$$
\begin{equation*}
t_{l}\left(p, p^{\prime} ; z\right)=v_{l}\left(p, p^{\prime}\right)+\int_{0}^{\infty} \frac{d p^{\prime \prime} v_{l}\left(p, p^{\prime \prime}\right) t_{l}\left(p^{\prime \prime}, p^{\prime} ; z\right)}{z-\epsilon_{p^{\prime \prime}}} \tag{5.14}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{t}_{l}\left(p, p^{\prime} ; \epsilon\right)=v_{l}\left(p, p^{\prime}\right)+\int_{0}^{\infty} \frac{d p^{\prime \prime} v_{l}\left(p, p^{\prime \prime}\right) \bar{t}_{l}\left(p^{\prime \prime}, p^{\prime} ; \epsilon\right)}{\epsilon-\epsilon_{p^{\prime \prime}}} \tag{5.15}
\end{equation*}
$$

where $v_{l}\left(p, p^{\prime}\right)$ is is the momentum-space/angular momentum representative of the interaction potential as given by (2.81).

The Lippmann-Schwinger equations (5.14) and (5.15) are one-dimensional linear Fredholm integral equations.

## 11. On-shell $T$ matrix

We show in Section 5.5 that

$$
\begin{equation*}
t(\vec{p}, \vec{q})=<\vec{q}\left|T\left(\epsilon_{p}+i 0\right)\right| \vec{p}> \tag{5.16}
\end{equation*}
$$

when $|\vec{p}|=|\vec{q}|$ where $t(\vec{p}, \vec{q})$ is the on-shell $T$ matrix (4.21).

## 12. Partial-wave scattering amplitude and phase shift

We show in Section 5.5 that

$$
\begin{equation*}
a_{l}(k)=-\frac{\pi \gamma m}{\hbar k^{2}} t_{l}\left(p, p ; \epsilon_{p}+i 0\right) \tag{5.17}
\end{equation*}
$$

and

$$
\begin{equation*}
\tan \delta_{l}(k)=-\frac{\pi \gamma m}{\hbar k} \bar{t}_{l}\left(p, p ; \epsilon_{p}\right) \tag{5.18}
\end{equation*}
$$

where $a_{l}(k)$ is the partial-wave scattering amplitude (4.28) and $\delta_{l}(k)$ is the phase shift (4.11).

### 5.4 Determining phase shifts and bound-state energies

## The method

## Phase shifts

(5.15) and (5.18) give a method for solving the scattering problem:

Solve (5.15); the phase shift $\delta_{l}(k)$ is given by (5.18).

## Bound-state energies

(5.14) and the analytic properties of $<\phi|T(z)| \chi>$ provide a method for determining bound-state energies of $H$ :

Solve (5.14); the bound-state energies of $H$ are the poles of $t_{l}\left(p, p^{\prime} ; z\right)$.

## Soluble example: separable potential

We show in Section 5.5 that (5.14) and (5.15) can be solved exactly when the potential is separable (2.94) to yield

$$
\begin{equation*}
a_{l}(k)=\frac{\pi \gamma m}{\hbar k^{2}}\left|v_{l}(k)\right|^{2}\left(1+\int_{0}^{\infty} \frac{d p^{\prime}\left|v_{l}\left(k^{\prime}\right)\right|^{2}}{\epsilon_{p}-\epsilon_{p^{\prime}}+i 0}\right)^{-1} \tag{5.19}
\end{equation*}
$$

$$
\begin{equation*}
\tan \delta_{l}(k)=\frac{\pi \gamma m}{\hbar k}\left|v_{l}(k)\right|^{2}\left(1+\int_{0}^{\infty} \frac{d p^{\prime}\left|v_{l}\left(k^{\prime}\right)\right|^{2}}{\epsilon_{p}-\epsilon_{p^{\prime}}}\right)^{-1} \tag{5.20}
\end{equation*}
$$

There is at most one bound state for each $l$ with this potential. The boundstate energy $\epsilon_{b}$ for a state with angular momentum $l \hbar$ is given by

$$
\begin{equation*}
1=\int_{0}^{\infty} \frac{d p\left|v_{l}(k)\right|^{2}}{\epsilon_{p}-\epsilon_{b}} \tag{5.21}
\end{equation*}
$$

## Inverse scattering problem: separable potential

We consider the inverse scattering problem for the separable potential (2.94). That is, we regard $\delta_{l}(k)$ as known and determine $v_{l}(k)$ in terms of $\delta_{l}(k)$.

We show in Section 5.5 that when there are no bound states

$$
\begin{equation*}
\left|v_{l}(k)\right|^{2}=\frac{\hbar k}{\pi \gamma m} \sin \delta_{l}(k) \exp \left[\frac{1}{\pi} \int_{m c^{2}}^{\infty} \frac{\delta_{l}\left(k^{\prime}\right) d \epsilon_{p^{\prime}}}{\epsilon_{p}-\epsilon_{p^{\prime}}}\right] \tag{5.22}
\end{equation*}
$$

## Comment

## 1. Constructed potential

(5.22) shows that is always possible to construct a separable potential (2.94) from a given phase shift $\delta_{l}(k)$ when there are no bound states.

## 2. Realistic potential

In view of (5.22), the separable potential (2.94) is a realistic potential because it reproduces the phase shift $\delta_{l}(k)$ exactly.

### 5.5 Some derivations

## Derivation of (5.11)

It follows from (2.115), (5.5) and (5.10) that

$$
\begin{gather*}
T(\epsilon+i 0)=V+V G_{0}(\epsilon+i 0) T(\epsilon+i 0) \\
=\bar{T}(\epsilon)+V \bar{G}_{0}(\epsilon)[T(\epsilon+i 0)-\bar{T}(\epsilon)]-i \pi V \delta\left(\epsilon-H_{0}\right) T(\epsilon+i 0) \tag{5.23}
\end{gather*}
$$

That is,

$$
\begin{equation*}
T(\epsilon+i 0)-\bar{T}(\epsilon)+i \pi\left[1-V \bar{G}_{0}(\epsilon)\right]^{-1} V \delta\left(\epsilon-H_{0}\right) T(\epsilon+i 0)=0 \tag{5.24}
\end{equation*}
$$

(5.11) then follows using (5.9).

## Derivation of (5.17)

It follows from (5.2), (5.3) and (2.108) that

$$
\begin{gather*}
<p l m_{l}|\{V, G(z)\}| p^{\prime} l m_{l}>=<p l m_{l}\left|\left\{T(z), G_{0}(z)\right\}\right| p^{\prime} l m_{l}> \\
=<p l m_{l}|T(z)| p^{\prime} l m_{l}>\left(\frac{1}{z-\epsilon_{p^{\prime}}}+\frac{1}{z-\epsilon_{p}}\right) \tag{5.25}
\end{gather*}
$$

(5.17) follows using (4.11), (4.28), (4.20) and

$$
\begin{align*}
\frac{1}{z-\epsilon_{p^{\prime}}} & +\frac{1}{z-\epsilon_{p}}=2\left(\frac{1}{\epsilon_{p}-\epsilon_{p^{\prime}}+i 0}-\frac{1}{\epsilon_{p}-\epsilon_{p^{\prime}}-i 0}\right)  \tag{5.26}\\
& =-4 \pi i \delta\left(\epsilon_{p}-\epsilon_{p^{\prime}}\right)=-\frac{4 \pi i \gamma m}{\hbar k} \delta\left(p-p^{\prime}\right)
\end{align*}
$$

when $z=\frac{1}{2}\left(\epsilon_{p}+\epsilon_{p^{\prime}}\right)+i 0$.
(5.16) is derived in a similar fashion using (4.19).

## Derivation of (5.18)

It follows from (5.11), (5.12) and (5.13) that

$$
\begin{gather*}
t_{l}\left(p, p ; \epsilon_{p}+i 0\right)-\bar{t}_{l}\left(p, p ; \epsilon_{p}\right) \\
=-i \pi \int_{0}^{\infty} d p^{\prime} \bar{t}_{l}\left(p, p^{\prime} ; \epsilon_{p}\right) \delta\left(\epsilon_{p}-\epsilon_{p^{\prime}}\right) t_{l}\left(p^{\prime}, p ; \epsilon_{p}+i 0\right)  \tag{5.27}\\
=-i \frac{\pi \gamma m}{\hbar k} \bar{t}_{l}\left(p, p ; \epsilon_{p}\right) t_{l}\left(p, p ; \epsilon_{p}+i 0\right)
\end{gather*}
$$

where the second line of (2.61) is used in middle line above. (5.18) then follows using (5.17).

## Derivation of (5.19) and (5.20)

Substituting (2.94) into (5.14) yields

$$
\begin{equation*}
t_{l}\left(p, p^{\prime} ; z\right)=-v_{l}(p) w_{l}\left(p^{\prime} ; z\right) \tag{5.28}
\end{equation*}
$$

where

$$
\begin{equation*}
w_{l}\left(p^{\prime} ; z\right)=v_{l}^{*}\left(p^{\prime}\right)+\int_{0}^{\infty} \frac{d p^{\prime \prime} v_{l}^{*}\left(p^{\prime \prime}\right) t_{l}\left(p^{\prime \prime}, p^{\prime} ; z\right)}{z-\epsilon_{p^{\prime \prime}}} \tag{5.29}
\end{equation*}
$$

and substituting (5.28) back into (5.29) yields

$$
\begin{equation*}
w_{l}\left(p^{\prime} ; z\right)=v_{l}^{*}\left(p^{\prime}\right)\left(1+\int_{0}^{\infty} \frac{d p^{\prime \prime}\left|v_{l}\left(p^{\prime \prime}\right)\right|^{2}}{z-\epsilon_{p^{\prime \prime}}}\right)^{-1} \tag{5.30}
\end{equation*}
$$

(5.28) then is

$$
\begin{equation*}
t_{l}\left(p, p^{\prime} ; z\right)=-v_{l}(p) v_{l}^{*}\left(p^{\prime}\right)\left(1+\int_{0}^{\infty} \frac{d p^{\prime \prime}\left|v_{l}\left(p^{\prime \prime}\right)\right|^{2}}{z-\epsilon_{p^{\prime \prime}}}\right)^{-1} \tag{5.31}
\end{equation*}
$$

and (5.19) follows using (5.17).
(5.20) is derived in a similar fashion; it also follows from (5.19) using (2.114).

## Derivation of (5.22)

We solve (5.20) to yield (5.22).

We write (5.20) as

$$
\begin{equation*}
\left|v_{l}(k)\right|^{2}=\frac{\hbar k}{\pi \gamma m} \tan \delta_{l}(k) \bar{D}_{l}\left(\epsilon_{p}\right) \tag{5.32}
\end{equation*}
$$

where

$$
\begin{equation*}
\bar{D}_{l}\left(\epsilon_{p}\right)=1+\int_{0}^{\infty} \frac{\left|v_{l}\left(k^{\prime}\right)\right|^{2} d p^{\prime}}{\epsilon_{p}-\epsilon_{p^{\prime}}} \tag{5.33}
\end{equation*}
$$

We define $D_{l}(z)$ for complex $z$ by

$$
\begin{equation*}
D_{l}(z)=1+\int_{0}^{\infty} \frac{\left|v_{l}(k)\right|^{2} d p}{z-\epsilon_{p}} \tag{5.34}
\end{equation*}
$$

then

$$
\begin{equation*}
\bar{D}_{l}\left(\epsilon_{p}\right)=\frac{1}{2}\left[D_{l}\left(\epsilon_{p}+i 0\right)+D_{l}\left(\epsilon_{p}-i 0\right)\right] \tag{5.35}
\end{equation*}
$$

We show below that

$$
\begin{equation*}
D_{l}(z)=\exp \left[\frac{1}{\pi} \int_{m c^{2}}^{\infty} \frac{\delta_{l}(k) d \epsilon_{p}}{z-\epsilon_{p}}\right] \tag{5.36}
\end{equation*}
$$

and, therefore, using (2.114),

$$
\begin{equation*}
D_{l}\left(\epsilon_{p} \pm i 0\right)=\exp \left[\frac{1}{\pi} \int_{m c^{2}}^{\infty} \frac{\delta_{l}\left(k^{\prime}\right) d \epsilon_{p^{\prime}}}{\epsilon_{p^{\prime}}-\epsilon_{p}}\right] e^{\mp i \delta_{l}(k)} \tag{5.37}
\end{equation*}
$$

(5.22) then follows from (5.32), (5.35) and (5.37).

## Derivation of (5.36)

It follows from (5.34) that $D_{l}(z)$ is analytic in the $z$-plane cut from $m c^{2}$ to $\infty$. Since the discontinuity of $\ln D_{l}(z)$ across this cut is known from

$$
\begin{equation*}
\frac{D_{l}\left(\epsilon_{p}-i 0\right)}{D_{l}\left(\epsilon_{p}+i 0\right)}=e^{2 i \delta_{l}(k)} \tag{5.38}
\end{equation*}
$$

which is derived below we write a Cauchy integral for $\ln D_{l}(z)$. We assume that $D_{l}(z)$ has no zeros in which case $\ln D_{l}(z)$ is analytic in the $z$-plane cut from $m c^{2}$ to $\infty$ so

$$
\begin{equation*}
\ln D_{l}(z)=\frac{1}{2 \pi i} \int_{C} \frac{\ln D_{l}\left(z^{\prime}\right) d z^{\prime}}{z^{\prime}-z} \tag{5.39}
\end{equation*}
$$

where $C$ is a closed contour about $z$ which avoids the branch cut.

Expanding the contour in (5.39) and assuming the contribution from the circle at infinity vanishes yields

$$
\begin{align*}
\ln D_{l}(z) & =\frac{1}{2 \pi i}\left[\int_{\infty}^{m c^{2}} \frac{\ln D_{l}\left(\epsilon_{p}-i 0\right) d \epsilon_{p}}{\epsilon_{p}-z}+\int_{m c^{2}}^{\infty} \frac{\ln D_{l}\left(\epsilon_{p}+i 0\right) d \epsilon_{p}}{\epsilon_{p}-z}\right]  \tag{5.40}\\
& =\frac{1}{2 \pi i} \int_{m c^{2}}^{\infty} \ln \frac{D_{l}\left(\epsilon_{p}+i 0\right)}{D_{l}\left(\epsilon_{p}-i 0\right)} \frac{d \epsilon_{p}}{\epsilon_{p}-z}=\frac{1}{\pi} \int_{m c^{2}}^{\infty} \frac{\delta_{l}(k) d \epsilon_{p}}{z-\epsilon_{p}}
\end{align*}
$$

taking the exponential of which gives (5.36).

## Derivation of (5.38)

It follows from (5.34) that

$$
\begin{gather*}
D_{l}\left(\epsilon_{p}+i 0\right)-D_{l}\left(\epsilon_{p}-i 0\right) \\
=\int_{0}^{\infty}\left[\frac{1}{\epsilon_{p}-\epsilon_{p^{\prime}}+i 0}-\frac{1}{\epsilon_{p}-\epsilon_{p^{\prime}}-i 0}\right]\left|v_{l}\left(k^{\prime}\right)\right|^{2} d p^{\prime}  \tag{5.41}\\
=-2 \pi i \int_{0}^{\infty} \delta\left(\epsilon_{p}-\epsilon_{p^{\prime}}\right)\left|v_{l}\left(k^{\prime}\right)\right|^{2} d p^{\prime}=-2 i \frac{\pi \gamma m}{\hbar k}\left|v_{l}(k)\right|^{2}
\end{gather*}
$$

use of which in (5.32) with (5.35) yields (5.38).

### 6.1 Introductory remarks

We return in this chapter to the scattering eigenkets $\mid \vec{p}+>$ and $\mid p l m_{l}+>$ and scattering functions $\psi_{\vec{p}}(\vec{x})$ and $\psi_{l}(r, k)$ defined in Section 3.4.

Further properties of the scattering eigenkets and scattering functions and in particular their relationship with the $T$ operator and phase shifts defined in Chapter 5 are given in Sections 6.2 and 6.3. Some results for the first Born approximation are given in Section 6.4 and standard coordinate-space methods and the variable phase method for solving the nonrelativistic scattering problem are given in Section 6.5. Some derivations are given in Section 6.6.

### 6.2 Properties of scattering eigenkets

1. Relationship to the $T$ operator

We show in Section 6.6 that

$$
\begin{equation*}
T\left(\epsilon_{p} \pm i 0\right)|\vec{p}\rangle=V \mid \vec{p} \pm> \tag{6.1}
\end{equation*}
$$

It then follows from (5.16) that

$$
\begin{equation*}
t(\vec{p}, \vec{q})=<\vec{q}|V| \vec{p}+> \tag{6.2}
\end{equation*}
$$

## 2. Relationship to the scattering amplitude

It follows from (4.23) and (6.2) that for a local potential (2.10)

$$
\begin{equation*}
f(\vec{p}, \vec{q})=-\frac{\gamma}{4 \pi} \int d^{3} x e^{-i \vec{q} \cdot \vec{x} / \hbar} U(\vec{x}) \psi_{\vec{p}}(\vec{x}) \tag{6.3}
\end{equation*}
$$

where $\psi_{\vec{p}}(\vec{x})$ is the scattering function (3.59).

## 3. Lippmann-Schwinger equation

We show in Section 6.6 that

$$
\begin{equation*}
\left|\vec{p}+>=\left|\vec{p}>+G_{0}\left(\epsilon_{p}+i 0\right) V\right| \vec{p}+>\right. \tag{6.4}
\end{equation*}
$$

(6.4) is the Lippmann-Schwinger equation for $\mid \vec{p}+>$.

## 4. Nonrelativistic Lippmann-Schwinger equation

It follows from (6.4) that the scattering function satisfies

$$
\begin{equation*}
\psi_{\vec{k}}(\vec{x})=e^{i \vec{k} \cdot \vec{x}}-\frac{1}{4 \pi} \int d^{3} y \frac{e^{i k|\vec{x}-\vec{y}|}}{|\vec{x}-\vec{y}|} U(\vec{y}) \psi_{\vec{k}}(\vec{y}) \tag{6.5}
\end{equation*}
$$

for nonrelativistic scattering by a local potential.

## 5. Scattering function at large distances

We show in Section 6.6 that for nonrelativistic scattering by a local potential
it follows from (6.5) that

$$
\begin{equation*}
\psi_{\vec{k}}(\vec{x}) \rightarrow e^{i \vec{k} \cdot \vec{x}}+f(k, \theta) \frac{e^{i k r}}{r} \quad \text { as } \quad r \rightarrow \infty \tag{6.6}
\end{equation*}
$$

where $r=|\vec{x}|$ and $\theta$ is the angle between $\vec{x}$ and $\vec{k}$.
6. Scattering wave function at large distances

It follows from (3.64) and (6.6) that

$$
\begin{equation*}
\psi(\vec{x}, t) \rightarrow \psi_{\text {in }}(\vec{x}, t)+\psi_{\text {scatt }}(\vec{x}, t) \quad \text { as } \quad r \rightarrow \infty \tag{6.7}
\end{equation*}
$$

where $\psi_{\text {in }}(\vec{x}, t)$ is given by (3.48) and where

$$
\begin{equation*}
\psi_{\mathrm{scatt}}(\vec{x}, t)=\left(\frac{\hbar}{2 \pi}\right)^{\frac{3}{2}} \int d^{3} k f(k, \theta) \frac{e^{i(k r-\omega t)}}{r} \psi_{\mathrm{in}}(\vec{k}) \tag{6.8}
\end{equation*}
$$

It follows from (6.7) that far from the target the scattering wave function consists of a free-particle incoming wave packet and a free-particle outgoing spherical wave packet, the amplitude of which depends on the value of the scattering amplitude.

### 6.3 Properties of partial-wave eigenkets

## 1. Relationship to the partial-wave scattering amplitude

It follows analogously from the derivation of (6.1) that

$$
\begin{equation*}
T\left(\epsilon_{p} \pm i 0\right)\left|p l m_{l}>=V\right| p l m_{l} \pm> \tag{6.9}
\end{equation*}
$$

It follows then from (5.17) that

$$
\begin{equation*}
a_{l}(k)=-\frac{\pi \gamma m}{\hbar k^{2}}<p l m_{l}|V| p l m_{l}+> \tag{6.10}
\end{equation*}
$$

## 2. Partial-wave amplitude for a central potential

It follows from (6.10) that for a central potential (2.11)

$$
\begin{equation*}
a_{l}(k)=-\frac{\gamma}{k^{2}} \int_{0}^{\infty} d r \widehat{j_{l}}(k r) U(r) \psi_{l}(r, k) \tag{6.11}
\end{equation*}
$$

where $\psi_{l}(r, k)$ is the partial-wave scattering function (3.60).

## 3. Phase shift for a central potential

It follows from (6.11) that for a central potential (2.11)

$$
\begin{equation*}
\tan \delta_{l}(k)=-\frac{\gamma}{k} \int_{0}^{\infty} d r \widehat{j}_{l}(k r) U(r) \bar{\psi}_{l}(r, k) \tag{6.12}
\end{equation*}
$$

where the real function $\bar{\psi}_{l}(r, k)$ is defined by

$$
\begin{equation*}
\psi_{l}(r, k)=e^{i \delta_{l}(k)} \cos \delta_{l}(k) \bar{\psi}_{l}(r, k) \tag{6.13}
\end{equation*}
$$

## 4. Lippmann-Schwinger equation

It follows analogously from the derivation of (6.4) that

$$
\begin{equation*}
\left|p l m_{l}+>=\left|p l m_{l}>+G_{0}\left(\epsilon_{p}+i 0\right) V\right| p l m_{l}+>\right. \tag{6.14}
\end{equation*}
$$

## 5. Nonrelativistic Lippmann-Schwinger equation

It follows from (6.14) and (B.29) that for nonrelativistic scattering by a central potential $\bar{\psi}_{l}(r, k)$ satisfies

$$
\begin{equation*}
\bar{\psi}_{l}(r, k)=\widehat{j}_{l}(k r)+\int_{0}^{\infty} d r^{\prime} \bar{G}_{0 l}\left(r, r^{\prime}, k\right) U\left(r^{\prime}\right) \bar{\psi}_{l}\left(r^{\prime}, k\right) \tag{6.15}
\end{equation*}
$$

where ${ }^{1}$

$$
\begin{equation*}
\bar{G}_{0 l}\left(r, r^{\prime}, k\right)=-\frac{1}{k} \widehat{j}_{l}\left(k r_{<}\right) \widehat{n}_{l}\left(k r_{>}\right) \tag{6.16}
\end{equation*}
$$

As shown by (2.124) the Green's function $\bar{G}_{0 l}\left(r, r^{\prime}, k\right)$ is proportional to the coordinate-space/angular momentum representative of the nonrelativistic principal-value Green's operator $\bar{G}_{0}\left(\epsilon_{p}\right)$.
or, more explicitly,

$$
\begin{align*}
& \bar{\psi}_{l}(r, k)= \widehat{j}_{l}(k r)-\frac{1}{k} \widehat{n}_{l}(k r) \int_{0}^{r} d r^{{ }_{\hat{j}}^{l}} \\
&  \tag{6.17}\\
&\left(k r^{\prime}\right) U\left(r^{\prime}\right) \bar{\psi}_{l}\left(r^{\prime}, k\right) \\
&-\frac{1}{k} \widehat{j}_{l}(k r) \int_{r}^{\infty} d r^{\prime} \widehat{n}_{l}\left(k r^{\prime}\right) U\left(r^{\prime}\right) \bar{\psi}_{l}\left(r^{\prime}, k\right)
\end{align*}
$$

## 6. Nonrelativistic Schrodinger equation

$\bar{\psi}_{l}(r, k)$ satisfies the nonrelativistic Schrodinger equation

$$
\begin{equation*}
\left[\frac{d^{2}}{d r^{2}}-\frac{l(l+1)}{r^{2}}+k^{2}-U(r)\right] \bar{\psi}_{l}(r, k)=0 \tag{6.18}
\end{equation*}
$$

7. Low-momentum behavior

It follows (6.15) that

$$
\begin{equation*}
\bar{\psi}_{l}(r, k) \propto k^{l+1} \quad \text { as } \quad k \rightarrow 0 \tag{6.19}
\end{equation*}
$$

8. Small- and large-distance behavior

It follows from (6.15) that

$$
\begin{equation*}
\bar{\psi}_{l}(r, k) \propto r^{l+1} \quad \text { as } \quad r \rightarrow 0 \tag{6.20}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{\psi}_{l}(r, k) \rightarrow \widehat{j}_{l}(k r)+\tan \delta_{l}(k) \widehat{n}_{l}(k r) \quad \text { as } \quad r \rightarrow \infty \tag{6.21}
\end{equation*}
$$

It follows from (6.21) that

$$
\begin{equation*}
\bar{\psi}_{l}(r, k) \rightarrow \sec \delta_{l}(k) \sin \left(k r-\pi l / 2+\delta_{l}(k)\right) \quad \text { as } \quad r \rightarrow \infty \tag{6.22}
\end{equation*}
$$

which equation shows that the scattering by the target has shifted the phase of the partial-wave scattering function by $\delta_{l}(k)$.

## 9. Finite-range potential

It follows from (6.21) and (6.22) that for values of $r$ outside the range of a finite-range potential

$$
\begin{equation*}
\bar{\psi}_{l}(r, k)=\widehat{j_{l}}(k r)+\tan \delta_{l}(k) \widehat{n}_{l}(k r) \tag{6.23}
\end{equation*}
$$

and

$$
\begin{equation*}
\bar{\psi}_{l}(r, k)=\sec \delta_{l}(k) \sin \left(k r-\pi l / 2+\delta_{l}(k)\right) \tag{6.24}
\end{equation*}
$$

## 10. Low-momentum behavior of the phase shifts

The effective range expansion

$$
\begin{equation*}
k^{2 l+1} \cot \delta_{l}(k)=-\frac{1}{a_{l}}+\frac{1}{2} r_{l} k^{2}+O\left(k^{4}\right) \tag{6.25}
\end{equation*}
$$

follows from (6.12) and (6.19). Then

$$
\begin{equation*}
\delta_{l}(k) \rightarrow n_{l} \pi-a_{l} k^{2 l+1} \quad \text { as } \quad k \rightarrow 0 \tag{6.26}
\end{equation*}
$$

As shown in Section 6.6, the constant $n_{l}$ is the number of bound states with angular momentum $l$.

The constant $a_{0}$ is the scattering length. It follows that the partial $S$-wave cross section at zero energy is

$$
\begin{equation*}
\sigma_{0}(0)=4 \pi a_{0}^{2} \tag{6.27}
\end{equation*}
$$

### 6.4 First Born approximation: some results

The first Born approximation is to take

$$
\begin{equation*}
T(z)=V \tag{6.28}
\end{equation*}
$$

as an approximation for the $T$ operator and

$$
\begin{align*}
\psi_{\vec{k}}(\vec{x}) & =e^{i \vec{k} \cdot \vec{x}}  \tag{6.29}\\
\bar{\psi}_{l}(r, k) & =\widehat{j}_{l}(k r) \tag{6.30}
\end{align*}
$$

as an approximation for the scattering functions.

## 1. Scattering amplitude for a local potential

It follows from (4.23), (5.16) and (6.28) that the first Born approximation for the scattering amplitude for a local potential (2.10) is

$$
\begin{equation*}
f(\vec{p}, \vec{q})=-\frac{\gamma}{4 \pi} \int d^{3} x e^{i(\vec{p}-\vec{q} \cdot \vec{x} / \hbar} U(\vec{x}) \tag{6.31}
\end{equation*}
$$

## 2. Scattering amplitude for a central potential

It follows from (2.11) and (6.31) that the first Born approximation for the scattering amplitude for a central potential is

$$
\begin{equation*}
f(k, \theta)=-\frac{\gamma}{\kappa} \int_{0}^{\infty} d r r \sin \kappa r U(r) \tag{6.32}
\end{equation*}
$$

where

$$
\begin{equation*}
\kappa=2 k \sin \frac{\theta}{2} \tag{6.33}
\end{equation*}
$$

## 3. Phase shift

It follows from (6.12) that the first Born approximation to the $l$-th partial wave phase shift $\delta_{l}(p)$ for a central potential is

$$
\begin{equation*}
\delta_{l}(k)=-\frac{\gamma}{k} \int_{0}^{\infty} d r \widehat{j}_{l}^{2}(k r) U(r) \tag{6.34}
\end{equation*}
$$

It follows from (6.34) that the first Born approximation to $\delta_{l}(k)$ is positive for an attractive potential.

It is shown in Topic 6.5.3 that this result follows more generally from the variable phase equations.

## 4. Yukawa potential

The first Born approximation for the scattering amplitude for the Yukawa potential (2.86) is

$$
\begin{equation*}
f(k, \theta)=\frac{2 g \gamma m}{\hbar^{2}\left(\mu^{2}+4 k^{2} \sin ^{2} \frac{\theta}{2}\right)} \tag{6.35}
\end{equation*}
$$

The first Born approximation for the phase shift for the Yukawa potential
(2.86) is

$$
\begin{equation*}
\delta_{l}(k)=\frac{g \gamma m}{\hbar^{2} k} Q_{l}\left(1+\frac{\mu^{2}}{2 k^{2}}\right) \tag{6.36}
\end{equation*}
$$

## 5. Rutherford formula

It follows using (2.89) and (2.90) that the first Born approximation for the cross-section for nonrelativistic scattering by the Coulomb potential (2.88) is

$$
\begin{equation*}
\frac{d \sigma}{d \Omega}(k, \theta)=\left(\frac{Z_{1} Z_{2} e^{2}}{4 E \sin ^{2} \frac{\theta}{2}}\right)^{2} \tag{6.37}
\end{equation*}
$$

where $E=p^{2} / 2 m$. (6.37) the Rutherford formula.
The above quantal derivation of the classical Rutherford formula was first given by G. Wentzel in 1926. Improved and exact derivations of the same result were given by J.R. Oppenheimer in 1927 and W. Gordon in 1928.

### 6.5 Solving the nonrelativistic scattering problem

Standard integral and differential methods

## Integral method

The standard integral method for solving the nonrelativistic scattering problem is:

Solve (6.15); the phase shift $\delta_{l}(k)$ is given by (6.12), (6.21) or (6.22).

## Differential method

The standard differential method for solving the nonrelativistic scattering problem is:

Solve (6.18) with boundary condition (6.20); the phase shift $\delta_{l}(k)$ is given by (6.12), (6.21) or (6.22).

## Qualitative picture

A qualitative picture of $\bar{\psi}_{0}(r, k)$ and $\delta_{0}(k)$ for an attractive potential is given in Figure 6.2.

An attractive potential increases the curvature of $\bar{\psi}_{0}(r, k)$; it pulls $\bar{\psi}_{0}(r, k)$ towards the origin and $\delta_{0}(k)$ is positive.

A repulsive potential decreases the curvature of $\bar{\psi}_{0}(r, k)$; it pushes $\bar{\psi}_{0}(r, k)$ away from the origin and $\delta_{0}(k)$ is negative.

Figure 6.2 Qualitative picture of $\bar{\psi}_{0}(r, k)$ for an attractive potential


## Some exact results

## 1. Phase shifts for the hard-sphere potential

It follows from (6.21) that the exact phase shifts for the hard-sphere potential (2.93) are given by

$$
\begin{equation*}
\tan \delta_{l}(k)=-\frac{\widehat{j}_{l}(k a)}{\widehat{n}_{l}(k a)} \tag{6.38}
\end{equation*}
$$

In particular,

$$
\begin{equation*}
\delta_{0}(k)=-k a \tag{6.39}
\end{equation*}
$$

and

$$
\begin{equation*}
a_{0}=a \tag{6.40}
\end{equation*}
$$

where $a_{0}$ is the scattering length.

## 2. Phase shifts for the square-well potential

The differential method can be solved exactly for the square-well potential (2.92) to yield

$$
\begin{equation*}
\tan \delta_{l}(k)=\frac{\widehat{j}_{l}^{\prime}(\alpha)-\beta_{l} \widehat{j}_{l}(\alpha)}{-\widehat{n}_{l}^{\prime}(\alpha)+\beta_{l} \widehat{n}_{l}(\alpha)} \tag{6.41}
\end{equation*}
$$

where prime means differentiation with respect to the argument and

$$
\begin{equation*}
\beta_{l}=\frac{\alpha_{0} \widehat{j}_{l}^{\prime}\left(\alpha_{0}\right)}{\alpha \widehat{j}_{l}\left(\alpha_{0}\right)} \tag{6.42}
\end{equation*}
$$

where

$$
\begin{equation*}
\alpha_{0}=\sqrt{\alpha^{2}+\frac{2 m a^{2} V_{0}}{\hbar^{2}}} \tag{6.43}
\end{equation*}
$$

and

$$
\begin{equation*}
\alpha=k a \tag{6.44}
\end{equation*}
$$

## 3. Phase shifts for the Boundary Condition Model

The Boundary Condition Model specifies the logarithmic derivative $f_{l} / r_{0_{l}}$ of $\bar{\psi}_{l}(r, k)$ at a boundary radius $r_{0_{l}}$. That is,

$$
\begin{equation*}
\left.\frac{d \bar{\psi}_{l}(r, k)}{d r}\right|_{r=r_{0_{l}}}=\frac{f_{l}}{r_{0_{l}}} \bar{\psi}_{l}\left(r_{0_{l}}, k\right) \tag{6.45}
\end{equation*}
$$

(6.45) is equivalent to the rank-2 separable potential (2.98).

It follows from (6.21) that exact phase shifts for the Boundary Condition model are given by

$$
\begin{equation*}
\tan \delta_{l}(k)=\frac{f_{l} \widehat{j}_{l}(\alpha)-\alpha \widehat{j}_{l}^{\prime}(\alpha)}{-f_{l} \widehat{n}_{l}(\alpha)+\alpha \widehat{n}_{l}^{\prime}(\alpha)} \tag{6.46}
\end{equation*}
$$

where prime means differentiation with respect to the argument and $\alpha=k r_{0_{i}}$.

The parameters $f_{l}$ and $r_{0_{l}}$ can be chosen to accommodate a phase shift which changes sign at some value of $k$.

## Variable phase differential method

The variable phase method for solving the nonrelativistic scattering problem is:

Determine the variable phase function $\delta_{l}(r, k)$ from

$$
\begin{equation*}
\frac{d \delta_{l}(r, k)}{d r}=-\frac{1}{k} U(r)\left[\widehat{j}_{l}(k r) \cos \delta_{l}(r, k)+\widehat{n}_{l}(k r) \sin \delta_{l}(r, k)\right]^{2} \tag{6.47}
\end{equation*}
$$

with boundary condition

$$
\begin{equation*}
\delta_{l}(0, k)=0 \tag{6.48}
\end{equation*}
$$

The phase shift $\delta_{l}(k)$ is determined from

$$
\begin{equation*}
\delta_{l}(\infty, k)=\delta_{l}(k) \tag{6.49}
\end{equation*}
$$

(6.47) for $S$-wave scattering is

$$
\begin{equation*}
\frac{d \delta_{0}(r, k)}{d r}=-\frac{1}{k} U(r) \sin ^{2}\left[k r+\delta_{0}(r, k)\right] \tag{6.50}
\end{equation*}
$$

## Comments

## 1. Calogero equation

(6.47), which we derive in Section 6.6, is the Calogero equation.

The Calogero equation is a nonlinear first-order ordinary differential equation for the variable phase function $\delta_{l}(r, k)$.

The variable phase method for solving the nonrelativistic scattering problem was invented by Calogero (1967).
2. Direct method for solving the scattering problem

In contrast with the standard integral and differential methods which give $\delta_{l}(k)$ indirectly via $\bar{\psi}_{l}(r, k)$, the variable phase method gives $\delta_{l}(k)$ directly via (6.49).

## 3. Sign of the phase shift

It follows from (6.47) to (6.49) that $\delta_{l}(k)$ is positive for an attractive potential and negative for a repulsive potential.

### 6.6 Some derivations

## Derivation of (6.1)

It follows from (3.25) and (3.22) that

$$
\begin{equation*}
\left|\vec{p} \pm>=\Omega_{ \pm}\right| \vec{p}>=\left[1+G\left(\epsilon_{p} \pm i 0\right) V\right] \mid \vec{p}> \tag{6.51}
\end{equation*}
$$

(6.1) then follows using (5.4).

## Derivation of (6.4)

It follows from (5.2) and (6.1) that

$$
\begin{equation*}
G(z) V\left|\vec{p}>=G_{0}(z) T(z)\right| \vec{p}>=G_{0}(z) V \mid \vec{p}+> \tag{6.52}
\end{equation*}
$$

where $z=\epsilon_{p} \pm i 0$. (6.4) then follows from (6.51).

## Derivation of (6.6)

We consider (6.5) for large $r=|\vec{x}|$. Now

$$
\begin{align*}
& |\vec{x}-\vec{y}|=\left(|\vec{x}|^{2}-2 \vec{x} \cdot \vec{y}+|\vec{y}|^{2}\right)^{\frac{1}{2}} \\
& \quad=r\left(1-\frac{\vec{x} \cdot \vec{y}}{r^{2}}\right)+O\left(\frac{|\vec{y}|^{2}}{r^{2}}\right) \tag{6.53}
\end{align*}
$$

so

$$
\begin{equation*}
\lim _{r \rightarrow \infty} \psi_{\vec{k}}(\vec{x})=e^{i \vec{k} \cdot \vec{x}}-\frac{e^{i k r}}{4 \pi r} \int d^{3} y e^{-i k \vec{x} \cdot \vec{y} / r} U(\vec{y}) \psi_{\vec{k}}(\vec{y}) \tag{6.54}
\end{equation*}
$$

which, using (6.3), is (6.6).

## Derivation of (6.47) to (6.49)

We write (6.17) as

$$
\begin{equation*}
\bar{\psi}_{l}(r, k)=\widehat{j}_{l}(k r) c_{l}(r, k)+\widehat{n}_{l}(k r) s_{l}(r, k) \tag{6.55}
\end{equation*}
$$

where

$$
\begin{equation*}
c_{l}(r, k)=1-\frac{1}{k} \int_{r}^{\infty} d r^{\prime} \widehat{n}_{l}\left(k r^{\prime}\right) U\left(r^{\prime}\right) \bar{\psi}_{l}\left(r^{\prime}, k\right) \tag{6.56}
\end{equation*}
$$

and

$$
\begin{equation*}
s_{l}(r, k)=-\frac{1}{k} \int_{0}^{r} d r^{\prime} \widehat{j}_{l}\left(k r^{\prime}\right) U\left(r^{\prime}\right) \bar{\psi}_{l}\left(r^{\prime}, k\right) \tag{6.57}
\end{equation*}
$$

Then

$$
\begin{equation*}
c_{l}(\infty, k)=1 \tag{6.58}
\end{equation*}
$$

and

$$
\begin{equation*}
s_{l}(\infty, k)=\tan \delta_{l}(k) \tag{6.59}
\end{equation*}
$$

Defining $t_{l}(r, k)$ by

$$
\begin{equation*}
t_{l}(r, k)=\frac{s_{l}(r, k)}{c_{l}(r, k)} \tag{6.60}
\end{equation*}
$$

it follows that

$$
\begin{equation*}
\frac{d t_{l}(r, k)}{d r}=-\frac{1}{k} U(r)\left[\widehat{j}_{l}(k r)+\widehat{n}_{l}(k r) t_{l}(r, k)\right]^{2} \tag{6.61}
\end{equation*}
$$

and

$$
\begin{gather*}
t_{l}(0, k)=0  \tag{6.62}\\
t_{l}(\infty, k)=\tan \delta_{l}(k) \tag{6.63}
\end{gather*}
$$

(6.47) to (6.49) follow on defining

$$
\begin{equation*}
\delta_{l}(r, k)=\tan ^{-1} t_{l}(r, k) \tag{6.64}
\end{equation*}
$$

## Chapter 7

## PARTIAL-WAVE AMPLITUDES REVISITED

### 7.1 Introductory remarks

We have previously considered the analytic properties of the matrix elements of the Green's operators and the $T$ operator as functions of complex energy. In this chapter we consider the nonrelativistic partial-wave amplitude as a function of complex momentum and energy and complex angular momentum.

We show in Section 7.3 how a Breit-Wigner resonance appears in the partialwave amplitude as a function of complex energy and we also develop another method (the $N / D$ method) for solving the scattering problem. We show in Section 7.4 how poles of the partial-wave amplitude as a function of complex angular momentum interpolate a family of bound states and resonances. Some derivations are given in Section 7.5.

### 7.2 Definition

The key to extending the definition of the partial-wave amplitude $a_{l}(k)$ to complex values of momentum and angular momentum is the fact that the Bessel function $J_{\lambda}(z)$ is defined for complex argument $z$ and complex order $\lambda$.

Thus, the function $a_{\lambda}(\kappa)$ is defined by analytic continuation from $a_{l}(k)$ by

$$
\begin{equation*}
a_{\lambda}(\kappa)=\frac{s_{\lambda}(\kappa)-1}{2 i \kappa} \tag{7.1}
\end{equation*}
$$

where

$$
\begin{equation*}
s_{\lambda}(\kappa)=e^{2 i \delta_{\lambda}(\kappa)} \tag{7.2}
\end{equation*}
$$

and where

$$
\begin{equation*}
\tan \delta_{\lambda}(\kappa)=-\frac{1}{\kappa} \int_{0}^{\infty} d r \widehat{j}_{l}(\kappa r) U(r) \bar{\psi}_{\lambda}(r, \kappa) \tag{7.3}
\end{equation*}
$$

where $\bar{\psi}_{\lambda}(r, \kappa)$ is determined from

$$
\begin{equation*}
\bar{\psi}_{\lambda}(r, \kappa)=\widehat{j}_{\lambda}(\kappa r)+\int_{0}^{\infty} d r^{\prime} \bar{G}_{0 \lambda}\left(r, r^{\prime}, \kappa\right) U\left(r^{\prime}\right) \bar{\psi}_{\lambda}\left(r^{\prime}, \kappa\right) \tag{7.4}
\end{equation*}
$$

with

$$
\begin{equation*}
\bar{G}_{0 \lambda}\left(r, r^{\prime}, \kappa\right)=-\frac{1}{\kappa} \widehat{j}_{\lambda}\left(\kappa r_{<}\right) \widehat{n}_{\lambda}\left(\kappa r_{>}\right) \tag{7.5}
\end{equation*}
$$

### 7.3 Complex momentum and energy

In this section we consider the partial-wave amplitude for integer angular momentum $l$ and complex momentum $\hbar \kappa$ and complex energy $z=\hbar^{2} \kappa^{2} / 2 m$. That is, we consider the functions $a_{l}(\kappa)$ and $a_{l}(z)$.

## Analytic properties

The analytic properties of $a_{l}(\kappa)$ and $a_{l}(z)$ for a potential which is a superposition of Yukawa potentials (2.91) are summarized on Figures 7.3 and 7.4.


Figure 7.3 Analytic properties of $a_{l}(\kappa)$

Figure 7.4 Analytic properties of $a_{l}(z)$


## Complex momentum

## 1. Physical region

The physical region in the complex $\kappa$-plane is the positive real axis. That is,

$$
\begin{equation*}
\kappa=k \quad \text { where } \quad k \geq 0 \tag{7.6}
\end{equation*}
$$

## 2. Branch cuts

$a_{l}(\kappa)$ has branch cuts on the imaginary axis. The branch points at $\pm i \mu / 2$ depend only on the range of the potential.

## 3. Branch cuts: first Born approximation

The branch points of $a_{l}(\kappa)$ can be seen by noting that the first Born approximation to $a_{l}(\kappa)$ for the Yukawa potential (2.86) is

$$
\begin{equation*}
a_{l}^{B}(\kappa)=\frac{g m}{\hbar^{2} \kappa^{2}} Q_{l}\left(1+\frac{\mu^{2}}{2 \kappa^{2}}\right) \tag{7.7}
\end{equation*}
$$

and recalling that the Legendre function $Q_{l}(z)$ has branch points at $z= \pm 1$.

## 4. Bound states, virtual states, resonances

$a_{l}(\kappa)$ has poles on the imaginary axis and in the lower-half $\kappa$-plane off the imaginary axis.

The poles on the positive real axis correspond to bound states.
The poles on the negative real axis correspond to virtual states.
The other poles in the lower-half plane correspond to resonances.

## Complex energy

## 1. Physical and unphysical sheets

$a_{l}(z)$ is a function on a two-sheeted Riemann surface.
The first sheet (the physical sheet) corresponds to the upper-half $\kappa$-plane.
The second sheet (the unphysical sheet) corresponds to the lower-half $\kappa$-plane.

## 2. Branch cuts, physical region

$a_{l}(z)$ has branch cuts on the real axis. The left-hand cut depends on the the potential; the right-hand cut arises because the Riemann surface is twosheeted.

The physical region in the complex $z$-plane is the upper lip of the right-hand cut on the physical sheet. That is,

$$
\begin{equation*}
z=\epsilon+i 0 \quad \text { where } \quad \epsilon=\frac{\hbar^{2} k^{2}}{2 m} \tag{7.8}
\end{equation*}
$$

## 3. Left-hand cut: first Born approximation

The branch points of $a_{l}(z)$ can be seen by noting that the first Born approximation to $a_{l}(z)$ for the Yukawa potential (2.86) is

$$
\begin{equation*}
a_{l}^{B}(z)=\frac{g}{2 z} Q_{l}\left(1-\frac{2 \epsilon_{L}}{z}\right) \tag{7.9}
\end{equation*}
$$

where

$$
\begin{equation*}
\epsilon_{L}=-\frac{\hbar^{2} \mu^{2}}{8 m} \tag{7.10}
\end{equation*}
$$

and recalling that the Legendre function $Q_{l}(z)$ has branch points at $z= \pm 1$.

## 4. Bound states, virtual states, resonances

Bound-state poles occur on the physical sheet.

Virtual-state poles and resonance poles occur on the unphysical sheet.

## Resonance

We suppose that $s_{l}(z)$ for some $l$ has a pole when

$$
\begin{equation*}
z=\bar{\epsilon}=\epsilon_{R}-i \frac{\Gamma}{2} \tag{7.11}
\end{equation*}
$$

where $\epsilon_{R}$ and $\Gamma$ are both positive. (7.11) characterizes a resonance pole of $s_{l}(z)$.

Now since

$$
\begin{equation*}
\delta_{l}^{*}(z)=\delta_{l}\left(z^{*}\right) \tag{7.12}
\end{equation*}
$$

it follows that $s_{l}(z)$ has a zero when $z=\bar{\epsilon}^{*}$ as per Figure 7.5. Accordingly, we write

$$
\begin{equation*}
e^{2 i \delta_{l}(z)}=\frac{\bar{\epsilon}^{*}-z}{\bar{\epsilon}-z} e^{2 i \delta_{b g}(z)} \tag{7.13}
\end{equation*}
$$

where $\delta_{b g}(z)$ is real on the real axis and where $e^{2 i \delta_{b g}(z)}$ is not singular when $z=\bar{\epsilon}$ or $\bar{\epsilon}^{*}$.

Figure 7.5 Resonance pole and zero of $s_{l}(z)$


It follows from (7.13) that

$$
\begin{equation*}
\delta_{l}(\epsilon)=\delta_{b g}(\epsilon)+\delta_{r e s}(\epsilon) \tag{7.14}
\end{equation*}
$$

where $\delta_{r e s}(\epsilon)$ is the angle shown in Figure 7.6. That is,

$$
\begin{equation*}
\sin \delta_{\operatorname{res}}(\epsilon)=\frac{\Gamma / 2}{\sqrt{\left(\epsilon-c_{R}\right)^{2}+(\Gamma / 2)^{2}}} \tag{7.15}
\end{equation*}
$$



## Comments

## 1. Experimental manifestation

Figure 7.6 shows that as $\epsilon$ is increased past $\epsilon_{R}, \delta_{\text {res }}(\epsilon)$ and hence $\delta_{l}(\epsilon)$ increases rapidly by $\pi$ provided $\bar{\epsilon}$ is close to the real axis, that is, if $\Gamma$ is sufficiently small.

The rapid increase of $\delta_{l}(\epsilon)$ by $\pi$ is the experimental manifestation of a resonance pole.

## 2. Breit-Wigner formula

If $\delta_{b g}(z)=0$, then

$$
\begin{equation*}
\sin \delta_{l}(\epsilon)=\frac{\Gamma / 2}{\sqrt{\left(\epsilon-\epsilon_{R}\right)^{2}+(\Gamma / 2)^{2}}} \tag{7.16}
\end{equation*}
$$

and

$$
\begin{equation*}
\sigma_{l}(p)=\frac{4 \pi(2 l+1)}{k^{2}} \frac{(\Gamma / 2)^{2}}{\left(\epsilon-\epsilon_{R}\right)^{2}+(\Gamma / 2)^{2}} \tag{7.17}
\end{equation*}
$$

for $\epsilon$ near $\epsilon_{R}$. (7.17) is the Breit-Wigner resonance formula.
The resonance factor

$$
\begin{equation*}
\frac{(\Gamma / 2)^{2}}{\left(\epsilon-\epsilon_{R}\right)^{2}+(\Gamma / 2)^{2}} \tag{7.18}
\end{equation*}
$$

has a maximum of unity when $\epsilon=\epsilon_{R}$ and is equal to one-half when $\epsilon=\epsilon_{R} \pm \frac{1}{2} \Gamma$.

The resonance occurs when $\epsilon=\epsilon_{R}$ and $\Gamma$ is the resonance width.

It follows from (7.16) that

$$
\begin{equation*}
\delta_{l}\left(\epsilon_{R}\right)=\frac{\pi}{2} \tag{7.19}
\end{equation*}
$$

## 3. Temporary capture of the particle

Far from the target the scattering wave function (6.7) consists of a free-particle incoming wave packet $\psi_{\text {in }}(\vec{x}, t)$ and a free-particle outgoing spherical wave packet $\psi_{\text {scatt }}(\vec{x}, t)$.

The behavior of $\psi_{\text {scatt }}(\vec{x}, t)$ depends on the relative sizes of $\Gamma$ and the energy uncertainty $\Delta \epsilon$ of $\psi_{\text {in }}(\vec{x}, t)$ when there is a resonance.

A bump in the cross-section corresponding to the resonance will be observed when $\Delta \epsilon \ll \Gamma$. In this case, the outgoing wave packet lags the incoming wave packet by a time interval

$$
\begin{equation*}
\frac{d \delta_{l}(\epsilon)}{d \epsilon} \tag{7.20}
\end{equation*}
$$

which may be arbitrarily large. The time lag corresponds to the temporary capture of the particle in a metastable state.

## 4. Exponential decay

The resonance must be very narrow ( $\Gamma \ll \Delta \epsilon$ ) for the decay of the metastable state to be observed directly. In this case, for a pure Breit-Wigner resonance, the time-dependence of $\left|\psi_{\text {scatt }}(\vec{x}, t)\right|^{2}$ has the form

$$
\begin{equation*}
e^{-\Gamma t} \tag{7.21}
\end{equation*}
$$

for sufficiently large values of $t .1 / \Gamma$ is the lifetime of the metastable state.

## Levinson's theorem

We show in Section 7.5 that $\delta_{l}(\epsilon)$ satisfies

$$
\begin{equation*}
\delta_{l}(0)-\delta_{l}(\infty)=\left(n_{b}-n_{p}\right) \pi \tag{7.22}
\end{equation*}
$$

where $n_{b}$ is the number of bound states with angular momentum $l$ and $n_{p}$ is the number of poles of $D_{l}(z)$ on the physical sheet.

## Comments

## 1. Levinson's theorem

(7.22) is Levinson's theorem; it is a remarkable connection between the phase shift and the number of bound states.

## 2. Poles

Poles of $D_{l}(z)$ on the physical sheet are called $C D D$ poles.

## $N / D$ equations

We show in Section 7.5 that $a_{l}(z)$ may be written in the form

$$
\begin{equation*}
a_{l}(z)=N_{l}(z) / D_{l}(z) \tag{7.23}
\end{equation*}
$$

where

$$
\begin{equation*}
D_{l}(z)=1-\frac{z-\epsilon_{0}}{\pi} \int_{0}^{\infty} \frac{d \epsilon k N_{l}(\epsilon)}{(\epsilon-z)\left(\epsilon-\epsilon_{0}\right)} \tag{7.24}
\end{equation*}
$$

and

$$
\begin{equation*}
N_{l}(z)=B_{l}(z)+\int_{0}^{\infty} d \epsilon K_{l}(z, \epsilon) N_{l}(\epsilon) \tag{7.25}
\end{equation*}
$$

where

$$
\begin{equation*}
K_{l}(z, \epsilon)=\frac{k}{\pi} \frac{\left(\epsilon-\epsilon_{0}\right) B_{l}(\epsilon)-\left(z-\epsilon_{0}\right) B_{l}(z)}{(\epsilon-z)\left(\epsilon-\epsilon_{0}\right)} \tag{7.26}
\end{equation*}
$$

and

$$
\begin{equation*}
B_{l}(z)=\frac{1}{\pi} \int_{-\infty}^{\epsilon_{L}} \frac{d \epsilon \operatorname{Im} a_{l}(\epsilon+i 0)}{\epsilon-z} \tag{7.27}
\end{equation*}
$$

where $\epsilon_{0}$ is a real constant.

We show also that

$$
\begin{equation*}
e^{2 i \delta_{l}(\epsilon)}=\frac{D_{l}(\epsilon-i 0)}{D_{l}(\epsilon+i 0)} \tag{7.28}
\end{equation*}
$$

## Comments

## 1. $N / D$ equations

(7.23) to (7.27) are the $N / D$ equations for $a_{l}(z)$.
(7.24) gives $D_{l}(z)$ in terms on $N_{l}(z)$ and (7.25) is a linear integral equation for $N_{l}(z)$.

Solving (7.25) requires values of $B_{l}(z)$; that is, values of $\operatorname{Im} a_{l}(\epsilon+i 0)$ on the left-hand cut.

The $N / D$ equations were first derived by Chew and Mandelstam (1960).
The $N / D$ equations are believed to hold for a wide class of interactions because the only direct dependence on the interaction potential is through $\epsilon_{L}$.

## 2. Self-consistent method

(7.23) to (7.27) provide a method to determine $a_{l}(z)$ :

Solve (7.25) for $N_{l}(z)$ for some choice of $\operatorname{Im} a_{l}(\epsilon+i 0)$ on the left-hand cut then compute $D_{l}(z)$ using (7.24). (7.23) then gives $a_{l}(z)$. Repeat the process until self-consistency is obtained.

## 3. Determinantal approximation

Rather than solving (7.25) for $N_{l}(z)$ a simple approximation (the determinan-
tal approximation) is to take

$$
\begin{equation*}
N_{l}(z)=a_{l}^{B}(z) \tag{7.29}
\end{equation*}
$$

where $a_{l}^{B}(z)$ is the first Born approximation to $a_{l}(z)$ [(7.9) gives the first Born approximation for the Yukawa potential] and to determine $D_{l}(z)$ from (7.24).

## 4. Separable potential

The partial-wave amplitude (5.19) for the separable potential (2.94) has the form (7.23) with

$$
\begin{equation*}
N_{l}(z)=\frac{\pi \hbar}{2 z}\left|v_{l}(\kappa)\right|^{2} \tag{7.30}
\end{equation*}
$$

and $\epsilon_{0}=\infty$. Thus the determinantal approximation (7.29) is equivalent to replacing the actual potential with the separable potential (2.94) where

$$
\begin{equation*}
\left|v_{l}(k)\right|^{2}=\frac{\hbar k^{2}}{\pi m} a_{l}^{B}(k) \tag{7.31}
\end{equation*}
$$

## 5. Bound states

A pole of $a_{l}(z)$ corresponds to a bound state. It follows from (7.23) that

$$
\begin{equation*}
D_{l}\left(\epsilon_{b}\right)=0 \tag{7.32}
\end{equation*}
$$

yields the bound-state energies.

## 6. Breit-Wigner resonance

We show in Section 7.5 that the resonance parameters specified by (7.11) for a pure Breit-Wigner resonance are determined by

$$
\begin{equation*}
\operatorname{Re} D_{l}\left(\epsilon_{R}+i 0\right)=0 \tag{7.33}
\end{equation*}
$$

and

$$
\begin{equation*}
\frac{1}{2} \Gamma=\left.\frac{\operatorname{Im} D_{l}(z)+\frac{d}{d z} \operatorname{Im} D_{l}(z)}{\frac{d}{d z} \operatorname{Re} D_{l}(z)}\right|_{z=\epsilon_{R}+i 0} \tag{7.34}
\end{equation*}
$$

### 7.4 Complex angular momentum; Regge poles

In this section we consider the function $a_{\lambda}(k)$ defined for complex angular momentum $\lambda$ and real momentum $\hbar k$ and real energy $\epsilon=\hbar^{2} k^{2} / 2 m$.

The physical region in the complex $\lambda$-plane is the set of non-negative integers.

$$
\begin{equation*}
\lambda=l=0,1,2, \cdots \tag{7.35}
\end{equation*}
$$

## Analytic properties

The analytic properties of $a_{\lambda}(k)$ have been investigated for a wide variety of potentials and in particular for a potential which is a superposition of Yukawa potentials (2.91). The first investigation was by Regge (1959); see also Bottino, Longoni and Regge (1962).

## Properties

1. $a_{\lambda}(k)$ vanishes exponentially as $\lambda \rightarrow \infty$.
2. $a_{\lambda}(k)$ has only a finite number of simple poles at $\alpha_{1}(k), \cdots, \alpha_{r_{\max }}(k)$ in the upper-half $\lambda$-plane when $\operatorname{Re} \lambda>-\frac{1}{2}$.
3. Each $\alpha_{r}(k)$ is real for negative $k^{2}$, complex for positive $k^{2}$ and satisfies

$$
\begin{equation*}
\lim _{k^{2} \rightarrow \pm \infty} \alpha_{r}(k)=-n_{r} \tag{7.36}
\end{equation*}
$$

where $n_{r}$ is a negative integer.
4. $\operatorname{Re} \alpha_{r}(k)$ is bounded above for every $r$.

## Comments

## 1. Regge poles

The poles of $a_{\lambda}(k)$ for complex $\lambda$ are called Regge poles.

## 2. Regge trajectory

A plot of $\operatorname{Re} \alpha_{r}(k)$ and $\operatorname{Im} \alpha_{r}(k)$ as a function of $\epsilon=\hbar^{2} k^{2} / 2 m$ is a Regge trajectory. A typical Regge trajectory is shown in Figure 7.7.

Figure 7.7 A typical Regge trajectory


## Regge expansion of the scattering amplitude

It follows from the theorem of residues in complex variable theory that the partial-wave expansion of the scattering amplitude $f(k, \theta)$ given by (4.26) may be written as

$$
\begin{equation*}
f(k, \theta)=\frac{i}{2} \int_{C} \frac{(2 \lambda+1) a_{\lambda}(k) P_{\lambda}(-\cos \theta) d \lambda}{\sin \pi \lambda} \tag{7.37}
\end{equation*}
$$

where $C$ encloses the zeros of $\sin \pi \lambda$ in the complex $\lambda$-plane.

It follows from the properties of $a_{\lambda}(k)$ given in Topic 7.4.1 that for a superposition of Yukawa potentials (2.91) the contour in (7.37) can be moved to yield

$$
\begin{gather*}
f(k, \theta)=-\pi \sum_{r=1}^{r_{\max }} \frac{\left(2 \alpha_{r}(k)+1\right) \beta_{r}(k) P_{\alpha_{r}(k)}(-\cos \theta)}{\sin \pi \alpha_{r}(k)} \\
\quad+\frac{i}{2} \int_{-\frac{1}{2}-i \infty}^{-\frac{1}{2}+i \infty} \frac{(2 \lambda+1) a_{\lambda}(k) P_{\lambda}(-\cos \theta) d \lambda}{\sin \pi \lambda} \tag{7.38}
\end{gather*}
$$

where $\alpha_{r}(k)$ is a (Regge) pole of $a_{\lambda}(k)$ for complex $\lambda$ and

$$
\begin{equation*}
\beta_{r}(k)=\lim _{\lambda \rightarrow r}\left(\lambda-\alpha_{r}(k)\right) a_{\lambda}(k) \tag{7.39}
\end{equation*}
$$

is the residue of the pole at $\alpha_{r}(k)$.

## Comments

## 1. Watson-Sommerfeld transformation

(7.37) is the Watson-Sommerfeld transformation.

Re-expressing an infinite summation as an integral in the complex plane is an old trick: it appears in Titchmarsh (1939); Sommerfeld (1925), Appendix to Chapter VI (application to propagation of radio waves); and was first applied to that problem by G.N. Watson in 1918.

## 2. Regge expansion of the scattering amplitude

(7.38) is the Regge expansion of the scattering amplitude.
3. Regge poles, bound states and resonances

It is seen from the first term in (7.38) that the scattering amplitude has a pole when $\alpha_{r}(k)$ is an integer.

Indeed, as indicated on Figure 7.7, each Regge pole $\alpha_{r}(k)$ interpolates a family of bound states and resonances:

The negative values of $\epsilon_{b}$ for which

$$
\begin{equation*}
\alpha_{r}\left(\epsilon_{b}\right)=l \tag{7.40}
\end{equation*}
$$

for $l=0,1,2, \cdots$ are the bound-state energies and $l \hbar$ is the angular momentum of the bound state.

The positive values of $\epsilon_{R}$ and $\Gamma$ given by

$$
\begin{gather*}
\operatorname{Re} \alpha_{r}\left(\epsilon_{R}\right)=l  \tag{7.41}\\
\frac{1}{2} \Gamma=\frac{\operatorname{Im} \alpha_{r}\left(\epsilon_{R}\right)}{\left.\frac{d}{d \epsilon} \operatorname{Re} \alpha_{r}(\epsilon) \right\rvert\, \epsilon=\epsilon_{R}} \tag{7.42}
\end{gather*}
$$

for $l=0,1,2, \cdots$ are the resonance parameters of a pure Breit-Wigner resonance with angular momentum $l \hbar$.

## 4. Separable potential

It follows from (5.19) that there is a single Regge pole $\alpha(k)$ for the separable potential (2.94) given by

$$
\begin{equation*}
1+\int_{0}^{\infty} \frac{d p^{\prime}\left|v_{\alpha(k)}\left(p^{\prime}\right)\right|^{2}}{\epsilon_{p}-\epsilon_{p^{\prime}}+i 0}=0 \tag{7.43}
\end{equation*}
$$

For further discussion see McMillan (1963, 1964).

### 7.5 Some derivations

## Derivation of (7.22)

It follows from Cauchy's Theorem that if a function $f(z)$ of the complex variable $z$ is analytic inside and on a closed contour $C$ then

$$
\begin{equation*}
\frac{1}{2 \pi i} \int_{C} d z \frac{f^{\prime}(z)}{f(z)}=n_{z e r o s}-n_{\text {poles }} \tag{7.44}
\end{equation*}
$$

where $n_{\text {zeros }}$ and $n_{\text {poles }}$ are the number of zeros and poles, respectively, of $f(z)$ inside $C$.

We consider $f(z)=D_{l}(z) . \quad D_{l}(z)$ has a branch cut along the real axis; the zeros of $D_{l}(z)$ correspond to bound states and the poles of $D_{l}(z)$ are $C D D$ poles.

Using

$$
\begin{equation*}
d z \frac{f^{\prime}(z)}{f(z)}=d z \frac{d}{d z} \ln f(z) \tag{7.45}
\end{equation*}
$$

and taking into account the analytic properties of $D_{l}(z)$ it follows that the contour in (7.44) can be moved to yield

$$
\begin{gather*}
\int_{\infty}^{0} d \epsilon \frac{d}{d \epsilon} \ln D_{l}(\epsilon-i 0)+\int_{0}^{\infty} d \epsilon \frac{d}{d \epsilon} \ln D_{l}(\epsilon+i 0)  \tag{7.46}\\
=2 \pi i\left(n_{b}-n_{p}\right)
\end{gather*}
$$

the left side of which is

$$
\begin{gather*}
\int_{0}^{\infty} d \epsilon \frac{d}{d \epsilon}\left[\ln D_{l}(\epsilon+i 0)-\ln D_{l}(\epsilon-i 0)\right]=\int_{0}^{\infty} d \epsilon \frac{d}{d \epsilon} \ln \frac{D_{l}(\epsilon+i 0)}{D_{l}(\epsilon-i 0)} \\
=2 i \int_{0}^{\infty} d \epsilon \frac{d}{d \epsilon} \delta_{l}(\epsilon)=2 i\left[\delta_{l}(\infty)-\delta_{l}(0)\right] \tag{7.47}
\end{gather*}
$$

using (7.28).

## Derivation of (7.23) to (7.27)

We write $a_{l}(z)$ as (7.23) where $N_{l}(z)$ is chosen to have only the left-hand cut and $D_{l}(z)$ the right-hand cut. In addition

$$
\begin{equation*}
N_{l}^{*}(z)=N_{l}\left(z^{*}\right) \tag{7.48}
\end{equation*}
$$

and

$$
\begin{equation*}
D_{l}^{*}(z)=D_{l}\left(z^{*}\right) \tag{7.49}
\end{equation*}
$$

We write Cauchy integrals for $N_{l}(z)$ and $D_{l}(z)$ :

$$
\begin{equation*}
N_{l}(z)=\frac{1}{2 \pi i} \int_{C} \frac{N_{l}\left(z^{\prime}\right) d z^{\prime}}{z^{\prime}-z} \tag{7.50}
\end{equation*}
$$

$$
\begin{equation*}
D_{l}(z)=\frac{1}{2 \pi i} \int_{C} \frac{D_{l}\left(z^{\prime}\right) d z^{\prime}}{z^{\prime}-z} \tag{7.51}
\end{equation*}
$$

Moving the contours and assuming that the contribution from the circle at infinity vanishes and using (7.48) and (7.49) yields

$$
\begin{gather*}
N_{l}(z)=\frac{1}{2 \pi i}\left[\int_{-\infty}^{\epsilon_{L}} \frac{N_{l}(\epsilon+i 0) d \epsilon}{\epsilon-z}+\int_{\epsilon_{L}}^{-\infty} \frac{N_{l}(\epsilon-i 0) d \epsilon}{\epsilon-z}\right] \\
=\frac{1}{\pi} \int_{-\infty}^{\epsilon_{L}} \frac{\operatorname{Im} N_{l}(\epsilon+i 0) d \epsilon}{\epsilon-z} \tag{7.52}
\end{gather*}
$$

and

$$
\begin{align*}
D_{l}(z)=\frac{1}{2 \pi i} & {\left[\int_{-\infty}^{0} \frac{D_{l}(\epsilon-i 0) d \epsilon}{\epsilon-z}+\int_{0}^{\infty} \frac{D_{l}(\epsilon+i 0) d \epsilon}{\epsilon-z}\right] } \\
& =\frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} D_{l}(\epsilon+i 0) d \epsilon}{\epsilon-z} \tag{7.53}
\end{align*}
$$

We choose to set $D_{l}\left(\epsilon_{0}\right)=1$ at some arbitrary real negative energy $\epsilon_{0}$ where $\epsilon_{L}<\epsilon_{0}<0$, that is,

$$
\begin{equation*}
\frac{1}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} D_{l}(\epsilon+i 0) d \epsilon}{\epsilon-\epsilon_{0}}=1 \tag{7.54}
\end{equation*}
$$

so that (7.51) may be written as

$$
\begin{equation*}
D_{l}(z)=1+\frac{z-\epsilon_{0}}{\pi} \int_{0}^{\infty} \frac{\operatorname{Im} D_{l}(\epsilon+i 0) d \epsilon}{(\epsilon-z)\left(\epsilon-\epsilon_{0}\right)} \tag{7.55}
\end{equation*}
$$

The constant $\epsilon_{0}$ is a subtraction constant. The subtraction at $\epsilon_{0}$ improves the convergence of the integral and introduces an arbitrary constant in the formalism.

We now re-express $\operatorname{Im} N_{l}(\epsilon+i 0)$ and $\operatorname{Im} D_{l}(\epsilon+i 0)$ in the intregrands in (7.50) and (7.55). For $-\infty<\epsilon \leq \epsilon_{L}$,

$$
\begin{gather*}
\operatorname{Im} N_{l}(\epsilon+i 0)=\operatorname{Im}\left[D_{l}(\epsilon+i 0) a_{l}(\epsilon+i 0)\right]  \tag{7.56}\\
=D_{l}(\epsilon) \operatorname{Im} a_{l}(\epsilon+i 0)
\end{gather*}
$$

and for $0 \leq \epsilon<\infty$,

$$
\begin{align*}
& \operatorname{Im} D_{l}(\epsilon+i 0)=\operatorname{Im} \frac{N_{l}(\epsilon+i 0)}{a_{l}(\epsilon+i 0)} \\
= & N_{l}(\epsilon) \operatorname{Im} \frac{1}{a_{l}(\epsilon+i 0)}=-k N_{l}(\epsilon) \tag{7.57}
\end{align*}
$$

so (7.50) and (7.55) may be written as

$$
\begin{gather*}
N_{l}(z)=\frac{1}{\pi} \int_{-\infty}^{\epsilon_{L}} \frac{D_{l}(\epsilon) \operatorname{Im} a_{l}(\epsilon+i 0) d \epsilon}{\epsilon-z}  \tag{7.58}\\
D_{l}(z)=1-\frac{z-\epsilon_{0}}{\pi} \int_{0}^{\infty} \frac{k N_{l}(\epsilon) d \epsilon}{(\epsilon-z)\left(\epsilon-\epsilon_{0}\right)} \tag{7.59}
\end{gather*}
$$

(7.59) is (7.24) and substituting (7.59) into (7.58) into yields (7.25).

## Derivation of (7.28)

It follows from (7.59) that

$$
\begin{gather*}
D_{l}(\epsilon+i 0)-D_{l}(\epsilon-i 0) \\
=-\frac{\epsilon-\epsilon_{0}}{\pi} \int_{0}^{\infty}\left[\frac{1}{\epsilon^{\prime}-\epsilon-i 0}-\frac{1}{\epsilon^{\prime}-\epsilon+i 0}\right] \frac{k^{\prime} N_{l}\left(\epsilon^{\prime}\right) d \epsilon^{\prime}}{\epsilon^{\prime}-\epsilon_{0}}  \tag{7.60}\\
=-2 \pi i \frac{\epsilon-\epsilon_{0}}{\pi} \int_{0}^{\infty} \delta\left(\epsilon^{\prime}-\epsilon\right) \frac{k^{\prime} N_{l}\left(\epsilon^{\prime}\right) d \epsilon^{\prime}}{\epsilon^{\prime}-\epsilon_{0}}=-2 i k N_{l}(\epsilon)
\end{gather*}
$$

That is,

$$
\begin{equation*}
N_{l}(\epsilon)=\frac{1}{2 i k}\left[D_{l}(\epsilon-i 0)-D_{l}(\epsilon+i 0)\right] \tag{7.61}
\end{equation*}
$$

(7.28) follows from (4.28), (7.23) and (7.61).

## Derivation of (7.33) and (7.34)

We suppose that $s_{l}(z)$ has a pole as specified by (7.11) and such that $\delta_{b g}(z)=0$.

It follows from (7.19) that $a_{l}\left(\epsilon_{R}\right)$ is imaginary; (7.33) then follows from (7.23) and (7.48).
(7.34) follows from (7.23) on expanding $D_{l}(\epsilon+i 0)$ about $\epsilon=\epsilon_{R}$.

## Appendix A Unitarily-equivalent potentials

## A. 1 Introductory remarks

We give the main methods for solving the scattering problem for a spinless particle by a fixed target in Chapters 3 to 6 . That is, the material in these chapters shows how to determine the scattering operator $S$ for a given interaction potential $V$.

Is there a unique $V$ for each $S$ ? Clearly not since a scattering experiment involves a measurement far from the target and so cannot for any finite energy be sensitive to every short-range detail of the target-projectile potential.

We show in this appendix how to generate a family of potentials all of which are equivalent as far as scattering is concerned. The potentials generated are nonlocal and differ from each other only at short distances.

## A. 2 The unitary transformation

We suppose that we have constructed a potential $V$ which via the Hamiltonian (2.7) yields the observed cross sections and binding energies. We now consider a new potential $\widetilde{V}$ defined so that the new Hamiltonian

$$
\begin{equation*}
\widetilde{H}=H_{0}+\widetilde{V} \tag{A.1}
\end{equation*}
$$

is related to the old Hamiltonian (2.7) by

$$
\begin{equation*}
\widetilde{H}=\mathcal{U} H \mathcal{U}^{\dagger} \tag{A.2}
\end{equation*}
$$

where $\mathcal{U}$ is unitary. That is,

$$
\begin{equation*}
\widetilde{V}=-H_{0}+\mathcal{U}\left(H_{0}+V\right) \mathcal{U}^{\dagger} \tag{A.3}
\end{equation*}
$$

where

$$
\begin{equation*}
\mathcal{U} \mathcal{U}^{\dagger}=\mathcal{U}^{\dagger} \mathcal{U}=1 \tag{A.4}
\end{equation*}
$$

## Claim

It is sufficient that

$$
\begin{equation*}
\mathcal{U}|\vec{x}>\rightarrow| \vec{x}>\quad \text { as } \quad|\vec{x}| \rightarrow \infty \tag{A.5}
\end{equation*}
$$

in order that $V$ and $\widetilde{V}$ be equivalent as far as scattering is concerned.

## Proof of claim

(A.5) follows on noting that $H$ and $\widetilde{H}$ have the same spectrum since $\mathcal{U}$ is unitary and it follows from (3.35) to (3.37) that

$$
\begin{equation*}
\widetilde{H}=\widetilde{H}_{s}+\widetilde{H}_{b} \tag{A.6}
\end{equation*}
$$

where

$$
\begin{equation*}
\widetilde{H}_{s}=\int d^{3} p\left|\widetilde{\vec{p} \pm}>\epsilon_{p}<\widetilde{\vec{p} \pm}\right| \tag{A.7}
\end{equation*}
$$

and

$$
\begin{equation*}
\widetilde{H}_{b}=\sum_{b=1}^{n_{b}}\left|\widetilde{b}>\epsilon_{b}<\widetilde{b}\right| \tag{A.8}
\end{equation*}
$$

where $\epsilon_{p}$ and $\epsilon_{b}$ are given by (2.38) and (3.4), respectively, and

$$
\begin{gather*}
|\widetilde{\vec{p} \pm}>=\mathcal{U}| \vec{p} \pm>  \tag{A.9}\\
|\widetilde{b}>=\mathcal{U}| b> \tag{A.10}
\end{gather*}
$$

Now since scattering involves $<\vec{x} \mid \widetilde{\vec{p} \pm}>$ and $<\vec{x} \mid \vec{p} \pm>$ for large distances it follows from the material of Chapter 6 that $H$ and $\widetilde{H}$ will give the same scattering wave function are large distances if

$$
\begin{equation*}
<\vec{x}|\widetilde{\vec{p} \pm}>\rightarrow<\vec{x}| \vec{p} \pm>\quad \text { as } \quad|\vec{x}| \rightarrow \infty \tag{A.11}
\end{equation*}
$$

which holds if (A.5) holds.

## A. 3 An example

The operator

$$
\begin{equation*}
\mathcal{U}=1-2|u><u| \tag{A.12}
\end{equation*}
$$

is unitary provided

$$
\begin{equation*}
<u \mid u>=1 \tag{A.13}
\end{equation*}
$$

and satisfies (A.5) if

$$
\begin{equation*}
u(\vec{x}) \rightarrow 0 \quad \text { as } \quad|\vec{x}| \rightarrow \infty \tag{A.14}
\end{equation*}
$$

The interaction potential then is

$$
\begin{equation*}
\widetilde{V}=V-2\{H,|u><u|\}+4|u><u| H|u><u| \tag{A.15}
\end{equation*}
$$

## Comments

## 1. Special case

(A.14) is satisfied if $\mid u>$ is a linear combination of the bound-state eigenvectors of $H$

$$
\begin{equation*}
\left|u>=\sum_{b=1}^{n_{b}} c_{b}\right| b> \tag{A.16}
\end{equation*}
$$

The simplest special case, namely, $\mid u>$ equal to one of the bound-state eigenvectors of $H$

$$
\begin{equation*}
|u>=| b^{\prime}> \tag{A.17}
\end{equation*}
$$

is, however, trivial since it yields

$$
\begin{gather*}
|\widetilde{\vec{p}+}>=| \vec{p}+>  \tag{A.18}\\
|\widetilde{b}>=| b>\quad \text { if } \quad b \neq b^{\prime}  \tag{A.19}\\
\left|\widetilde{\tilde{b}^{\prime}}>=-\right| b^{\prime}> \tag{A.20}
\end{gather*}
$$

and

$$
\begin{equation*}
\widetilde{V}=V \tag{A.21}
\end{equation*}
$$

## 2. Generalization

A generalization of (A.12) is

$$
\begin{equation*}
\mathcal{U}=1-\sum_{j, k=1}^{n} a_{j k}\left|u_{j}><u_{k}\right| \tag{A.22}
\end{equation*}
$$

where

$$
\begin{equation*}
<u_{j} \mid u_{k}>=\delta_{j k} \tag{A.23}
\end{equation*}
$$

and where the complex numbers $\delta_{j k}-a_{j k}$ are form a unitary $n \times n$ matrix.

## Appendix B Riccati functions

## Definitions

As with Taylor (1972) we define the Riccati-Bessel function $\widehat{j}_{l}(z)$, the RiccatiNeumann function $\widehat{n}_{l}(z)$ and the Riccati-Hankel functions $\widehat{h}_{l}^{ \pm}(z)$ by

$$
\begin{gather*}
\widehat{j}_{l}(z)=\sqrt{\frac{\pi z}{2}} J_{l+\frac{1}{2}}(z)  \tag{B.1}\\
\widehat{n}_{l}(z)=(-)^{l} \sqrt{\frac{\pi z}{2}} J_{-l-\frac{1}{2}}(z)  \tag{B.2}\\
\widehat{h}_{l}^{ \pm}(z)=\widehat{n}_{l}(z) \pm i \widehat{j}_{l}(z) \tag{B.3}
\end{gather*}
$$

where $J_{\lambda}(z)$ is the Bessel function.

## Relationship to spherical functions

The Riccati functions (B.1) to (B.3) are related to the spherical Bessel $j_{l}(z)$, spherical Neumann $n_{l}(z)$ and spherical Hankel functions $h_{l}^{( \pm)}(z)$ defined in Messiah (1958) by

$$
\begin{gather*}
\widehat{j}_{l}(z)=z j_{l}(z)  \tag{B.4}\\
\widehat{n}_{l}(z)=z n_{l}(z)  \tag{B.5}\\
\widehat{h}_{l}^{ \pm}(z)=z h_{l}^{( \pm)}(z) \tag{B.6}
\end{gather*}
$$

As noted in Messiah (1958), most authors denote by $n_{l}(z)$ the same function with a change of sign and define spherical Hankel functions $h_{l}^{(1)}(z)=-i h_{l}^{(+)}(z)$ and $h_{l}^{(2)}(z)=i h_{l}^{(-)}(z)$.

## General forms

The Riccati functions (B.1) to (B.3) have the general form

$$
\begin{gather*}
\widehat{j}_{l}(z)=R_{l}(z) \sin z+S_{l}(z) \cos z  \tag{B.7}\\
\widehat{n}_{l}(z)=R_{l}(z) \cos z-S_{l}(z) \sin z  \tag{B.8}\\
\widehat{h}_{l}^{ \pm}(z)=\left[R_{l}(z) \pm i S_{l}(z)\right] e^{ \pm i z} \tag{B.9}
\end{gather*}
$$

where $R_{l}(z)$ is a polynomial in $1 / z$ of degree $l$ with real coefficients and $S_{l}(z)$ is a polynomial in $1 / z$ of degree $l-1$ with real coefficients. In particular, $\hat{j}_{l}(z)$ and $\widehat{n}_{l}(z)$ are real when $z$ is real.

## Some derivatives

If $f_{l}(z)$ is a linear combination of $\widehat{j}_{l}(z)$ and $\widehat{n}_{l}(z)$ with coefficients independent of $l$ and $z$ then

$$
\begin{align*}
\frac{d f_{l}(z)}{d z} & =\frac{l+1}{z} f_{l}(z)-f_{l+1}(z)  \tag{B.10}\\
\frac{d f_{l+1}(z)}{d z} & =-\frac{l+1}{z} f_{l+1}(z)+f_{l}(z) \tag{B.11}
\end{align*}
$$

$$
\begin{equation*}
\left[\frac{d^{2}}{d z^{2}}+1-\frac{l(l+1)}{z^{2}}\right] f_{l}(z)=0 \tag{B.12}
\end{equation*}
$$

(B.10) provides $f_{l+1}(z)$ given $f_{l}(z)$ and (B.12) follows from (B.10) and (B.11).

## Some values

$$
\begin{align*}
& \widehat{j}_{0}(z)=\sin z  \tag{B.13}\\
& \widehat{n}_{0}(z)=\cos z  \tag{B.14}\\
& \widehat{h}_{0}^{ \pm}(z)=e^{ \pm i z} \tag{B.15}
\end{align*}
$$

$$
\begin{gather*}
\widehat{j}_{1}(z)=\frac{1}{z} \sin z-\cos z  \tag{B.16}\\
\widehat{n}_{1}(z)=\frac{1}{z} \cos z+\sin z  \tag{B.17}\\
\widehat{h}_{1}^{ \pm}(z)=\left(1 \pm \frac{i}{z}\right) e^{ \pm i(z-\pi / 2)} \tag{B.18}
\end{gather*}
$$

$$
\begin{align*}
& \widehat{j}_{2}(z)=\left(\frac{3}{z^{2}}-1\right) \sin z-\frac{3}{z} \cos z  \tag{B.19}\\
& \widehat{n}_{2}(z)=\left(\frac{3}{z^{2}}-1\right) \cos z+\frac{3}{z} \sin z  \tag{B.20}\\
& \widehat{h}_{2}^{ \pm}(z)=\left(1 \pm \frac{3 i}{z}-\frac{3}{z^{2}}\right) e^{ \pm i(z-\pi)} \tag{B.21}
\end{align*}
$$

## Behavior at the origin

$$
\begin{array}{ll}
\widehat{j}_{l}(z)=\frac{z^{l+1}}{(2 l+1)!!}\left[1+O\left(z^{2}\right)\right] & \text { as } \\
\widehat{n}_{l}(z)=\frac{(2 l-1)!!}{z^{l}}\left[1+O\left(z^{2}\right)\right] & \text { as } \tag{B.23}
\end{array} \quad z \rightarrow 0
$$

## Behavior at infinity

$$
\begin{align*}
& \widehat{j}_{l}(z)=\sin (z-l \pi / 2)+O\left(z^{-1}\right) \text { as } \quad z \rightarrow \infty, \text { real }  \tag{B.24}\\
& \widehat{n}_{l}(z)=\cos (z-l \pi / 2)+O\left(z^{-1}\right) \quad \text { as } \quad z \rightarrow \infty, \text { real }  \tag{B.25}\\
& \quad \widehat{h}_{l}^{ \pm}(z)=e^{ \pm i(z-l \pi / 2)}\left[1+O\left(z^{-1}\right)\right] \quad \text { as } \quad z \rightarrow \infty \tag{B.26}
\end{align*}
$$

## Some integrals

$$
\begin{equation*}
\int_{0}^{\infty} d r \widehat{j}_{l}(k r) \widehat{j}_{l}\left(k^{\prime} r\right)=\frac{\pi}{2} \delta\left(k-k^{\prime}\right) \tag{B.27}
\end{equation*}
$$

$$
\begin{equation*}
\int_{0}^{\infty} d r \frac{e^{-\mu r}}{r} \widehat{j}_{l}(k r) \widehat{j}_{l}\left(k^{\prime} r\right)=\frac{1}{2} Q_{l}\left(\frac{k^{2}+k^{\prime 2}+\mu^{2}}{2 k k^{\prime}}\right) \tag{B.28}
\end{equation*}
$$

where $Q_{l}(z)$ is a Legendre function of the second kind.

$$
\begin{equation*}
\int_{0}^{\infty} \frac{d \kappa \widehat{j}_{l}(\kappa r) \widehat{j}_{l}\left(\kappa r^{\prime}\right)}{k^{2}-\kappa^{2} \pm i 0}=-\frac{\pi}{2 k} \widehat{j}_{l}\left(k r_{<}\right) h_{l}^{ \pm}\left(k r_{>}\right) \tag{B.29}
\end{equation*}
$$

where

$$
\begin{align*}
& f\left(x_{<}\right) g\left(y_{>}\right)=f(x) g(y) \quad \text { if } \quad x<y  \tag{B.30}\\
&=f(y) g(x) \quad \text { if } \quad y<x
\end{align*}
$$

## SELECTED REFERENCE BOOKS

M. Abramowitz and I. A. Stegun, eds., Handbook of Mathematical Functions,
U.S. National Bureau of Standards, 1964 .
G. Arfken, Mathematical Methods for Physicists, Academic Press, Inc., 3rd. ed., 1985.
H. A. Bethe and E. E. Salpeter, Quantum Mechanics of One- and Two-Electron Atoms, Academic Press, Inc., 1957.
L. Biedenharn and H. van Dam, eds., Quantum Theory of Angular Momentum, Academic Press, Inc., 1965.
J. D. Bjorken and S. D. Drell, Relativistic Quantum Mechanics, McGrawHill. Inc., 1964.
__, Relativistic Quantum Fields, McGraw-Hill, Inc., 1965.
N. N. Bogliubov, A. A. Logunov, and I. T. Todorov, Introduction to Axiomatic Quantum Field Theory, W. A. Benjamin, Inc., 1975.
F. Calogero, Variable Phase Approach to Potential Scattering, Academic Press, Inc., 1967.
P. A. M. Dirac, The Principles of Quantum Mechanics, Oxford University Press, 4th. ed., 1958.
M. L. Goldberger and K. M. Watson, Collision Theory, John Wiley \& Sons, Inc., 1964.
R. I. G. Hughes, The Stucture and Interpretation of Quantum Mechanics, Harvard University Press, 1989.
C. Itzykson and J. B. Zuber, Quantum Field Theory, McGraw-Hill, Inc., 1980.
J. M. Jauch, Foundations of Quantum Mechanics, Addison-Wesley Publishing Company, Inc., 1965.
J. M. Jauch and F. Rohrlich, The Theory of Electrons and Positrons: The Relativistic Quantum Field Theory of Charged Particles with Spin One-Half, Addison-Wesley Publishing Company, Inc., 1955.
T. F. Jordan, Linear Operators for Quantum Mechanics, John Wiley \& Sons, Inc., 1969.
F. A. Kaempffer, Concepts in Quantum Mechanics, Academic Press, Inc., 1965.
Y. S. Kim and M. E. Noz, Theory and Applications of the Poincare Group, D. Reidel Publishing Company, 1986.
R. H. Landau, Quantum Mechanics II, John Wiley \& Sons, Inc., 2nd. ed., 1996.
R. D. Mattuck, A Guide to Feynman Diagrams in the Many-Body Problem, Dover Publications, Inc., 2nd. ed., 1976.
A. Messiah, Quantum Mechanics I \& II, John Wiley \& Sons, Inc., 1958.
P. L. Morse and H. Feshbach, Methods for Theoretical Physics, McGrawHill, Inc., 1953.
R. G. Newton, Scattering Theory of Waves and Particles, McGraw-Hill, Inc., 1966.
R. Omnes, The Interpretation of Quantum Mechanics, Princeton University Press, 1994.
D. Pines, ed., The Many-Body Problem, W. A. Benjamin, Inc., 1961.
F. Riesz and B. Sz.-Nagy, Functional Analysis, Frederick Ungar Publishing Company, Inc., 1955.
M. E. Rose, Elementary Theory of Angular Momentum, John Wiley \& Sons, Inc., 1957.
S. S. Schweber, An Introduction to Relativistic Quantum Field Theory, Row, Peterson and Company, 1961.
J. Schwinger, ed., Selected Papers on Quantum Electrodynamics, Dover Publications, Inc., 1958.
——, Particles, Sources and Fields, Addison-Wesley Publishing Company, Inc.. 1970.
A. E. Sommerfeld, Partial Differential Equations in Physics, Academic Press, Inc., 1949.
R. F. Streater and A. S. Wightman, PCT, Spin and Statistics, and All That, Benjamin/Cummings Publishing Company, Inc., 1964.
J. R. Taylor, Scattering Theory: The Quantum Theory on Nonrelativistic Collisions, John Wiley \& Sons, Inc., 1972.
E. C. Titchmarsh, The Theory of Functions, Oxford University Press, ..... 1939.W.-K. Tung, Group Theory in Physics, World Scientific Publishing Company,Inc.,1994.D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, Quantum Theoryof Angular Momentum, World Scientific Publishing Company, Inc., 1988.
J. von Neumann, Mathematical Foundations of Quantum Mechanics,Princeton University Press, 1955.
S. Weinberg, The Quantum Theory of Fields, Cambridge University Press, 1995.
E. P. Wigner. Group Theory and its Application to the Quantum Mechanics of Atomic Spectra, Academic Press, Inc., 1959.

## SELECTED REFERENCE JOURNAL ARTICLES

B. Bakamjian and L. H. Thomas, Relativistic Particle Dynamics, Phys. Rev. 92, 1300 (1953).
J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Theory of Superconductivity, Phys. Rev. 108, 1175 (1957).
V. Bargmann, Note on Wigner's Theorem on Symmetry Operators, J. Math. Phys. 5, 862 (1964).
_-. On Unitary Ray Representations of Continuous Groups, Ann. Math. 59, 1 (1954).
N. N. Bogoliubov, A New Method in the Theory of Superconductivity, Soviet Physics, JETP 7, 41 (1958).
A. Bottino, A. Longoni, and T. Regge, Potential Scattering for Complex Energy and Complex Angular Momentum, Nuovo Cimento 23, 954 (1962).
G. Breit and W. G. Bouricius, Boundary Conditions and Range of Force for S State of Two Nucleons, Phys. Rev. 75, 1029 (1949).
G. F. Chew and S. Mandelstam, Theory of Low-Energy Pion-Pion Interaction, Phys. Rev. 119, 467 (1960).
L. N. Cooper, Bound Electron Pairs in a Degenerate Fermi Gas, Phys. Rev. 104, 1189 (1956).
D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, Relativistic Invariance and Hamiltonian Theories of Interacting Particles, Rev. Mod. Phys. 35, 350 (1963).
P. A. M. Dirac, A Theory of Electrons and Protons, Proc. Roy. Soc. A126, 360 (1930).
_-, Forms of Relativistic Dynamics, Rev. Mod. Phys. 21, 392 (1949).
_-, The Conditions for a Quantum Field Theory to be Relativistic, Rev. Mod. Phys. 34, 592 (1962).
__, The Quantum Theory of the Electron, Proc. Roy. Soc. A117, 610 (1928).
U. Fano, Description of States in Quantum Mechanics by Density Matrix and Operator Techniques, Rev. Mod. Phys. 29, 74 (1957).
H. Feshbach and E. Lomon, Nucleon-Nucleon Scattering, Phys. Rev. 102, 891 (1956).
L. L. Foldy, Relativistic Particle Systems with Interaction, Phys. Rev. 122, 275 (1961).
L. L. Foldy and S. A. Wouthuysen, On the Dirac Theory of Spin 1/2 Particles and Its Nonrelativistic Limit, Phys. Rev. 78, 29 (1950).
R. Fong and J. Sucher, Relativistic Particle Dynamics and the S Matrix, J. Math. Phys. 5, 456 (1964).
J. Goldstone, Derivation of the Brueckner Many-Body Theory, Proc. Roy. Soc. A239, 267 (1957).
O. W. Greenberg, Haag's Theorem and Clothed Operators, Phys. Rev. 115, 706 (1959).
O. W. Greenberg and S. S. Schweber, Clothed Particle Operators in Simple. Models of Quantum Field Theory, Nuovo Cimento 8, 378 (1958).
D. H. Hearn, M. McMillan, and A. Raskin, Dressing the Cloudy Bag Model: Second-order Nucleon-Nucleon Potential, Phys. Rev. C28, 2489 (1983).
T. F. Jordan, A. J. MacFarlane, and E. C. G. Sudarshan, Hamiltonian Model of Lorentz Invariant Particle Interactions, Phys. Rev. 133, B487 (1964).
W. E. Lamb and R. C. Retherford, Fine Structure of the Hydrogen Atom by a Microwave Method, Phys. Rev. 72, 241 (1947).
M. McMillan, Regge Cuts in a Nonrelativistic Three-Body Problem Involving Separable Potentials, Nuovo Cimento 32, 919 (1964).
—_Separable Nonlocal Potentials and Regge Poles, Nuovo Cimento 29, 1043 (1963).
M. McMillan and E. L. Lomon, The Nuclear Many-Body Problem Using the Boundary Condition Model, Ann. Phys. (N.Y.) 23, 439 (1963).
T. D. Newton and E. P. Wigner, Localized States for Elementary Systems, Rev. Mod. Phys. 21, 400 (1949).
W. Pauli, The Connection Between Spin and Statistics, Phys. Rev. 58, 716 (1940).
M. H. L. Pryce, Commuting Coodinates in the New Field Theory, Proc. Roy. Soc. A150, 166 (1935).
_._The Mass-Centre in the Restricted Theory of Relativity and its Connexion with the Quantum Theory of Elementary Particles, Proc. Roy. Soc. A195, 621 (1948).
T. Regge, Introduction to Complex Orbital Momentum, Nuovo Cimento 14, 951, (1959).
F. Rohrlich, Many-Body Forces and the Cluster Decompositon, Phys. Rev. D23, 1305 (1981).
J. Schwinger, Non-Abelian Gauge Fields. Relativistic Invariance, Phys. Rev. 127, 324 (1962).
E. C. G. Sudarshan, The Nature of the Axioms of Relativistic Quantum Field Theory, J. Math. Phys. 4, 1029 (1963).
E. C. G. Sudarshan and K. Bardacki, Nature of the Axioms of Relativistic Quantum Field Theory, J. Math. Phys. 2, 767 (1961).
L. H. Thomas, The Relativistic Dynamics of a System of Particles Interacting at a Distance, Phys. Rev. 85, 868 (1952).
J. G. Valatin, Comments on the Theory of Superconductivity, Nuovo Cimento 7, 843 (1958).
S. Weinberg, Feynman Rules for Any Spin, Phys. Rev. B133, 1318 (1964).
_ Feynman Rules for Any Spin. Massless Particles, Phys. Rev. B134, 882 (1964).
E. P. Wigner, Relativistic Invariance and Quantum Phenomena, Rev. Mod. Phys. 29, 255 (1957).
__, Relativistic Invariance in Quantum Mechanics, Nuovo Cimento 3, 517 (1956).

## SELECTED REFERENCE THESES

R. Cropp, Dressing a Model of Interacting Fermions, Antifermions and Bosons, B.Sc. thesis, U.B.C., 1996.
D. H. Hearn, The Dressing Transformation and its Application to a FermionBoson Trilinear Interaction, M.Sc. thesis, U.B.C., 1981.
W. W. Hsieh, A New Hamiltonian for Systems of Pions and Nucleons, M.Sc. thesis, U.B.C., 1979.
H. N. James, The Dressing Transformation and Scattering of Dressed Particles in a Model of Fermions and Bosons, M.Sc. thesis, U.B.C., 1982.
P. A. Kalyniak, Consequences of Space-Time Invariances in Quantum Mechanics and Direct Interactions Theories, M.Sc. thesis, U.B.C., 1978.
M. McMillan, The Nuclear Many-Body Problem Using the Boundary Condition Model, Ph.D. thesis, McGill, 1961.
A. H. Monahan, Relativistic Few-Body Quantum Mechanics, M.Sc. thesis, U.B.C., 1995.


[^0]:    Some properties of Riccati functions are given in Appendix B.
    2 The second equality in (2.69) follows using the addition theorem for spherical harmonics.

[^1]:    $1 \quad r_{0_{l}} \pm 0$ means that $v_{l}\left(r, r^{\prime}\right)$ is evaluated at $r_{0_{l}} \pm \eta$ where $\eta>0$ and the limit $\eta \rightarrow 0$ then taken.

[^2]:    1 Some properties of Riccati functions are given in Appendix B.

[^3]:    2 There is no dependence on the azimuthal angle $\varphi$ because of rotational invariance.

