QUANTUM LEAPS AND BOUNDS

Some Lorentz Invariant Systems

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Contents

Preface

The six volumes of notes *Quantum Leaps and Bounds (QLB)* form the basis of the introductory graduate quantum mechanics course I have given in the Department of Physics at the University of British Columbia at various times since 1973.

The six volumes of *QLB* are

- *Introductory Topics*: a collection of miscellaneous topics in introductory quantum mechanics
- *Scattering Theory*: an introduction to the basic ideas of quantum scattering theory by considering the scattering of a relativistic spinless particle from a fixed target
- *Quantum Mechanics in Fock Space*: an introduction to the second-quantization description of nonrelativistic many-body systems
- *Relativistic Quantum Mechanics*: an introduction to incorporating special relativity in quantum mechanics
- *Some Lorentz Invariant Systems*: some examples of systems incorporating special relativity in quantum mechanics
- *Relativistic Quantum Field Theory*: an elementary introduction to the relativistic quantum field theory of spinless bosons, spin $\frac{1}{2}$ fermions and antifermions and to quantum electrodynamics, the relativistic quantum field theory of electrons, positrons and photons

QLB assumes no familiarity with relativistic quantum mechanics. It does assume that students have taken undergraduate courses in nonrelativistic quantum mechanics which include discussion of the nonrelativistic Schrodinger equation and the solutions of some standard problems (e.g., the one-dimensional harmonic oscillator and the hydrogen atom) and perturbation theory and other approximation methods.

QLB assumes also that students will take other graduate courses in condensed matter physics, nuclear and particle physics and relativistic quantum field theory. Accordingly, our purpose in *QLB* is to introduce some basic ideas and formalism and thereby give students sufficient background to read the many excellent texts on these subjects.

I am happy to have this opportunity to thank my friends and colleagues R. Barrie, B. Bergersen, M. Bloom, J. Feldman, D.H. Hearn, W.W. Hsieh, R.I.G. Hughes, F.A. Kaempffer, P.A. Kalyniak, R.H. Landau, E.L. Lomon, A.H. Monahan, W. Opechowski, M.H.L. Pryce, A. Raskin, P. Rastall, L. Rosen, L. Sobrino, F. Tabakin, A.W. Thomas, E.W. Vogt and G.M. Volkoff for sharing their knowledge of quantum mechanics with me.

I also thank my wife, Henrietta, for suggesting the title for these volumes of notes. Quite correctly, she found my working title *Elements of Intermediate Quantum Mechanics* a bore.

SOME LORENTZ INVARIANT SYSTEMS

Chapter 1 INTRODUCTORY REMARKS

As discussed in *QLB: Relativistic Quantum Mechanics*, in order to describe a Lorentz invariant physical system using quantum mechanics one must:

1. specify a set of fundamental dynamical variables for the system;

2. specify the fundamental algebra of the set of fundamental dynamical variables;

3. select a complete set of compatible observables for the system;

4. specify the Hilbert space of the system through spectral resolution of the complete set of compatible observables;

5. determine the Poincare generators $H, \vec{P}, \vec{J}, \vec{K}$ for the system in terms of the fundamental dynamical variables.

For convenience we give some elements of relativistic quantum mechanics in Section 1.1.

We give a number of examples of Lorentz invariant systems in this volume of *QLB*. We follow the above steps in each case. This procedure differs from the historical one for the Dirac particle discussed in Chapter 4 but it yields all the usual results.

We consider a single spinless particle in Chapter 2, a particle with spin in Chapter 3, a Dirac particle in Chapter 4, a system of particles with spin in Chapter 5 and a simple system involving particle creation and annihilation in Chapter 6. Lists of selected reference books, journal articles and theses follow Chapter 6.

1.1 Some relativistic quantum mechanics

In this section we give some elements of relativistic (and nonrelativistic) quantum mechanics which are used later in this volume and which are discussed more fully in *QLB: Relativistic Quantum Mechanics.*

Poincare Generators

The Hermitian operators $H, \vec{P}, \vec{J}, \vec{K}$ are the Poincare generators for a Lorentz invariant physical system. H is the Hamiltonian; $\vec{P}$ is the total momentum; $\vec{J}$ is the total angular momentum; $\vec{K}$ is the Lorentz booster. These operators generate time translations, spatial displacements, rotations and Lorentz boosts, respectively.

Poincare Algebra

The Poincare Algebra is the following set of commutation relations for the Poincare generators:

$$\left[P^j, P^k\right] = 0 \tag{1.1}$$

$$\left[P^j, H\right] = 0 \tag{1.2}$$

$$\left[J^j, P^k\right] = i\hbar\epsilon_{jkl}P^l \tag{1.3}$$

$$\left[J^j, H\right] = 0 \tag{1.4}$$

$$\left[J^j, J^k\right] = i\hbar\epsilon_{jkl}J^l \tag{1.5}$$

$$\left[K^j, P^k\right] = -i\hbar\delta_{jk}H/c^2 \tag{1.6}$$

$$\left[K^j, H\right] = -i\hbar P^j \tag{1.7}$$

$$\left[K^j, J^k\right] = i\hbar\epsilon_{jkl}K^l \tag{1.8}$$

$$\left[K^j, K^k\right] = -i\hbar\epsilon_{jkl}J^l/c^2 \tag{1.9}$$

where $\hbar = h/2\pi$, h is Planck's constant, c is the speed of light, δ_{jk} is the Kronecker delta symbol and ϵ_{jkl} is the Levi-Civita permutation symbol.

Galilei Algebra

The Galilei Algebra is a set of commution relations appropriate for describing a Galilei invariant physical system. The Galilei Algebra is identical to the Poincare Algebra except for (1.6) and (1.9) which are the only equations in the Poincare Algebra which involve the speed of light c. More specifically, the Galilei Algebra is identical to the Poincare Algebra except in having (1.6) and (1.9) replaced by

$$\left[K^j, P^k\right] = -i\hbar m\delta_{jk} \tag{1.10}$$

$$\left[K^j, K^k\right] = 0 \tag{1.11}$$

where m is the mass of the system. The Galilei booster $\vec{K}$ generates Galilei boosts.

Unitary Poincare operators

The unitary Poincare operators for a Lorentz invariant system are

$$U(t) = e^{-iHt/\hbar} \tag{1.12}$$

$$D^j(a) = e^{-iP^j a/\hbar} \tag{1.13}$$

$$R^j(\theta) = e^{-iJ^j\theta/\hbar} \tag{1.14}$$

$$L^j(u) = e^{-icK^j u/\hbar} \tag{1.15}$$

As discussed in *QLB: Relativistic Quantum Mechanics*, these operators correspond to space-time transformations in a fixed inertial frame as follows:

We consider a state $\mid \psi >$ of the system prepared by a preparation apparatus in a fixed inertial frame S at time zero.

$U(t)$ is the evolution operator for the system. The state

$$\mid \psi(t) >= U(t) \mid \psi > \tag{1.16}$$

is the state of the system in S at time t. It follows from (1.16) that

$$H \mid \psi(t) >= i\hbar\frac{d}{dt} \mid \psi(t) > \tag{1.17}$$

(1.17) is the Schrodinger equation for the system.

$D^j(a)$ is the displacement operator along the j-axis of S for the system. The state

$$\mid \psi_{disp} >= D^j(a) \mid \psi > \tag{1.18}$$

is the state of the system prepared in S at time zero by the apparatus displaced by a along the j-axis of S.

$R^j(\theta)$ is the rotation operator about the j-axis in S for the system. The state

$$\mid \psi_{rot} >= R^j(\theta) \mid \psi > \tag{1.19}$$

is the state of the system prepared in S at time zero by the apparatus rotated by θ about the j-axis of S.

$L^j(u)$ is the Lorentz boost operator along the j-axis in S for the system. The state

$$\mid \psi_{boost} >= L^j(u) \mid \psi > \tag{1.20}$$

is the state of the system prepared in S at time zero by the apparatus boosted with rapidity u along the j-axis of S.

The states $\mid \psi_{disp} >$, $\mid \psi_{rot} >$ and $\mid \psi_{boost} >$ evolve under the influence of the Hamiltonian H and at time t in S are

$$| \psi_{disp}(t) >= U(t) \mid \psi_{disp} > \tag{1.21}$$

$$| \psi_{rot}(t) >= U(t) \mid \psi_{rot} > \tag{1.22}$$

$$| \psi_{boost}(t) >= U(t) \mid \psi_{boost} > \tag{1.23}$$

It follows from the Poincare Algebra (1.1) to (1.9) that the above states are related to the state $| \psi(t) >$ as follows:

$$| \psi_{disp}(t) >= D^j(a) \mid \psi(t) > \tag{1.24}$$

$$| \psi_{rot}(t) >= R^j(\theta) \mid \psi(t) > \tag{1.25}$$

$$D^j(x') \mid \psi_{boost}(t') >= L^j(u) D^j(x) \mid \psi(t) > \tag{1.26}$$

where

$$x' = \gamma(x - vt) \tag{1.27}$$

$$t' = \gamma\left(t - \frac{vx}{c^2}\right) \tag{1.28}$$

where $\gamma = \cosh u$.

Space inversion and time reversal

As discussed in *QLB: Relativistic Quantum Mechanics*, the Poincare generators transform under space inversion and time reversal as follows:

$$\mathrm{P} H \mathrm{P}^\dagger = H \tag{1.29}$$

$$\mathrm{P} \vec{P} \mathrm{P}^\dagger = -\vec{P} \tag{1.30}$$

$$\mathrm{P} \vec{J} \mathrm{P}^\dagger = \vec{J} \tag{1.31}$$

$$\mathrm{P} \vec{K} \mathrm{P}^\dagger = -\vec{K} \tag{1.32}$$

$$\mathrm{T} H \mathrm{T}^\dagger = H \tag{1.33}$$

$$\mathrm{T} \vec{P} \mathrm{T}^\dagger = -\vec{P} \tag{1.34}$$

$$\mathrm{T} \vec{J} \mathrm{T}^\dagger = -\vec{J} \tag{1.35}$$

$$\mathrm{T} \vec{K} \mathrm{T}^\dagger = \vec{K} \tag{1.36}$$

where the linear operator P is the space inversion operator and the antilinear operator T is the time reversal operator for the system.

P and T correspond to space inversion and time reversal in a fixed inertial frame as follows:

We consider a state $\mid \psi >$ of the system prepared by a preparation apparatus in a fixed inertial frame S at time zero.

The state

$$\boxed{\mid \psi_{inv} >= \mathrm{P} \mid \psi > \qquad (1.37)}$$

is the state of the system prepared in S at time zero by the space-inverted apparatus and the state

$$\boxed{\mid \psi_{rev} >= \mathrm{T} \mid \psi > \qquad (1.38)}$$

is the state of the system prepared in S at time zero by the time-reversed apparatus.

The states $\mid \psi_{inv} >$ and $\mid \psi_{rev} >$ evolve under the influence of the Hamiltonian H and at time t in S are

$$\mid \psi_{inv}(t) >= U(t) \mid \psi_{inv} > \qquad (1.39)$$

$$\mid \psi_{rev}(t) >= U(t) \mid \psi_{rev} > \qquad (1.40)$$

It follows from (1.29) and (1.33) that the above states are related to the state $\mid \psi(t) >$ as follows:

$$\mid \psi_{inv}(t) >= \mathrm{P} \mid \psi(t) > \tag{1.41}$$

$$\mid \psi_{rev}(t) >= \mathrm{T} \mid \psi(-t) > \tag{1.42}$$

Invariant mass

The invariant mass M of a Lorentz invariant system is

$$M = \frac{1}{c}\sqrt{P.P} \tag{1.43}$$

That is,

$$Mc^2 = \sqrt{H^2 - P^2c^2} \tag{1.44}$$

Pauli-Lubanski four-vector

The Pauli-Lubanski four-vector W^μ for a Lorentz invariant system is

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\sigma\tau} P_\nu M_{\sigma\tau} \tag{1.45}$$

where $\epsilon^{\mu\nu\sigma\tau}$ is the unit antisymmetric tensor. That is,

$$W^0 = \vec{J} \cdot \vec{P} \tag{1.46}$$

$$\vec{W} = \frac{1}{c} H \vec{J} + c\vec{K} \times \vec{P} \tag{1.47}$$

Centre of mass position

The centre of mass position $\vec{\hat{X}}$ of a Lorentz invariant system is

$$\vec{\hat{X}} = -\frac{c^2}{2}\left(\frac{1}{H}\vec{K} + \vec{K}\frac{1}{H}\right) - \frac{c}{(E + Mc^2)MH}\vec{P} \times \vec{W} \tag{1.48}$$

where

$$E = \sqrt{H^2} = \sqrt{P^2c^2 + M^2c^4} \tag{1.49}$$

Centre of mass velocity

The centre of mass velocity $\vec{V}$ of a Lorentz invariant physical system is defined as

$$i\hbar\vec{V} = \left[\vec{\hat{X}}, H\right] \tag{1.50}$$

It follows from (1.48) that

$$\boxed{\vec{V} = \frac{c^2 \vec{P}}{H} \qquad (1.51)}$$

Internal angular momentum

The internal angular momentum $\vec{\hat{S}}$ of a Lorentz invariant system is

$$\boxed{\vec{\hat{S}} = \frac{1}{Mc}\left(\frac{H}{E}\vec{W} - \frac{c}{E + Mc^2} W^0 \vec{P}\right) \qquad (1.52)}$$

It follows from (1.52) that

$$\left[\vec{\hat{S}}, H\right] = 0 \qquad (1.53)$$

$$W.W = -(Mc)^2\, \vec{\hat{S}} \cdot \vec{\hat{S}} \qquad (1.54)$$

$\vec{\hat{S}}$ is a constant of the motion and $\vec{\hat{S}} \cdot \vec{\hat{S}}$ is a Lorentz invariant.

Helicity

The helicity Λ of a Lorentz invariant system is

$$\boxed{\Lambda = \frac{\vec{\hat{S}} \cdot \vec{P}}{P} \qquad (1.55)}$$

where $P = \sqrt{\vec{P} \cdot \vec{P}}$.

It follows from on calculation that

$$[\Lambda, H] = 0 \qquad (1.56)$$

$$\left[\Lambda, \vec{P}\right] = 0 \qquad (1.57)$$

$$\left[\Lambda, \vec{J}\right] = 0 \qquad (1.58)$$

$$\left[\Lambda, \vec{K}\right] = i\hbar M c\left(P\vec{\hat{S}} - \Lambda\vec{P}\right) \qquad (1.59)$$

Λ is a Lorentz invariant for a system with $M = 0$.

Chapter 2 SPINLESS PARTICLE

In this chapter we consider a Lorentz invariant physical system consisting of a single spinless particle.

We follow the steps given in Chapter 1: we specify the physical system by a set of fundamental dynamical variables and we construct Poincare generators in terms of these variables.

Fundamental dynamical variables for the system are given in Section 2.1, Poincare generators are given in Section 2.2, various space-time transformation operators are given in Section 2.3, coordinate- and momentum-space wave functions are given in Section 2.4, some transformed coordinate-space wave functions are given in Section 2.5, the integro-differential equation for the coordinate-space wave function is given in Section 2.6 and the Klein-Gordon equation for the coordinate-space wave function is given and discussed in Section 2.7. Some derivations are given in Section 2.8.

2.1 Fundamental dynamical variables

Fundamental dynamical variables for a physical system consisting of a spinless particle of rest mass m are the Cartesian coordinates and momenta[1]

$$\boxed{X^1, X^2, X^3, P^1, P^2, P^3 \qquad (2.1)}$$

[1] (2.16) shows that P^j is a Poincare generator. We have anticipated this result in order not to proliferate the number of symbols.

which satisfy the fundamental quantum conditions

$$\left[X^j, X^k\right] = 0 \tag{2.2}$$

$$\left[P^j, P^k\right] = 0 \tag{2.3}$$

$$\left[X^j, P^k\right] = i\hbar\delta_{jk} \tag{2.4}$$

The operators

$$X^1, X^2, X^3 \tag{2.5}$$

$$P^1, P^2, P^3 \tag{2.6}$$

each form a complete set of compatible observables. We denote their simultaneous eigenkets by

$$\mid \vec{x} >=\mid x, y, z >=\mid x^1, x^2, x^3 > \tag{2.7}$$

$$\mid \vec{p} >=\mid p^1, p^2, p^3 > \tag{2.8}$$

These eigenkets may be used as bases for the Hilbert space. That is,

$$X^j = \int d^3x \mid \vec{x} > x^j < \vec{x} \mid \tag{2.9}$$

$$P^j = \int d^3p \mid \vec{p} > p^j < \vec{p} \mid \tag{2.10}$$

and

$$1 = \int d^3x \mid \vec{x} >< \vec{x} \mid = \int d^3p \mid \vec{p} >< \vec{p} \mid \tag{2.11}$$

$$< \vec{x} \mid \vec{x'} > = \delta\left(\vec{x} - \vec{x'}\right) \tag{2.12}$$

$$< \vec{p} \mid \vec{p'} > = \delta\left(\vec{p} - \vec{p'}\right) \tag{2.13}$$

where $\delta\left(\vec{x} - \vec{x'}\right)$ and $\delta\left(\vec{p} - \vec{p'}\right)$ are 3-dimensional Dirac delta functions.

It follows from (2.2) to (2.4) that

$$< \vec{x} \mid \vec{p} > = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar} \tag{2.14}$$

2.2 Poincare generators

We require the particle to be a Lorentz invariant system. We must, therefore, construct the Poincare generators $H, \vec{P}, \vec{J}, \vec{K}$ in terms of (2.1). We quote the final results. The Poincare Algebra (1.1) to (1.9) is satisfied when

$$H = \sqrt{P^2c^2 + m^2c^4} \tag{2.15}$$

$$\vec{P} = \vec{P} \tag{2.16}$$

$$\vec{J} = \vec{X} \times \vec{P} \tag{2.17}$$

$$\vec{K} = -\frac{1}{2c^2}\left(\vec{X}H + H\vec{X}\right) \tag{2.18}$$

Comments

1. **Spectral decomposition of the Hamiltonian**

H is a function of momentum so

$$H = \int d^3p \mid \vec{p} > \epsilon_p < \vec{p} \mid \tag{2.19}$$

where

$$\epsilon_p = \sqrt{p^2c^2 + m^2c^4} \tag{2.20}$$

2. Comparison with the nonrelativistic case

The Galilei Algebra is satisfied when

$$H = \frac{P^2}{2m} \tag{2.21}$$

$$\vec{P} = \vec{P} \tag{2.22}$$

$$\vec{J} = \vec{X} \times \vec{P} \tag{2.23}$$

$$\vec{K} = -m\vec{X} \tag{2.24}$$

The Hamiltonian H and the booster $\vec{K}$ are different in relativistic and non-relativistic quantum mechanics. The total momentum $\vec{P}$ and total angular momentum $\vec{J}$ are the same.

3. Invariant mass

It follows from (1.44) that

$$M = m \tag{2.25}$$

4. Centre of mass position and internal angular momentum

It follows from (1.48) and (1.52) that

$$\vec{\hat{X}} = \vec{X} \tag{2.26}$$

$$\vec{\hat{S}} = 0 \tag{2.27}$$

The centre of mass position of the particle is the Cartesian position; the internal angular momentum of the particle is zero.

5. Centre of mass velocity

(2.15) corresponds to the classical expression for the energy of a particle in terms of its momentum. It follows from (1.51) and (2.15) that the velocity of the particle is

$$\vec{V} = \frac{c^2\vec{P}}{\sqrt{P^2c^2 + m^2c^4}} \tag{2.28}$$

Furthermore,

$$\vec{P} = \frac{m\vec{V}}{\sqrt{1 - V^2/c^2}} \tag{2.29}$$

$$E = \frac{mc^2}{\sqrt{1 - V^2/c^2}} \tag{2.30}$$

(2.30) is the quantal version of Einstein's equation in classical special relativity.

6. **Particle with zero rest mass**

It follows from (2.15) and (2.28) that when $m = 0$

$$H = Pc \tag{2.31}$$

$$\vec{V} = \frac{c\vec{P}}{P} \tag{2.32}$$

$$V = c \tag{2.33}$$

A particle with zero rest mass travels at the speed of light.

2.3 Space-time transformation operators

In this section we give expressions in terms of position and momentum eigenkets for the unitary Poincare operators (1.12) to (1.15) and for the space inversion and time reversal operators.

Unitary Poincare operators

The evolution operator (1.12) is

$$U(t) = \int d^3p \mid \vec{p} > e^{-i\epsilon_p t/\hbar} < \vec{p} \mid \tag{2.34}$$

The displacement operator (1.13), rotation operator (1.14) and boost operator (1.15) may be written as

$$D^j(a) = \int d^3x \mid \vec{x}_{D^j} >< \vec{x} \mid \tag{2.35}$$

$$R^j(\theta) = \int d^3x \mid \vec{x}_{R^j} >< \vec{x} \mid = \int d^3p \mid \vec{p}_{R^j} >< \vec{p} \mid \tag{2.36}$$

$$L^j(u) = \int d^3p \sqrt{\frac{\epsilon_{pL^j}}{\epsilon_p}} \mid \vec{p}_{L^j} >< \vec{p} \mid \tag{2.37}$$

where, for example,

$$\mid \vec{x}_{D^1} > = \mid x^1 + a, x^2, x^3 > \tag{2.38}$$

$$\mid x_{R^1} > = \mid x^1, x^2 \cos\theta - x^3 \sin\theta, x^2 \sin\theta + x^3 \cos\theta > \tag{2.39}$$

$$\mid p_{R^1} > = \mid p^1, p^2 \cos\theta - p^3 \sin\theta, p^2 \sin\theta + p^3 \cos\theta > \tag{2.40}$$

$$\mid \vec{p}_{L^1} > = \mid p^1 \cosh u + \frac{1}{c}\epsilon_p \sinh u, p^2, p^3 > \tag{2.41}$$

$$\epsilon_{pL^1} = \epsilon_p \cosh u + cp^1 \sinh u \tag{2.42}$$

(2.35), (2.36) and (2.37) give a direct correspondence with transformations of space-time points; they also allow simple proofs of

$$D^j(a)X^kD^{j\dagger}(a) = X^k - a\delta_{jk} \tag{2.43}$$

$$R^1(\theta)X^1R^{1\dagger}(\theta) = X^1 \tag{2.44}$$

$$R^1(\theta)X^2R^{1\dagger}(\theta) = X^2\cos\theta + X^3\sin\theta \tag{2.45}$$

$$R^1(\theta)X^3R^{1\dagger}(\theta) = -X^2\sin\theta + X^3\cos\theta \tag{2.46}$$

$$L^j(u)HL^{j\dagger}(u) = H\cosh u - cP^j\sinh u \tag{2.47}$$

$$L^j(u)cP^jL^{j\dagger}(u) = cP^j\cosh u - H\sinh u \tag{2.48}$$

$$L^j(u)cP^kL^{j\dagger}(u) = cP^k \qquad (j \neq k) \tag{2.49}$$

Space inversion and time reversal operators

The operator P corresponding to space inversion is

$$\mathrm{P} = \int d^3x \mid -\vec{x} >< \vec{x} \mid = \int d^3p \mid -\vec{p} >< \vec{p} \mid \tag{2.50}$$

The operator T corresponding to time reversal is

$$\mathrm{T} \mid \phi >= \int d^3x \mid \vec{x} >< \phi \mid \vec{x} >= \int d^3p \mid -\vec{p} >< \phi \mid \vec{p} > \tag{2.51}$$

where $\mid \phi >$ is any vector or ket in the Hilbert space.

It follows from (2.50) and (2.51) that for any linear operator A

$$\begin{aligned} \mathrm{PAP}^{\dagger} &= \int d^3x d^3x' \mid \vec{x} >< -\vec{x} \mid A \mid -\vec{x'} >< \vec{x'} \mid \\ &= \int d^3p d^3p' \mid \vec{p} >< -\vec{p} \mid A \mid -\vec{p'} >< \vec{p'} \mid \end{aligned} \tag{2.52}$$

$$\begin{aligned} \mathrm{TAT}^{\dagger} &= \int d^3x d^3x' \mid \vec{x} >< \vec{x} \mid A \mid \vec{x'} >^{*} < \vec{x'} \mid \\ &= \int d^3p d^3p' \mid \vec{p} >< -\vec{p} \mid A \mid -\vec{p'} >^{*} < \vec{p'} \mid \end{aligned} \tag{2.53}$$

Thus,

$$\mathrm{PAP}^{\dagger} = A \tag{2.54}$$

if

$$\boxed{< -\vec{x} \mid A \mid -\vec{x'} > = < \vec{x} \mid A \mid \vec{x'} >} \tag{2.55}$$

or

$$\boxed{< -\vec{p} \mid A \mid -\vec{p'} > = < \vec{p} \mid A \mid \vec{p'} >} \tag{2.56}$$

and

$$\boxed{\mathrm{T}A\mathrm{T}^{\dagger} = A} \tag{2.57}$$

if

$$\boxed{< \vec{x} \mid A \mid \vec{x'} >^{*} = < \vec{x} \mid A \mid \vec{x'} >} \tag{2.58}$$

or

$$\boxed{< -\vec{p} \mid A \mid -\vec{p'} >^{*} = < \vec{p} \mid A \mid \vec{p'} >} \tag{2.59}$$

In particular, if

$$< \vec{x} \mid A \mid \vec{x'} >= a(\vec{x})\delta\left(\vec{x} - \vec{x'}\right) \tag{2.60}$$

then (2.55) and (2.58) become

$$a(-\vec{x}) = a(\vec{x}) \tag{2.61}$$

$$a^*(\vec{x}) = a(\vec{x}) \tag{2.62}$$

(2.62) is satisfied if A is Hermitian; accordingly, a local observable is invariant under time-reversal, as follows also from (2.66).

Finally, (2.52) and (2.53) allow simple proofs of

$$\mathrm{P}\vec{X}\mathrm{P}^{\dagger} = -\vec{X} \tag{2.63}$$

$$\mathrm{P}\vec{P}\mathrm{P}^{\dagger} = -\vec{P} \tag{2.64}$$

$$\mathrm{P}H\mathrm{P}^{\dagger} = H \tag{2.65}$$

$$\mathrm{T}\vec{X}\mathrm{T}^{\dagger} = \vec{X} \tag{2.66}$$

$$\mathrm{T}\vec{P}\mathrm{T}^{\dagger} = -\vec{P} \tag{2.67}$$

$$\mathrm{T}H\mathrm{T}^{\dagger} = H \tag{2.68}$$

2.4 Wave functions

The coordinate-space wave function $\psi(\vec{x}, t)$ for the particle is defined as

$$\psi(\vec{x}, t) =< \vec{x} \mid \psi(t) > \tag{2.69}$$

where $\mid \psi(t) >$ given by (1.16) is the state of the particle at time t, and

$$\mid \psi(\vec{x}, t) \mid {}^2 d^3x \tag{2.70}$$

is the probability that the particle is in the volume d^3x about $\vec{x}$ at time t.

The momentum-space wave function $\psi(\vec{p}, t)$ for the particle is defined as

$$\psi(\vec{p}, t) =< \vec{p} \mid \psi(t) > \tag{2.71}$$

and

$$
\mid \psi(\vec{p}, t) \mid {}^2 d^3p \tag{2.72}
$$

is the probability that the particle has momentum in the volume d^3p about $\vec{p}$ at time t.

It follows from (2.14) that

$$
\psi(\vec{x}, t) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p e^{i\vec{p}\cdot\vec{x}/\hbar} \psi(\vec{p}, t) \tag{2.73}
$$

$$
\psi(\vec{p}, t) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p e^{-i\vec{p}\cdot\vec{x}/\hbar} \psi(\vec{x}, t) \tag{2.74}
$$

It follows from (2.34) that

$$
\psi(\vec{p}, t) = e^{-i\epsilon_p t/\hbar} < \vec{p} \mid \psi > \tag{2.75}
$$

and therefore

$$
\psi(\vec{x}, t) = \left(\frac{\hbar}{2\pi}\right)^{\frac{3}{2}} \int d^3k e^{i\left(\vec{k}\cdot\vec{x} - \omega_k t\right)} \psi\left(\vec{k}\right) \tag{2.76}
$$

where $\vec{p} = \hbar\vec{k}$, $\epsilon_p = \hbar\omega_k$ and $\psi\left(\vec{k}\right) =< \vec{p} \mid \psi >$.

Comments

1. **Plane wave**

   The function

$$e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} \tag{2.77}$$

   describes a plane sine wave moving with phase speed ω_k/k and group speed

$$\frac{d\omega_k}{dk} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} \tag{2.78}$$

   in the direction $\vec{k}$.

2. **Wave packet**

   (2.76) describes a wave packet; its shape is determined by the probability amplitude for having prepared the particle at time zero with momentum $\hbar\vec{k}$.

2.5 Some transformed wave functions

In this section we give expressions for some space-time transformed coordinate-space wave functions which follow using the space-time transformation operators given in Section 2.3.

Displaced wave function

The displaced coordinate-space wave function $\psi_{disp}(\vec{x}, t)$ for the particle is at time t is defined as

$$\boxed{\psi_{disp}(\vec{x}, t) = < \vec{x} \mid \psi_{disp}(t) > \qquad (2.79)}$$

where, following (1.21),

$$| \psi_{disp} >= D(\vec{a}) \mid \psi > \tag{2.80}$$

where

$$D(\vec{a}) = D^1(a^1)D^2(a^2)D^3(a^3) \tag{2.81}$$

It follows from (2.35) that

$$\psi_{disp}(\vec{x},t) = \psi(\vec{x}-\vec{a},t) \tag{2.82}$$

<u>**Space-inverted wave function**</u>

The space-inverted coordinate-space wave function $\psi_{inv}(\vec{x},t)$ for the particle is at time t is defined as

$$\psi_{inv}(\vec{x},t) =< \vec{x} \mid \psi_{inv}(t) > \tag{2.83}$$

where $| \psi_{inv}(t) >$ is given by (1.39). It follows from (2.50) that

$$\psi_{inv}(\vec{x},t) = \psi(-\vec{x},t) \tag{2.84}$$

Time-reversed wave function

The time-reversed coordinate-space wave function $\psi_{inv}(\vec{x},t)$ for the particle is at time t is defined as

$$\boxed{\psi_{rev}(\vec{x},t) = <\vec{x} \mid \psi_{rev}(t)> \qquad (2.85)}$$

where $\mid \psi_{rev}(t) >$ is given by (1.40). It follows from (2.51) that

$$\boxed{\psi_{rev}(\vec{x},t) = \psi^*(\vec{x},-t) \qquad (2.86)}$$

The complex conjugated wave function appears on the right side of (2.86) because the time reversal operator is antiunitary.

2.6 Equation for the coordinate-space wave function

We show in Section 2.8 that the coordinate space wave function $\psi(\vec{x},t)$ satisfies the following integro-differential equation:

$$\boxed{\left(\frac{1}{2\pi\hbar}\right)^3 \int d^3x' d^3p \; \epsilon_p e^{i\vec{p}\cdot(\vec{x}-\vec{x}')/\hbar} \psi\left(\vec{x}',t\right) = i\hbar \frac{\partial\psi(\vec{x},t)}{\partial t} \qquad (2.87)}$$

Nonrelativistic Schrodinger equation

The nonrelativistic limit of (2.87) follows on replacing ϵ_p by $p^2/2m$. This replacement yields

$$\boxed{-\frac{\hbar^2}{2m}\nabla^2\psi(\vec{x},t) = i\hbar\frac{\partial}{\partial t}\psi(\vec{x},t) \qquad (2.88)}$$

(2.88) is the nonrelativistic Schrodinger equation.

2.7 Klein-Gordon equation

As is seen from performing the required differentiations, the wave packet (2.76) satisfies

$$\boxed{\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi(\vec{x},t) = 0 \qquad (2.89)}$$

where

$$\boxed{\Box = \frac{1}{c^2}\frac{\partial^2}{\partial t^2} - \nabla^2 \qquad (2.90)}$$

(2.89) is the Klein-Gordon equation. We derive (2.89) directly in Section 2.8.

Comments

1. **Negative energy solutions**

The coordinate representation of every solution of the Schrodinger equation (1.17) satisfies (2.89). The converse, however, is not true. This follows since the derivation of (2.89) involves the square of H so (2.89) could also have been derived by taking H to be equal to $-\sqrt{P^2c^2+m^2c^4}$. One says that (2.89) possesses both positive energy solutions and negative energy solutions.

The negative energy solutions of (2.89) have no sensible interpretation in a one-particle theory.

2. **Lorentz invariant solutions**

The d'Alembertian (2.90) is invariant under all Poincare transformations. That is,

$$\Box' = \Box \tag{2.91}$$

where

$$\Box' = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \nabla'^2 = \frac{1}{c^2}\frac{\partial^2}{\partial t'^2} - \frac{\partial^2}{\partial x'^2} - \frac{\partial^2}{\partial y'^2} - \frac{\partial^2}{\partial z'^2} \tag{2.92}$$

where

$$x'^{\mu} = \Lambda^{\mu}{}_{\nu}x^{\nu} + a^{\mu} \tag{2.93}$$

is a Poincare transformation. It follows that if $f(x,y,z,t)$ is a solution of (2.89) then so is $f(x',y',z',t')$. One can construct solutions $f(x,y,z,t)$ of (2.89) which satisfy

$$f(x',y',z',t') = f(x,y,z,t) \tag{2.94}$$

Such solutions are called Lorentz invariant solutions of the Klein—Gordon equation.

3. **Lorentz invariant wave-packet solutions**

The wave packet (2.76) is a Lorentz invariant solution of (2.89) if

$$\omega'_{k'}\psi\left(\vec{k'}\right) = \omega_k\psi\left(\vec{k}\right) \tag{2.95}$$

where

$$k'^{\mu} = \Lambda^{\mu}{}_{\nu}k^{\nu} \tag{2.96}$$

and

$$k^0 = \omega_k/c \tag{2.97}$$

A manifestly covariant form for this wave packet is

$$\boxed{\psi(\vec{x},t) = \int d^4k\delta\left(k.k - \left(\frac{mc}{\hbar}\right)^2\right)\theta\left(k^0\right)e^{ik.x}a(k^{\mu}) \qquad (2.98)}$$

where $\theta\left(k^0\right)$ is the positive step function and

$$a(k^{\mu}) = 2\left(\frac{\hbar}{2\pi}\right)^{\frac{3}{2}}ck^0\psi\left(\vec{k}\right) \tag{2.99}$$

4. Quantizing the Klein-Gordon equation

(2.98) is the traditional starting point for the development of the relativistic quantum field theory of uncharged bosons with zero spin.

The function $a(k^{\mu})$ is reinterpreted as an annihilation operator for a boson with energy-momentum k^{μ} and (2.98) is reinterpreted as an expression for a field operator in the Heisenberg picture. The field operator obeys the Klein-Gordon equation (2.89)..

The above reinterpretation is called quantizing the Klein-Gordon equation.

The negative energy solutions of (2.89) are reinterpreted as field operators for particles with negative charge.

We develop the relativistic quantum field theory of spinless bosons from first principles in *QLB: Relativistic Quantum Field Theory*. The traditional approach of quantizing the Klein-Gordon equation will not be followed.

2.8 Some derivations

Derivation of (2.87)

$\psi(\vec{x},t)$ satisfies

$$< \vec{x} \mid H \mid \psi(t) >= i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} \tag{2.100}$$

(2.87) follows using

$$< x \mid H \mid \psi(t) >= \int d^3x' < \vec{x} \mid H \mid \vec{x'} > \psi\left(\vec{x'},t\right) \tag{2.101}$$

and

$$\begin{aligned} < \vec{x} \mid H \mid \vec{x'} > &= \int d^3p d^3p' < \vec{x} \mid \vec{p} >< \vec{p} \mid H \mid \vec{p'} >< \vec{p'} \mid \vec{x'} > \\ &= \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3p \; \epsilon_p e^{i\vec{p}\cdot(\vec{x}-\vec{x'})/\hbar} \end{aligned} \tag{2.102}$$

which yields (2.87).

Derivation of (2.89)

Operating on both sides of (1.17) with H yields

$$H^2 \mid \psi(t) >= -\hbar^2 \frac{d^2}{dt^2} \mid \psi(t) > \tag{2.103}$$

the coordinate representation of which is

$$< \vec{x} \mid H^2 \mid \psi(t) >= -\hbar^2 \frac{\partial^2 \psi(\vec{x},t)}{\partial t^2} \tag{2.104}$$

Using (2.15), the left side of (2.104) is

$$c^2 < \vec{x} \mid P^2 \mid \psi(t) > +m^2c^4\psi(\vec{x},t) \tag{2.105}$$

and since

$$< \vec{x} \mid \vec{P} \mid \psi(t) >= -i\hbar\vec{\nabla}\psi(\vec{x},t) \tag{2.106}$$

it follows that

$$< \vec{x} \mid H^2 \mid \psi(t) >= -\hbar^2c^2\nabla^2\psi(\vec{x},t) + m^2c^4\psi(\vec{x},t) \tag{2.107}$$

(2.89) is (2.104) in more compact form.

Chapter 3 PARTICLE WITH SPIN

In this chapter we extend the material of Chapter 2 to consider a Lorentz invariant physical system consisting of a single particle with spin s where s is any positive integer or positive half-odd integer. As we shall see, inclusion of spin is simple and straightforward.

We follow the steps given in Chapter 1: we specify the physical system by a set of fundamental dynamical variables and we construct Poincare generators in terms of these variables.

Fundamental dynamical variables for the system are given in Section 3.1, Poincare generators are given in Section 3.2, coordinate- and momentum-space wave functions are given in Section 3.3 and helicity wave functions are given in Section 3.4.

3.1 Fundamental dynamical variables

Fundamental dynamical variables for a physical system consisting of a particle of rest mass m and spin s where s is any positive integer or positive half-odd integer are the Cartesian coordinates, momenta and spin[1]

$$\boxed{X^1, X^2, X^3, P^1, P^2, P^3, S^1, S^2, S^3 \qquad (3.1)}$$

which satisfy

[1] (3.20) shows that P^j is a Poincare generator. We have anticipated this result in order not to proliferate the number of symbols.

$$\boxed{\begin{aligned} \left[X^j, X^k\right] &= 0 && (3.2)\\ \left[P^j, P^k\right] &= 0 && (3.3)\\ \left[X^j, P^k\right] &= i\hbar\delta_{jk} && (3.4) \end{aligned}}$$

$$\boxed{\begin{aligned} \left[S^j, S^k\right] &= i\hbar\epsilon_{jkl}S^l && (3.5)\\ \vec{S}\cdot\vec{S} &= s(s+1)\hbar^2 && (\mathrm{I}.6) \end{aligned}}$$

$$\boxed{\left[S^j, X^k\right] = \left[S^j, P^k\right] = 0 \qquad (3.7)}$$

The operators

$$\boxed{\begin{aligned} &X^1, X^2, X^3, S^3 && (3.8)\\ &P^1, P^2, P^3, S^3 && (3.9) \end{aligned}}$$

each form a complete set of compatible observables. We denote their simultaneous eigenkets by

$$| \vec{x} m_s >=| x, y, z, m_s >=| x^1, x^2, x^3, m_s > \tag{3.10}$$

$$| \vec{p} m_s >=| p^1, p^2, p^3, m_s > \tag{3.11}$$

These eigenkets may be used as bases for the Hilbert space. That is,

$$X^j = \sum_{m_s=-s}^{+s} \int d^3x \mid \vec{x} m_s > x^j < \vec{x} m_s \mid \tag{3.12}$$

$$P^j = \sum_{m_s=-s}^{+s} \int d^3p \mid \vec{p} m_s > p^j < \vec{p} m_s \mid \tag{3.13}$$

$$\begin{aligned} S^3 &= \sum_{m_s=-s}^{+s} \int d^3x \mid \vec{x} m_s > m_s \hbar < \vec{x} m_s \mid \\ &= \sum_{m_s=-s}^{+s} \int d^3p \mid \vec{p} m_s > m_s \hbar < \vec{p} m_s \mid \end{aligned} \tag{3.14}$$

and

$$\begin{aligned} 1 &= \sum_{m_s=-s}^{+s} \int d^3x \mid \vec{x} m_s >< \vec{x} m_s \mid \\ &= \sum_{m_s=-s}^{+s} \int d^3p \mid \vec{p} m_s >< \vec{p} m_s \mid \end{aligned} \tag{3.15}$$

$$< \vec{x}m_s \mid \vec{x}'m_s' >= \delta\left(\vec{x} - \vec{x}'\right)\delta_{m_s m_s'} \tag{3.16}$$

$$< \vec{p}m_s \mid \vec{p}'m_s' >= \delta\left(\vec{p} - \vec{p}'\right)\delta_{m_s m_s'} \tag{3.17}$$

It follows from (3.2) to (3.4) that

$$< \vec{x}m_s \mid \vec{p}m_s' >= \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar}\delta_{m_s m_s'} \tag{3.18}$$

3.2 Poincare generators

We require the particle to be a Lorentz invariant system. We must, therefore, construct the Poincare generators $H, \vec{P}, \vec{J}, \vec{K}$ in terms of (3.1). We quote the final results. The Poincare Algebra (1.1) to (1.9) is satisfied when

$$H = \sqrt{P^2c^2 + m^2c^4} \tag{3.19}$$

$$\vec{P} = \vec{P} \tag{3.20}$$

$$\vec{J} = \vec{X} \times \vec{P} + \vec{S} \tag{3.21}$$

$$\vec{K} = -\frac{1}{2c^2}\left(\vec{X}H + H\vec{X}\right) + \frac{1}{H + mc^2}\left(\vec{S} \times \vec{P}\right) \tag{3.22}$$

Comments

1. Comparison with the spinless particle case

H is the same as in the spinless particle case: H does not depend upon $\vec{S}$.

(3.21) is the obvious generalization of (2.17); $\vec{K}$ therefore depends upon $\vec{S}$ because of the Poincare Algebra equation (1.9).

Modifying $\vec{J}$ and $\vec{K}$ from the spinless case and leaving H and $\vec{P}$ unchanged corresponds to the point form of dynamics defined by Dirac (1949).

2. Comparison with the nonrelativistic case

The Galilei Algebra is satisfied when

$$H = \frac{P^2}{2m} \tag{3.23}$$

$$\vec{P} = \vec{P} \tag{3.24}$$

$$\vec{J} = \vec{X} \times \vec{P} + \vec{S} \tag{3.25}$$

$$\vec{K} = -m\vec{X} \tag{3.26}$$

Only $\vec{J}$ depends upon $\vec{S}$ in the nonrelativistic case.

3. Energy spectrum

H is a function of momentum so

$$H = \sum_{m_s=-s}^{+s} \int d^3p \mid \vec{p}m_s > \epsilon_p < \vec{p}m_s \mid \tag{3.27}$$

where ϵ_p is given by (2.20).

4. Invariant mass

It follows from (1.44) that

$$M = m \tag{3.28}$$

5. Centre of mass position and internal angular momentum

It follows from (1.48) and (1.52) that

$$\vec{\hat{X}} = \vec{X} \tag{3.29}$$

$$\vec{\hat{S}} = \vec{S} \tag{3.30}$$

The centre of mass position of the particle is the Cartesian position; the internal angular momentum of the particle is the Cartesian spin.

6. Centre of mass velocity

It follows from (1.51) that

$$\vec{V} = \frac{c^2 \vec{P}}{\sqrt{P^2 c^2 + m^2 c^4}} \tag{3.31}$$

$$\vec{P} = \frac{m\vec{V}}{\sqrt{1 - V^2/c^2}} \tag{3.32}$$

$$E = \frac{mc^2}{\sqrt{1 - V^2/c^2}} \tag{3.33}$$

(3.33) is, for a particle with spin, the quantal version of Einstein's equation in classical special relativity.

7. Wigner rotation of spin

The dependence of $\vec{K}$ upon $\vec{S}$ in the Poincare case yields a rotation of spin under Lorentz boosts. This is the Wigner rotation for a particle of spin s.

The Wigner rotation is a purely relativistic effect. There is no Wigner rotation in the Galilei case because, according to (3.26), $\vec{K}$ in that case does not depend upon $\vec{S}$.

8. Particle with zero rest mass

It follows from (3.19) and (3.31) that when $m = 0$

$$H = Pc \tag{3.34}$$

$$\vec{V} = \frac{c\vec{P}}{P} \tag{3.35}$$

$$V = c \tag{3.36}$$

A particle with zero rest mass and spin s travels at the speed of light.

3.3 Wave functions

The coordinate-space/spin wave function $\psi_{m_s}(\vec{x}, t)$ for the particle is defined as

$$\psi_{m_s}(\vec{x}, t) = < \vec{x} m_s \mid \psi(t) > \tag{3.37}$$

where $\mid \psi(t) >$ given by (1.16) is the state of the particle at time t, and

$$\mid \psi_{m_s}(\vec{x}, t) \mid {}^2 d^3 x \tag{3.38}$$

is the probability that the particle is in the volume d^3x about $\vec{x}$ at time t with 3-component of spin equal to m_s.

The momentum-space/spin wave function $\psi_{m_s}(\vec{p}, t)$ for the particle is defined as

$$\boxed{\psi_{m_s}(\vec{p},t) = <\vec{p}m_s \mid \psi(t)> \qquad (3.39)}$$

and

$$\boxed{\mid \psi_{m_s}(\vec{p},t) \mid {}^2 d^3p \qquad (3.40)}$$

is the probability that the particle has momentum in the volume d^3p about $\vec{p}$ at time t with 3-component of spin equal to m_s.

The coordinate-space/spin and momentum-space/spin wave functions are related according to

$$\psi_{m_s}(\vec{x},t) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p e^{i\vec{p}\cdot\vec{x}/\hbar} \psi_{m_s}(\vec{p},t) \qquad (3.41)$$

$$\psi_{m_s}(\vec{p},t) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p e^{-i\vec{p}\cdot\vec{x}/\hbar} \psi_{m_s}(\vec{x},t) \qquad (3.42)$$

It follows from (3.27) that

$$\boxed{\psi_{m_s}(\vec{p},t) = e^{-i\epsilon_p t/\hbar} < \vec{p}m_s \mid \psi > \qquad (3.43)}$$

and therefore

$$\psi_{m_s}(\vec{x},t) = \left(\frac{\hbar}{2\pi}\right)^{\frac{3}{2}} \int d^3k e^{i\left(\vec{k}\cdot\vec{x}-\omega_k t\right)} \psi_{m_s}\left(\vec{k}\right) \tag{3.44}$$

where $\vec{p} = \hbar\vec{k}$, $\epsilon_p = \hbar\omega_k$ and $\psi_{m_s}\left(\vec{k}\right) =< \vec{p}m_s \mid \psi >$.

(3.44) describes a wave packet in terms of the probability amplitude for having prepared the particle at time zero with momentum $\hbar\vec{k}$ and 3-component of spin equal to m_s.

$\psi_{m_s}(\vec{x},t)$ given by (3.44) satisfies the Klein-Gordon equation (2.89).

3.4 Helicity eigenkets

It follows from (1.55) and (3.21) that the helicity Λ of the particle is

$$\Lambda = \frac{\vec{S}\cdot\vec{P}}{\sqrt{\vec{P}\cdot\vec{P}}} \tag{3.45}$$

Λ is the projection of the intrinsic spin of the particle along the direction of the momentum of the particle.

The eigenvalues of Λ are $\lambda\hbar$ where $\lambda = s, s-1, \cdots, -s$.

When expressed in terms of the eigenkets $\mid \vec{p}m_s >$,

$$\boxed{\Lambda = \sum_{m_s=-s}^{+s} \sum_{m_s'=-s}^{+s} \int d^3p \mid \vec{p}m_s > \left(\vec{s}\cdot\vec{\hat{p}}\right)_{m_s m_s'} < \vec{p}m_s' \mid \qquad (3.46)}$$

where $\vec{\hat{p}}$ is the unit vector $\vec{p}/\sqrt{\vec{p}\cdot\vec{p}}$ and s^1, s^2, s^3 are $(2s+1)\times(2s+1)$ matrices satisfying

$$\boxed{\left[s^j, s^k\right] = i\hbar\epsilon_{jkl}s^l \qquad (3.47)}$$

with s^3 diagonal.

For a particle with spin $\frac{1}{2}$,

$$\boxed{\vec{s} = \frac{1}{2}\hbar\vec{\sigma} \qquad (3.48)}$$

where the $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices, in which case,

$$\boxed{\vec{\sigma}\cdot\vec{\hat{p}} = \begin{pmatrix} \cos\theta & \sin\theta e^{-i\varphi} \\ \sin\theta e^{i\varphi} & -\cos\theta \end{pmatrix} \qquad (3.49)}$$

where (θ, φ) are the spherical polar coordinates of $\vec{p}$ in the fixed inertial reference frame.

The operators

$$P^1, P^2, P^3, \Lambda \tag{3.50}$$

are a complete set of compatible observables. We denote their simultaneous eigenkets by

$$\mid h^\lambda(\vec{p}) > \tag{3.51}$$

These eigenkets may be used as a basis for the Hilbert space. That is,

$$P^j = \sum_{\lambda=-s}^{+s} \int d^3p \mid h^\lambda(\vec{p}) > p^j < h^\lambda(\vec{p}) \mid \tag{3.52}$$

$$\Lambda = \sum_{\lambda=-s}^{+s} \int d^3p \mid h^\lambda(\vec{p}) > \lambda\hbar < h^\lambda(\vec{p}) \mid \tag{3.53}$$

and

$$1 = \sum_{\lambda=-s}^{+s} \int d^3p \mid h^\lambda(\vec{p}) >< h^\lambda(\vec{p}) \mid \tag{3.54}$$

$$< h^\lambda(\vec{p}) \mid h^{\lambda'}\left(\vec{p'}\right) >= \delta\left(\vec{p} - \vec{p'}\right)\delta_{\lambda\lambda'} \tag{3.55}$$

Furthermore,

$$< \vec{p}'m_s \mid h^\lambda(\vec{p}) >= \delta\left(\vec{p} - \vec{p}'\right) h^\lambda_{m_s}(\vec{p}) \tag{3.56}$$

which defines the functions $h^\lambda_{m_s}(\vec{p})$, and, using (3.18),

$$< \vec{x}m_s \mid h^\lambda(\vec{p}) >= \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar} h^\lambda_{m_s}(\vec{p}) \tag{3.57}$$

The functions $h^\lambda_{m_s}(\vec{p})$ are determined by solving the eigenvalue problem for Λ in the $\mid \vec{p}m_s >$ representation. For a particle with spin $\frac{1}{2}$ this yields

$$\begin{pmatrix} h^{+\frac{1}{2}}_{+\frac{1}{2}}(\vec{p}) & h^{-\frac{1}{2}}_{+\frac{1}{2}}(\vec{p}) \\ h^{+\frac{1}{2}}_{-\frac{1}{2}}(\vec{p}) & h^{-\frac{1}{2}}_{-\frac{1}{2}}(\vec{p}) \end{pmatrix} = \begin{pmatrix} \cos\frac{\theta}{2} & -e^{-i\varphi}\sin\frac{\theta}{2} \\ e^{i\varphi}\sin\frac{\theta}{2} & \cos\frac{\theta}{2} \end{pmatrix} \tag{3.58}$$

That is,

$$h^\lambda_{m_s}(\vec{p}) = D^{\frac{1}{2}}_{m_s\lambda}(\varphi, \theta, 0) \tag{3.59}$$

where $D^j_{m'm}(\alpha, \beta, \gamma)$ is the rotation matrix given by Rose (1957), page 234.

For general s

$$h^{\lambda}_{m_s}(\vec{p}) = D^{s}_{m_s\lambda}(\varphi, \theta, 0) \tag{3.60}$$

3.4.1 Momentum-space/helicity wave function

The momentum-space/helicity wave function $\psi^{\lambda}(\vec{p}, t)$ for the particle is defined as

$$\psi^{\lambda}(\vec{p}, t) = < h^{\lambda}(\vec{p}) \mid \psi(t) > \tag{3.61}$$

and

$$\mid \psi^{\lambda}(\vec{p}, t) \mid ^2 d^3p \tag{3.62}$$

is the probability that the particle has momentum in the volume d^3p about $\vec{p}$ at time t with helicity λ.

Helicity is a Lorentz invariant for a system with $M = 0$, that is, a particle with zero rest mass. In this case, where the Hamiltonian (3.19) is invariant under space inversion,

$$\psi^{\lambda}(\vec{p}, t) = 0 \qquad \text{unless } \lambda = s \quad \text{or} \quad -s \tag{3.63}$$

It follows from (3.56) and (3.59) that the momentum-space/spin and momentum-space/helicity wave functions are related according to

$$\psi_{m_s}(\vec{p},t) = \sum_{\lambda=-s}^{+s} D^{s}_{m_s\lambda}(\varphi,\theta,0)\psi^{\lambda}(\vec{p},t) \tag{3.64}$$

$$\psi^{\lambda}(\vec{p},t) = \sum_{m_s=-s}^{+s} D^{s*}_{m_s\lambda}(\varphi,\theta,0)\psi_{m_s}(\vec{p},t) \tag{3.65}$$

It follows from (3.52) that

$$\psi^{\lambda}(p,t) = e^{-i\epsilon t/\hbar} < h^{\lambda}(p) \mid \psi > \tag{3.66}$$

The coordinate-space/spin wave function (3.44) may be written as

$$\psi_{m_s}(\vec{x},t) = \left(\frac{\hbar}{2\pi}\right)^{\frac{3}{2}} \sum_{\lambda=-s}^{+s} \int d^3k e^{i(\vec{k}\cdot\vec{x}-\omega_k t)} D^{s}_{m_s\lambda}(\varphi,\theta,0)\psi^{\lambda}\left(\vec{k}\right) \tag{3.67}$$

where $\psi^{\lambda}\left(\vec{k}\right) =< h^{\lambda}(\vec{p}) \mid \psi >$.

$\psi_{m_s}(x,t)$ given by (3.67) describes a wave packet in terms of the probability amplitude for having prepared the particle at time zero with momentum $\hbar\vec{k}$ and helicity $\lambda\hbar$.

Chapter 4 DIRAC PARTICLE

In Chapter 3 we considered a Lorentz invariant system consisting of a particle of rest mass m and spin s where s is any positive integer or positive half-odd integer. We showed that inclusion of spin was simple and straightforward; we also showed that the energy of the particle was independent of spin and always nonnegative.

In this chapter we consider the physical system to be a Dirac particle of rest mass m. The quantum mechanics of this system was first considered by Dirac (1928). As will be shown, the Dirac particle has spin $\frac{1}{2}$ and possesses both positive and negative energy states. Dirac assumed the particle to be an electron.

The negative energy states have no physical interpretation in a one-particle theory. Dirac's bold interpretation of these states (hole theory) predicted the existence of antiparticles and led to the invention of relativistic quantum field theory. It is one of the greatest achievements in the history of quantum physics.[1]

Dirac's hole theory, as brilliant as it was in 1930, is not the modern view of antiparticles. In *QLB: Relativistic Quantum Field Theory* we construct a relativistic quantum field theory of electrons and positrons where particles and antiparticles appear on equal footing and with positive energies.

Some of the ideas of hole theory do, however, appear in a modified and correct form in the modern view of many-body physics as discussed in *QLB: Quantum Mechanics in Fock Space*.

As in Chapters 2 and 3 we follow the steps given in Chapter 1: we specify the physical system by a set of fundamental dynamical variables and we construct

[1] Nonrelativistic quantum mechanics was invented by Heisenberg and Schrodinger in 1925. Dirac proposed his Hamiltonian (4.10) in 1928. He was 25 at the time. Dirac proposed his interpretation of the negative energy states in 1930. The positron was discovered by Anderson in 1932. Relativistic quantum field theory was invented by Pauli, Jordan and others between 1933 and 1935.

Poincare generators in terms of these variables. This differs from the usual procedure for the Dirac particle but it yields all the usual results.

Fundamental dynamical variables for the system are given in Section 4.1 and Dirac's expressions for the Poincare generators are given in Section 4.2. The Pryce-Foldy-Wouthuysen transformation is given in Section 4.3 and zitterbewegung is discussed in Section 4.4. It is shown that the Dirac particle has spin $\frac{1}{2}$ in Section 4.5. Dirac's discovery of the Dirac Hamiltonian is described in Section 4.6. Some properties of Dirac matrices and γ-matrices are given in Section 4.7. The coordinate-space wave function of the particle and the Dirac equation are given in Section 4.8 and the momentum-space wave function is discussed in Section 4.9. Space inversion and time reversal are considered in Section 4.10. Energy/helicity eigenkets are defined in Section 4.11 and a transformation similar to the Pryce-Foldy-Wouthuysen transformation is given in Section 4.12. The role of the negative energy states in zitterbewegung is discussed in Section 4.13. Energy/helicity spinors are given in Section 4.14 and the most general solution of the Dirac equation is given in terms of these spinors in Section 4.15. A Dirac particle in a electromagnetic field is discussed in Section 4.16. The g factor for a nonrelativistic Dirac particle is derived and the energy eigenvalues for the Dirac hydrogen atom are given. Dirac's interpretation of the negative energy states is given in Section 4.17. Finally, some derivations are given in Section 4.18.

4.1 Fundamental dynamical variables

Fundamental dynamical variables for a physical system consisting of a Dirac particle of rest mass m are the Cartesian coordinates and momenta and the four Dirac operators[1]

$$\boxed{X^1, X^2, X^3, P^1, P^2, P^3, \alpha^1, \alpha^2, \alpha^3, \beta \qquad (4.1)}$$

[1] (4.11) shows that P^j is a Poincare generator. We have anticipated this result in order not to proliferate the number of symbols.

which satisfy

$$\left[X^j, X^k\right] = 0 \tag{4.2}$$

$$\left[P^j, P^k\right] = 0 \tag{4.3}$$

$$\left[X^j, P^k\right] = i\hbar\delta_{jk} \tag{4.4}$$

$$\left\{\alpha^j, \alpha^k\right\} = 2\delta_{jk} \tag{4.5}$$

$$\left\{\alpha^j, \beta\right\} = 0 \tag{4.6}$$

$$\beta^2 = 1 \tag{4.7}$$

$$\left[\alpha^j, X^k\right] = \left[\alpha^j, P^k\right] = 0 \tag{4.8}$$

$$\left[\beta, X^k\right] = \left[\beta, P^k\right] = 0 \tag{4.9}$$

4.2 Poincare generators

We require the Dirac particle to be a Lorentz invariant system. We must, therefore, construct the Poincare generators $H, \vec{P}, \vec{J}, \vec{K}$ in terms of (4.1). We quote the final results. The Poincare Algebra (1.1) to (1.9) is satisfied when

$$H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2 \tag{4.10}$$

$$\vec{P} = \vec{P} \tag{4.11}$$

$$\vec{J} = \vec{X} \times \vec{P} + \vec{S} \tag{4.12}$$

$$\vec{K} = -\frac{1}{2c^2}\left(\vec{X}H + H\vec{X}\right) \tag{4.13}$$

where

$$\vec{S} = \frac{1}{2}\hbar\vec{\Sigma} \tag{4.14}$$

$$\vec{\Sigma} = -\frac{i}{2}\vec{\alpha} \times \vec{\alpha} \tag{4.15}$$

Comments

1. Comparison with the spinless particle and spin particle cases

We recall from Chapter 3 that comparison of the Poincare generators (3.20) to (3.22) for a particle of spin s with the Poincare generators (2.16) to (2.18) for a spinless particle shows that the former are arrived at from the latter by modifying $\vec{J}$ and $\vec{K}$. This corresponds to the point form of dynamics defined in Dirac (1949).)

Comparison of the Poincare generators (4.11) to (4.13) for a Dirac particle with the Poincare generators (2.16) to (2.18) for a spinless particle shows that the former are arrived at from the latter by modifying H and $\vec{J}$. This corresponds to Dirac's instant form of dynamics.

We leave it to the reader to construct Poincare generators corresponding to Dirac's front form of dynamics.

2. **Energy spectrum**

It follows using (4.5) to (4.7) that

$$H^2 = P^2c^2 + m^2c^4 \tag{4.16}$$

The spectrum of H^2 is continuous in the interval (m^2c^4, ∞). The spectrum of H is continuous in the intervals $(-\infty, -mc^2)$ and (mc^2, ∞).

That is, the Dirac particle possesses negative energy states.

The negative energy states have no physical interpretation for a one-particle system. We outline Dirac's interpretation of these states in Section 4.17.

3. **Other sets of generators**

We refer to the generators $H, \vec{P}, \vec{J}, \vec{K}$ given by (4.10) to (4.13) as the Dirac generators. They are not a unique solution to the Poincare Algebra (1.1) to (1.9).

The Poincare Algebra is satisfied by the generators (3.19) to (3.22) for a particle with spin where $\vec{S}$ is given by (4.14). This set of generators is not unitarily related to the Dirac generators because the energy spectra are different.

The Poincare Algebra is also satisfied by the Pryce-Foldy-Wouthuysen (PFW) generators given in Section 4.3. The PFW generators are unitarily related to the Dirac generators.

4. **Invariant mass**

It follows from (1.44) that

$$\boxed{M = m \qquad (4.17)}$$

5. **Centre of mass position and internal angular momentum**

It follows from (1.48) and (1.52) that

$$\vec{\bar{X}} = \vec{X} + \frac{i\hbar c\beta\vec{\alpha}}{2E} - \frac{c^2}{E(E+mc^2)}\left[\vec{S}\times\vec{P} + \frac{i\hbar c}{2E}\beta\left(\vec{\alpha}\cdot\vec{P}\right)\vec{P}\right] \qquad (4.18)$$

$$\vec{\bar{S}} = \vec{S} - \frac{i\hbar c\beta}{E}\vec{\alpha}\times\vec{P} - \frac{c^2}{E(E+mc^2)}\vec{P}\times\left(\vec{S}\times\vec{P}\right) \qquad (4.19)$$

where

$$\boxed{E = \sqrt{P^2c^2 + m^2c^4} \qquad (4.20)}$$

That is, $\vec{X}$ is not the centre of mass position and $\vec{S}$ is not the spin. Indeed, one finds on calculation that

$$\left[\vec{X}, H\right] = i\hbar c\vec{\alpha} \tag{4.21}$$

$$\left[\vec{S}, H\right] \neq 0 \tag{4.22}$$

That is, $\vec{S}$ is not a constant of the motion, and interpreting $\vec{X}$ as the position of the particle implies that the speed of the particle is equal to the speed of light.

The above results may seem disturbing. The situation will be clarified in Section 4.3 when we view the Dirac particle in the Pryce-Foldy-Wouthuysen picture. We note here, however, that the numerical differences between matrix elements of $\vec{\vec{X}}$ and $\vec{X}$, and $\vec{\vec{S}}$ and $\vec{S}$, are small; they are of the order of the Compton wavelength $\hbar/mc$ of the particle.

4.3 Pryce-Foldy-Wouthuysen picture

We recall that the physical content of quantum mechanics is unchanged if each state $\mid \psi >$ is replaced by $U \mid \psi >$ and each observable A is replaced by $UAU^{\dagger}$ where U is a unitary or antiunitary operator. Every U provides a picture of quantum mechanics.

So far in this chapter we have used the Dirac picture of the Dirac particle. We now consider the Pryce-Foldy-Wouthuysen (PFW) picture of the Dirac particle. This picture is provided by the unitary operator[1]

[1] The PFW picture of the Dirac particle was introduced by Pryce (1948) and Foldy and Wouthuysen (1950). It is usually called the Foldy-Wouthuysen picture.

$$F = \frac{E + \beta H}{\sqrt{2E(E + mc^2)}} \tag{4.23}$$

If $\mid \psi >$ is a state of the particle in the Dirac picture and A is an observable of the particle in the Dirac picture, then

$$F \mid \psi > \tag{4.24}$$

$$FAF^{\dagger} \tag{4.25}$$

are the corresponding state and observable of the particle in the PFW picture.

It follows on calculation that

$$FHF^{\dagger} = \beta\sqrt{P^2c^2 + m^2c^4} \tag{4.26}$$

$$F\vec{P}F^{\dagger} = \vec{P} \tag{4.27}$$

$$F\vec{\hat{X}}F^{\dagger} = \vec{X} \tag{4.28}$$

$$F\vec{\hat{S}}F^{\dagger} = \vec{S} \tag{4.29}$$

Comments

1. **Hamiltonian**

The Hamiltonian in the PFW picture is $\beta\sqrt{P^2c^2 + m^2c^4}$.

The PFW picture thus shows the relationship between Dirac's Hamiltonian (4.10) and the Hamiltonian (3.19) for a particle of rest mass m and spin s.

By introducing the operator β, which has eigenvalues are ± 1, Dirac seems merely to have introduced negative energy states along with positive energy states.

2. **Centre of mass position and internal angular momentum**

The centre of mass position in the PFW picture is $\vec{X}$ and the internal angular momentum is $\vec{S}$.

As pointed out above, unphysical effects follow if one interprets $\vec{X}$ as the centre of mass position and $\vec{S}$ as the internal angular momentum in the Dirac picture.

One such unphysical effect, zitterbewegung, is discussed in Section 4.4.

4.4 Zitterbewegung

We view the Dirac particle in the Dirac picture, that is, using the Dirac generators (4.10) to (4.13).

The centre of mass position $\vec{\tilde{X}}(t)$ of the particle in the Heisenberg picture is

$$\vec{\tilde{X}}(t) = \vec{\tilde{X}} + \vec{V}t \tag{4.30}$$

where

$$\vec{V} = \frac{c^2\vec{P}}{H} \tag{4.31}$$

where H is the Dirac Hamiltonian (4.10).

We show in Section 4.18 that the Cartesian position $\vec{X}(t)$ of the particle in the Heisenberg picture is

$$\vec{X}(t) = \vec{X} + \vec{V}t + \vec{Z}(t) \tag{4.32}$$

where

$$\vec{Z}(t) = e^{iHt/\hbar} \sin\left(\frac{Ht}{\hbar}\right)\frac{\hbar}{H}\left(c\vec{\alpha} - \vec{V}\right) \tag{4.33}$$

Comments

1. **Zitterbewegung operator**

$\vec{Z}(t)$ is the zitterbewegung ("jittering motion") operator. It describes a high frequency, small amplitude displacement.

2. **Zitterbewegung: not an observable position operator**

   Zitterbewegung is not an observable position of the Dirac particle because the Cartesian position is not the position operator in the Dirac picture.

3. **Negative energy states**

   We show in Section 4.13 that nonzero matrix elements $\vec{Z}(t)$ arise only if a state has both positive and negative energy components.

4.5 Spin

$\vec{\hat{S}}$ is the internal angular momentum of the particle in the Dirac picture and $\vec{S}$ is the internal angular momentum of the particle in the PFW picture. $\vec{\hat{S}} \cdot \vec{\hat{S}}$ and $\vec{S} \cdot \vec{S}$ have the same eigenvalues because they are related by a unitary transformation.

It follows on calculation that

$$\boxed{\vec{S} \cdot \vec{S} = s(s+1)\hbar^2} \tag{4.34}$$

where

$$\boxed{s = \frac{1}{2}} \tag{4.35}$$

A Dirac particle has spin $\frac{1}{2}$.

Comments

1. **Value of spin**

   The value of the spin of the Dirac particle has been derived from properties of $\vec{\alpha}$; it has not specified *ab initio* as was the case for the particle with spin discussed in Chapter 3.

   This was considered the most remarkable feature of Dirac's approach to the Lorentz invariant description of a single particle when it was invented in 1928.

2. **Electron**

   Dirac considered the particle described by his Hamiltonian to be an electron.

4.6 Dirac's discovery of the Dirac Hamiltonian

The Poincare Algebra approach had not been invented when Dirac discovered his wave equation for the electron in 1928. How then did he proceed?

The Klein-Gordon equation (2.89) was considered to be the appropriate relativistic wave equation for a particle with zero spin. Dirac sought to derive a relativistic wave equation which would describe the electron, that is, a spin $\frac{1}{2}$ particle.

He started with the coordinate representation of the Schrodinger equation for a single particle with spin

$$< \vec{x}m_s \mid H \mid \psi(t) >= i\hbar\frac{\partial\psi_{m_s}(\vec{x},t)}{\partial t} \tag{4.36}$$

and determined a form for H such that space and time would be on equal footing in (4.36).[1] Since (4.36) is a first-order partial differential equation in time and

[1] "Equal footing" was imposed because of the mixing of space and time variables by a Lorentz boost. The Poincare Algebra, however, does not imply "equal footing".

since

$$< \vec{x} m_s \mid \vec{P} \mid \psi(t) >= -i\hbar\vec{\nabla}\psi_{m_s}(\vec{x},t) \tag{4.37}$$

Dirac therefore sought a form for H which was linear in $\vec{P}$; he also required that

$$H^2 = P^2c^2 + m^2c^4 \tag{4.38}$$

in order to incorporate the classical relationship between energy and momentum.

Dirac generalized

$$\left(\vec{\sigma} \cdot \vec{P}\right)^2 = P^2 \tag{4.39}$$

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices by writing (4.10) where $\alpha^1, \alpha^2, \alpha^3, \beta$ are to be determined such that (4.38) holds. That is, such that

$$\boxed{\left(c\vec{\alpha} \cdot \vec{P} + \beta mc^2\right)^2 = P^2c^2 + m^2c^4 \qquad (4.40)}$$

(4.40) holds provided $\alpha^1, \alpha^2, \alpha^3, \beta$ satisfy (4.5) to (4.7).

4.7 Matrix representation

We give some properties of Dirac matrices and γ-matrices in this section. Further properties are given in *QLB: Relativistic Quantum Mechanics.*

Dirac matrices

The Dirac algebra (4.5) to (4.7) is satisfied by $n \times n$ matrices, the smallest value of n being $n = 4$. The Dirac representation of 4×4 matrices $\vec{\alpha}$ and β is

$$\vec{\alpha} = \begin{pmatrix} 0 & \vec{\sigma} \\ \vec{\sigma} & 0 \end{pmatrix} \tag{4.41}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{4.42}$$

In this representation,

$$\vec{S} = \frac{1}{2}\hbar \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix} \tag{4.43}$$

Comments

1. **Dimension of matrices**

   It is necessary to use 4×4 matrices to describe the spin of a Dirac particle because the particle has up and down spin states for both positive energy states and negative energy states.

2. **Diagonal matrices**

   The matrices β (4.42) and S^3 (4.43) are diagonal. We use this fact in Sections 4.8 and 4.9 to construct simultaneous eigenkets of sets of compatible operators

γ-matrices

The four γ-matrices

$$\gamma^0 = \beta \tag{4.44}$$

$$\gamma^j = \beta\alpha^j \tag{4.45}$$

transform as components of a contravariant vector under Lorentz boosts and rotations. That is,

$$S(\Lambda)^{-1}\gamma^\mu S(\Lambda) = \Lambda^\mu{}_\nu \gamma^\nu \tag{4.46}$$

where $S(\Lambda)$ is a 4×4 representation of $SL(2,c)$ and the $\Lambda^\mu{}_\nu$ characterize rotations and Lorentz boosts.[1]

Constructing the Lorentz scalar

$$\gamma.P = mc \tag{4.47}$$

yields (4.10). This procedure provides an alternative derivation of (4.10).

[1] $SL(2,c)$ is the group of complex 2×2 matrices with determinant equal to unity; $SL(2,c)$ is the covering group for the restricted Lorentz group.

4.8 Position eigenkets

In this section we construct eigenkets appropriate for describing the coordinate-space wave function of a Dirac particle.

The operators

$$X^1, X^2, X^3, S^3, \beta \tag{4.48}$$

form a complete set of compatible observables. We denote their simultaneous eigenkets by

$$\mid \vec{x}d >=\mid x, y, z, d >=\mid x^1, x^2, x^3, d > \qquad (d = 1, 2, 3, 4) \tag{4.49}$$

These eigenkets may be used as a basis for the Hilbert space. That is,

$$X^j = \sum_{d=1}^{4} \int d^3x \mid \vec{x}d > x^j < \vec{x}d \mid \tag{4.50}$$

$$\alpha^j = \sum_{d,d'=1}^{4} \int d^3x \mid \vec{x}d > \alpha^j_{dd'} < \vec{x}d' \mid \tag{4.51}$$

$$\beta = \sum_{d,d'=1}^{4} \int d^3x \mid \vec{x}d > \beta_{dd'} < \vec{x}d' \mid \tag{4.52}$$

$$S^j = \frac{1}{2}\hbar \sum_{d,d'=1}^{4} \int d^3x \mid \vec{x}d > \Sigma^j_{dd'} < \vec{x}d' \mid \tag{4.53}$$

and

$$1 = \sum_{d=1}^{4} \int d^3x \mid \vec{x}d >< \vec{x}d \mid \tag{4.54}$$

$$< \vec{x}d \mid \vec{x'}d >= \delta\left(\vec{x} - \vec{x'}\right)\delta_{dd'} \tag{4.55}$$

where $\alpha^j_{dd'}$, $\beta_{dd'}$ and $\Sigma^j_{dd'}$ are elements of the matrices (4.41), (4.42) and (4.43).

The correspondence between the values of d and the eigenvalues of S^3 and β is given in the following table:

Table 1 **Values of d and the eigenvalues of S^3 and β**

Value of d	*Eigenvalue of* S^3	*Eigenvalue of* β
1	$+\frac{1}{2}\hbar$	+1
2	$-\frac{1}{2}\hbar$	+1
3	$+\frac{1}{2}\hbar$	-1
4	$-\frac{1}{2}\hbar$	-1

Coordinate-space wave function

The coordinate-space wave function $\psi_d(\vec{x}, t)$ for the particle is defined as

$$\boxed{\psi_d(\vec{x}, t) =< \vec{x}d \mid \psi(t) > \qquad (4.56)}$$

where $\mid \psi(t) >$ given by (1.16) is the state of the particle at time t, and

$$\boxed{\mid \psi_d(\vec{x}, t) \mid {}^2 d^3x \qquad (4.57)}$$

is the probability that the particle is in the volume d^3x about $\vec{x}$ at time t with values of S^3 and β corresponding to the value of d in Table 1.

Dirac equation

We show in Section 4.18 that

$$\left(-i\partial\!\!\!/ + \frac{mc}{\hbar}\right)\psi(\vec{x},t) = 0 \tag{4.58}$$

where

$$\psi(\vec{x},t) = \begin{pmatrix} \psi_1(\vec{x},t) \\ \psi_2(\vec{x},t) \\ \psi_3(\vec{x},t) \\ \psi_4(\vec{x},t) \end{pmatrix} \tag{4.59}$$

$$\partial\!\!\!/ = \gamma^\mu \partial_\mu \tag{4.60}$$

$$\partial_\mu = \frac{\partial}{\partial x^\mu} \tag{4.61}$$

Comments

1. **Dirac spinor**

The matrix (4.59) is called a Dirac spinor.

We will generally call a column matrix of the form (4.59) a Dirac spinor.

2. **Dirac equation**

(4.58) is the Dirac equation; it is a first-order partial differential matrix equation for the Dirac spinor (4.59).

3. **Klein-Gordon equation**

In view of (4.38), it follows that every solution of the Dirac equation (4.58) is also a solution of the Klein-Gordon equation, that is,

$$\boxed{\left[\Box + \left(\frac{mc}{\hbar}\right)^2\right]\psi_d(\vec{x}, t) = 0 \qquad (4.62)}$$

4. **Form invariance of the Dirac equation**

The Dirac equation (4.58) is form invariant under a Lorentz transformation. That is,[2]

$$\boxed{\left(-i\partial' + \frac{mc}{\hbar}\right)\psi'(x') = 0 \qquad (4.63)}$$

where

$$\boxed{x'^{\mu} = \Lambda^{\mu}{}_{\nu} x^{\nu} \qquad (4.64)}$$

---

2 In the following x stands for x, y, z, t

is a restricted Lorentz transformation and

$$\psi'(x') = S(\Lambda)\psi(\Lambda x) \tag{4.65}$$

$$\partial' = \gamma'^{\mu}\partial'_{\mu} \tag{4.66}$$

$$\partial'_{\mu} = \frac{\partial}{\partial x'^{\mu}} \tag{4.67}$$

$$\gamma'^{\mu} = S(\Lambda)^{-1}\gamma^{\mu}S(\Lambda) = \Lambda^{\mu}{}_{\nu}\gamma^{\nu} \tag{4.68}$$

4.9 Momentum eigenkets

In this section we define a momentum eigenket $\mid \vec{p}d >$ analogous to the position eigenket $\mid \vec{x}d >$.

The operators

$$P^1, P^2, P^3, S^3, \beta \tag{4.69}$$

form a complete set of compatible observables. We denote their simultaneous eigenkets by

$$\mid \vec{p}d >=\mid p^1, p^2, p^3, d > \qquad (d = 1, 2, 3, 4) \tag{4.70}$$

These eigenkets may be used as a basis for the Hilbert space. That is,

$$P^j = \sum_{d=1}^{4} \int d^3p \mid \vec{p}d > p^j < \vec{p}d \mid \tag{4.71}$$

$$\alpha^j = \sum_{d,d'=1}^{4} \int d^3p \mid \vec{p}d > \alpha^j_{dd'} < \vec{p}d' \mid \tag{4.72}$$

$$\beta = \sum_{d,d'=1}^{4} \int d^3p \mid \vec{p}d > \beta_{dd'} < \vec{p}d' \mid \tag{4.73}$$

$$S^j = \frac{1}{2}\hbar \sum_{dd'=1}^{4} \int d^3p \mid \vec{p}d > \Sigma^j_{dd'} < \vec{p}d' \mid \tag{4.74}$$

and

$$1 = \sum_{d=1}^{4} \int d^3p \mid \vec{p}d >< \vec{p}d \mid \tag{4.75}$$

$$< \vec{p}d \mid \vec{p}'d' >= \delta\left(\vec{p} - \vec{p}'\right)\delta_{dd'} \tag{4.76}$$

It follows from (4.4) that

$$< \vec{x}d \mid \vec{p}d' >= \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar}\delta_{dd'} \tag{4.77}$$

Comments

1. Dirac Hamiltonian

The Dirac Hamiltonian (4.10) is expressed in terms of momentum eigenkets as

$$H = \sum_{d,d'=1}^{4} \int d^3p \mid \vec{p}d > h_{dd'}(\vec{p}) < \vec{p}d' \mid \tag{4.78}$$

where

$$(h_{dd'}(\vec{p})) = \begin{pmatrix} mc^2 & c\vec{\sigma}\cdot\vec{p} \\ c\vec{\sigma}\cdot\vec{p} & -mc^2 \end{pmatrix} \tag{4.79}$$

The matrix $h(\vec{p})$ is not diagonal; $\mid \vec{p}d >$ is not an eigenket of the Dirac Hamiltonian (4.10).

2. Comparison with the spin particle case

The Hamiltonian (3.19) for a particle of arbitrary spin is a function of momentum independent of spin. It follows that every eigenket of momentum is also an eigenket of (3.19).

$\mid \vec{p}d >$, on the other hand, is not an eigenket of the Dirac Hamiltonian (4.10) because (4.10) depends upon $\vec{\alpha}$ and β as well as upon momentum.

Eigenkets of the Dirac Hamiltonian (4.10) will be the subject of Section 4.11.

4.10 Space inversion and time reversal

In this section we express the space inversion operator P and time reversal operator T for a Dirac particle in terms of position and momentum eigenkets.

Space inversion

The space inversion operator P is linear and the Poincare generators transform under space inversion according to (1.29) to (1.32).

We show in Section 4.18 that

$$\boxed{\mathrm{P} = \beta \mathrm{P}' \qquad (4.80)}$$

where

$$\boxed{\mathrm{P}' = \sum_{d=1}^{4} \int d^3x \mid -\vec{x}d > < \vec{x}d \mid = \sum_{d=1}^{4} \int d^3p \mid -\vec{p}d > < \vec{p}d \mid \qquad (4.81)}$$

Time reversal

The time reversal operator T is antilinear and the Poincare generators transform under time reversal according to (1.33) to (1.36).

We show in Section 4.18 that

$$\mathrm{T} = i\Sigma^2\mathrm{T}' \tag{4.82}$$

where

$$\mathrm{T}' \mid \phi >= \sum_{d=1}^{4} \int d^3x \mid \vec{x}d >< \phi \mid \vec{x}d >= \sum_{d=1}^{4} \int d^3p \mid -\vec{p}d >< \phi \mid \vec{p}d > \tag{4.83}$$

4.11 Energy/helicity eigenkets

We have seen from (4.78) that the momentum eigenket $\mid \vec{p}d >$ is not an eigenket of the Dirac Hamiltonian (4.10). In this section we construct an energy/helicity eigenket $\mid u^r(\vec{p}) >$ which is an eigenket of (4.10).

The operators

$$P^1, P^2, P^3, \Lambda, H \tag{4.84}$$

are a complete set of compatible observables where Λ is the helicity (1.55) and H is the Dirac Hamiltonian (4.10). Λ has eigenvalues $\pm\frac{1}{2}\hbar$.

It follows from (4.19) that

$$\boxed{\Lambda = \frac{\vec{S} \cdot \vec{P}}{P} \qquad (4.85)}$$

When expressed in terms of the momentum eigenkets,

$$\boxed{\Lambda = \frac{1}{2}\hbar \sum_{d,d'=1}^{4} \int d^3p \mid \vec{p}d > \left(\vec{\Sigma} \cdot \vec{\hat{p}}\right)_{dd'} < \vec{p}d' \mid \qquad (4.86)}$$

where the 4×4 matrix $\vec{\Sigma} \cdot \vec{\hat{p}}$ is

$$\boxed{\vec{\Sigma} \cdot \vec{\hat{p}} = \begin{pmatrix} \vec{\sigma} \cdot \vec{\hat{p}} & 0 \\ 0 & \vec{\sigma} \cdot \vec{\hat{p}} \end{pmatrix} \qquad (4.87)}$$

where $\vec{\sigma} \cdot \vec{\hat{p}}$ is the 2×2 matrix (3.49).

We denote the simultaneous eigenkets of (4.84) by

$$\boxed{\mid u^r(\vec{p}) > \qquad (r = 1, 2, 3, 4) \qquad (4.88)}$$

These eigenkets may be used as a basis for the Hilbert space. That is,

$$\boxed{\begin{aligned} P^j &= \sum_{r=1}^{4} \int d^3p \mid u^r(\vec{p}) > p^j < u^r(\vec{p}) \mid && (4.89) \\ \Lambda &= \sum_{r=1}^{4} \int d^3p \mid u^r(\vec{p}) > \lambda_r \hbar < u^r(\vec{p}) \mid && (4.90) \\ H &= \sum_{r=1}^{4} \int d^3p \mid u^r(\vec{p}) > \epsilon_{pr} < u^r(\vec{p}) \mid && (4.91) \end{aligned}}$$

and

$$\boxed{\begin{aligned} 1 &= \sum_{r=1}^{4} \int d^3p \mid u^r(\vec{p}) >< u^r(\vec{p}) \mid && (4.92) \\ < u^r(\vec{p}) \mid u^{r'}\left(\vec{p'}\right) > &= \delta\left(\vec{p} - \vec{p'}\right)\delta_{rr'} && (4.93) \end{aligned}}$$

where

$$\boxed{\begin{pmatrix} \lambda_1 \\ \lambda_2 \\ \lambda_3 \\ \lambda_4 \end{pmatrix} = \begin{pmatrix} +\frac{1}{2} \\ -\frac{1}{2} \\ +\frac{1}{2} \\ -\frac{1}{2} \end{pmatrix} \qquad (4.94)}$$

$$\begin{pmatrix} \epsilon_{p1} \\ \epsilon_{p2} \\ \epsilon_{p3} \\ \epsilon_{p4} \end{pmatrix} = \begin{pmatrix} \epsilon_{p} \\ \epsilon_{p} \\ -\epsilon_{p} \\ -\epsilon_{p} \end{pmatrix} \tag{4.95}$$

where ϵ_p is given by (2.20).

Furthermore,

$$< \vec{p}d \mid u^r\left(\vec{p'}\right) >= \delta\left(\vec{p} - \vec{p'}\right) u_d^r(\vec{p}) \tag{4.96}$$

which defines $u_d^r(\vec{p})$, and, using (4.77),

$$< \vec{x}d \mid u^r(\vec{p}) >= \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar} u_d^r(\vec{p}) \tag{4.97}$$

The functions $u_d^r(p)$ are given in Section 4.14.

The correspondence between values of r and spectral values of Λ and H is given in the following table:

Table 2 **Values of r and the spectral values of Λ and H**

Value of r	***Eigenvalue of*** Λ	***Spectral value of*** H
1	$+\frac{1}{2}\hbar$	$+\epsilon_p$
2	$-\frac{1}{2}\hbar$	$+\epsilon_p$

Table 2 (Continued) **Values of r and the spectral values of Λ and H**

3	$+\frac{1}{2}\hbar$	$-\epsilon_p$
4	$-\frac{1}{2}\hbar$	$-\epsilon_p$

Energy/helicity wave function

The energy/helicity wave function $\psi_r(\vec{p}, t)$ for the particle is defined as

$$\psi_r(\vec{p}, t) = < u^r(\vec{p}) \mid \psi(t) > \tag{4.98}$$

where $\mid \psi(t) >$ given by (1.16) is the state of the particle at time t, and

$$\mid \psi_r(\vec{p}, t) \mid {}^2 d^3p \tag{4.99}$$

is the probability that the particle has momentum in the volume d^3p about $\vec{p}$ at time t with values of helicity and energy corresponding to the value of r as in Table 2.

It follows from (4.91) that

$$\psi_r(\vec{p}, t) = e^{-i\epsilon_{pr}t/\hbar} < u^r(\vec{p}) \mid \psi > \tag{4.100}$$

Negative energy components

A physical particle has positive energy. The negative energy components (the $r = 3$ and $r = 4$ components) of $\mid \psi(t) >$ are discussed in Section 4.17.

4.12 Pryce-Foldy-Wouthuysen picture revisited

We have constructed momentum eigenkets $\mid \vec{p}d >$ and $\mid u^r(\vec{p}) >$. The unitary operator G defined by

$$G = \sum_{r=1}^{4} \int d^3p \mid \vec{p}r >< u^r(\vec{p}) \mid \tag{4.101}$$

transforms one eigenket to the other, that is,

$$\mid \vec{p}d >= G \mid u^d(\vec{p}) > \tag{4.102}$$

$$\mid u^r(\vec{p}) >= G^\dagger \mid \vec{p}r > \tag{4.103}$$

It follows from (4.91) that

$$\begin{aligned} GHG^\dagger &= \sum_{r=1}^{4} \int d^3pG \mid u^r(\vec{p}) > \epsilon_{pr} < u^r(\vec{p}) \mid G^\dagger \\ &= \sum_{d=1}^{4} \int d^3p \mid \vec{p}d > \epsilon_{dr} < \vec{p}d \mid \end{aligned} \tag{4.104}$$

That is,

$$\boxed{GHG^{\dagger} = \beta\sqrt{P^2c^2 + m^2c^4} \qquad (4.105)}$$

Similarly, it follows from (4.90) that

$$\boxed{G\Lambda G^{\dagger} = S^3 \qquad (4.106)}$$

Comments

1. **Pryce-Foldy-Wouthuysen transformation**

   (4.101) is similar (but not identical) to the Pryce-Foldy-Wouthuysen transformation operator (4.23).

2. **Pryce-Foldy-Wouthuysen Hamiltonian**

   (4.104) provides a simple derivation of the Hamiltonian $\beta\sqrt{P^2c^2 + m^2c^4}$.

3. **Matrix elements**

   In the $\mid \vec{p}d >$ representation,

$$\boxed{< \vec{p}d \mid G^{\dagger} \mid \vec{p'}r >= \delta\left(\vec{p} - \vec{p'}\right) u_d^r(\vec{p}) \qquad (4.107)}$$

4.13 Energy projectors

We define a complete set of orthogonal projection operators Γ_+ and Γ_- which project onto the positive and negative energy states, respectively:

$$\boxed{\begin{aligned}\Gamma_+ &= \sum_{r=1,2} \int d^3p \mid u^r(\vec{p}) >< u^r(\vec{p}) \mid \qquad (4.108)\\ \Gamma_- &= \sum_{r=3,4} \int d^3p \mid u^r(\vec{p}) >< u^r(\vec{p}) \mid \qquad (4.109)\end{aligned}}$$

Alternatively,

$$\boxed{\Gamma_\pm = \frac{1}{2}\left(1 \pm \frac{H}{\sqrt{P^2c^2 + m^2c^4}}\right) \qquad (4.110)}$$

A general state $\mid \psi >$ of the Dirac particle can be written as

$$\boxed{\mid \psi >=\mid \psi_+ > + \mid \psi_- > \qquad (4.111)}$$

where

$$\boxed{\mid \psi_\pm >= \Gamma_\pm \mid \psi > \qquad (4.112)}$$

$| \psi_+ >$ and $| \psi_- >$ are the positive and negative energy components of $| \psi >$, respectively.

Comments

1. **Projection of velocity**

It follows on calculation that

$$\boxed{\Gamma_\pm c\vec{\alpha}\Gamma_\pm = \Gamma_\pm \vec{V}\Gamma_\pm \qquad (4.113)}$$

where $\vec{V}$ is the velocity operator (4.31).

2. **Zitterbewegung**

It follows from (4.32) and (4.113) that

$$\boxed{\Gamma_\pm \vec{Z}(t)\Gamma_\pm = 0 \qquad (4.114)}$$

and therefore that

$$\boxed{< \psi \mid \vec{Z}(t) \mid \psi > = < \psi_- \mid \vec{Z}(t) \mid \psi_+ > + < \psi_+ \mid \vec{Z}(t) \mid \psi_- > \qquad (4.115)}$$

As discussed in Section 4.4, zitterbewegung is an unphysical phenomenon associated with regarding the Cartesian position as the centre of mass position in the Dirac picture. Zitterbewgung arises because the Dirac Hamiltonian

(4.10) allows both positive and negative energy states; there is no zitterbewegung if $|\psi>$ possesses only a positive energy component or only a negative energy component.

4.14 Energy/helicity spinors

In this section we specify the 16 functions $u_d^r(\vec{p})$ defined in (4.96). The functions are given as four Dirac spinors $u^r(\vec{p}) = \left(u_d^r(\vec{p})\right)$; the derivations are given in Section 4.18. We quote the final results:

Positive energy, positive helicity

$$u^1(\vec{p}) = \sqrt{\frac{\epsilon_p + mc^2}{2\epsilon_p}} \begin{pmatrix} h^{\frac{1}{2}}(\vec{p}) \\ \zeta h^{\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.116}$$

Positive energy, negative helicity

$$u^2(\vec{p}) = \sqrt{\frac{\epsilon_p + mc^2}{2\epsilon_p}} \begin{pmatrix} h^{-\frac{1}{2}}(\vec{p}) \\ -\zeta h^{-\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.117}$$

Negative energy, positive helicity

$$u^3(\vec{p}) = \sqrt{\frac{\epsilon_p + mc^2}{2\epsilon_p}} \begin{pmatrix} -\zeta h^{\frac{1}{2}}(\vec{p}) \\ h^{\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.118}$$

Negative energy, negative helicity

$$u^4(\vec{p}) = \sqrt{\frac{\epsilon_p + mc^2}{2\epsilon_p}} \begin{pmatrix} \zeta h^{-\frac{1}{2}}(\vec{p}) \\ h^{-\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.119}$$

where

$$h^{\lambda}(\vec{p}) = \begin{pmatrix} h^{\lambda}_{\frac{1}{2}}(\vec{p}) \\ h^{\lambda}_{-\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.120}$$

$$\zeta = \frac{pc}{\epsilon_p + mc^2} \tag{4.121}$$

The functions $h^{\lambda}_{m_s}(\vec{p})$ are given by (3.59); $h^{\lambda}_{m_s}(\vec{p})$ is the probability amplitude for a particle with spherical polar coordinates of momentum equal to (θ, φ) and helicity λ to have 3-component of spin equal to m_s.

Comments

1. **Energy/helicity spinors**

   Matrices (4.116) to (4.119) are called energy/helicity spinors.

2. **Eigenvalue equation**

   The energy/helicity spinors satisfy the eigenvalue equations:

$$(\not p - mc)u^r(\vec{p}) = 0 \qquad (r = 1, 2) \tag{4.122}$$

$$(\not p + mc)u^r(-\vec{p}) = 0 \qquad (r = 3, 4) \tag{4.123}$$

   where

$$\not p = \gamma . p \tag{4.124}$$

$$p^\mu = \left(\frac{\epsilon_p}{c}, p^1, p^2, p^3\right) \tag{4.125}$$

   (4.122) and (4.123) follow directly using (4.47) and the eigenvalue equation for P^μ.

3. **Normalization**

(4.116) to (4.119) are normalized such that

$$\sum_{d=1}^{4} u_d^{r^*}(\vec{p}) u_d^{r'}(\vec{p}) = \delta_{rr'} \tag{4.126}$$

$$\sum_{r=1}^{4} u_d^{r^*}(\vec{p}) u_{d'}^{r}(\vec{p}) = \delta_{dd'} \tag{4.127}$$

4. **Properties of ζ**

It follows from (4.121) that

$$0 \leq \zeta \leq 1 \tag{4.128}$$

$$\lim_{c\to\infty} \zeta = 0 \tag{4.129}$$

$$\lim_{m\to 0} \zeta = 1 \tag{4.130}$$

$$\zeta < 1 \qquad \text{when} \qquad m \neq 0 \tag{4.131}$$

5. **Nonrelativistic limit**

It follows from (4.116), (4.117) and (4.129) that

$$\boxed{\lim_{c\to\infty} u^r(\vec{p}) = \begin{pmatrix} h^{\frac{3}{2}-r}(\vec{p}) \\ 0 \end{pmatrix} \qquad (r = 1,2)} \tag{4.132}$$

The two positive energy spinors reduce to the appropriate two-component spinors in the nonrelativistic limit.

6. Large and small components

In view of (4.131), when $r = 1,2$, components of $u^r(\vec{p})$ proportional to ζ are called the small components and components of $u^r(\vec{p})$ not proportional to ζ are called the large components.

7. Dirac particle with zero rest mass

It follows from (4.116) to (4.119) and (4.130) that

$$\lim_{m=0} u^1(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{\frac{1}{2}}(\vec{p}) \\ h^{\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.133}$$

$$\lim_{m=0} u^2(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{-\frac{1}{2}}(\vec{p}) \\ -h^{-\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.134}$$

$$\lim_{m=0} u^3(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} -h^{\frac{1}{2}}(\vec{p}) \\ h^{\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.135}$$

$$\lim_{m=0} u^4(\vec{p}) = \frac{1}{\sqrt{2}}\begin{pmatrix} h^{-\frac{1}{2}}(\vec{p}) \\ h^{-\frac{1}{2}}(\vec{p}) \end{pmatrix} \tag{4.136}$$

4.15 Most general solution of the Dirac equation

It follows from (4.92), (4.98) and (4.100) that a general state $\mid \psi(t) >$ of a Dirac particle may be written as

$$\boxed{\mid \psi(t) >= \sum_{r=1}^{4} \int d^3p \mid u^r(\vec{p}) > e^{-i\epsilon_r t/\hbar} \psi_r(\vec{p}, 0) \qquad (4.137)}$$

from which it follows using (4.97) that the Dirac spinor (4.59) may be written as

$$\psi(\vec{x}, t) =$$

$$\sum_{r=1,2} \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p \Big[e^{-ip.x/\hbar} u^r(\vec{p})\psi_r(\vec{p}) + e^{ip.x/\hbar} u^{r+2}(-\vec{p})\psi_{r+2}(-\vec{p})\Big] \qquad (4.138)$$

where

$$\boxed{\psi_r(\vec{p}) =< u^r(\vec{p}) \mid \psi > \qquad (4.139)}$$

is the probability amplitude that the Dirac particle at time zero has momentum $\vec{p}$ and values of helicity and energy corresponding to the value of r in Table 2.

Comments

1. **Most general solution of the Dirac equation**

(4.138) gives the most general solution of the Dirac equation (4.58).

2. **Quantizing the Dirac equation**

(4.138) is the traditional starting point for the development of the relativistic quantum field theory of electrons and positrons.

The functions $\psi_r(\vec{p})$ and $\psi_{r+2}(-\vec{p})$ are reinterpreted, respectively, as annihilation operators for electrons and creation operators for positrons and (4.138) is reinterpreted as an equation for a field operator $\psi(\vec{x}, t)$ in the Heisenberg picture. The field operator obeys the Dirac equation (4.58).

The above reinterpretation is called quantizing the Dirac equation.

We develop the relativistic quantum field theory of electrons and positrons from first principles in *QLB: Relativistic Quantum Field Theory*. The traditional approach of quantizing the Dirac equation will not be followed.

4.16 Electron in an electromagnetic field

In this section we explore some consequences of the interaction of the Dirac particle with an external electromagnetic field. In *QLB: Relativistic Quantum Field Theory* we characterize the electromagnetic field in terms of photons; here we describe it in terms of potentials which are functions of the Cartesian coordinates X^1, X^2, X^3 and the time t. We assume the Dirac particle to be an electron.

Electromagnetic field

We specify a scalar potential $\Phi\left(\vec{X}, t\right)$ and a vector potential $\vec{A}\left(\vec{X}, t\right)$ in terms of which an electric field $\vec{E}$ and a magnetic field $\vec{B}$ are defined by

$$i\hbar\vec{E} = \left[\vec{P}, \Phi\right] - i\hbar\frac{1}{c}\frac{\partial\vec{A}}{\partial t} \tag{4.140}$$

$$i\hbar\vec{B} = -\vec{P} \times \vec{A} \tag{4.141}$$

Comments

1. **Coordinate-space representation**

It follows using (4.191) that

$$< \vec{x}d \mid \vec{E} \mid \psi(t) >= \vec{E}(\vec{x},t)\psi_d(\vec{x},t) \tag{4.142}$$

$$< \vec{x}d \mid \vec{B} \mid \psi(t) >= \vec{B}(\vec{x},t)\psi_d(\vec{x},t) \tag{4.143}$$

where

$$\vec{E}(\vec{x},t) = -\vec{\nabla}\Phi(\vec{x},t) - \frac{1}{c}\frac{\partial\vec{A}(\vec{x},t)}{\partial t} \tag{4.144}$$

$$\vec{B}(\vec{x},t) = \vec{\nabla} \times \vec{A}(\vec{x},t) \tag{4.145}$$

2. **Maxwell's equations**

(4.144) and (4.145) ensure that

$$\vec{\nabla} \times \vec{E}(\vec{x}, t) = -\frac{1}{c}\frac{\partial \vec{B}(\vec{x}, t)}{\partial t} \tag{4.146}$$

$$\vec{\nabla} \cdot \vec{B}(\vec{x}, t) = 0 \tag{4.147}$$

that is, Faraday's Law holds and there are no free magnetic poles. It is required also that

$$\vec{\nabla} \times \vec{B}(\vec{x}, t) = \frac{4\pi}{c}\vec{J}_c(\vec{x}, t) - \frac{1}{c}\frac{\partial \vec{E}(\vec{x}, t)}{\partial t} \tag{4.148}$$

$$\vec{\nabla} \cdot \vec{E}(\vec{x}, t) = 4\pi \rho_c(\vec{x}, t) \tag{4.149}$$

that is, Gauss' Law holds. $\rho_c(\vec{x}, t)$ and $\vec{J}_c(\vec{x}, t)$ are the coordinate representatives of the charge density and current density, respectively, of the source of the electric and magnetic fields.

(4.146) to (4.149) are Maxwell's equations.

3. Lorentz condition; wave equations

Maxwell's equations (4.146) to (4.149) hold if the potentials $\Phi(\vec{x}, t)$ and $\vec{A}(\vec{x}, t)$ satisfy the Lorentz condition

$$\vec{\nabla} \cdot \vec{A}(\vec{x}, t) + \frac{1}{c}\frac{\partial \Phi(\vec{x}, t)}{\partial t} = 0 \tag{4.150}$$

and the inhomogeneous wave equations

$$\Box\Phi(\vec{x},t) = 4\pi\rho_c(\vec{x},t) \tag{4.151}$$

$$\Box\vec{A}(\vec{x},t) = \frac{4\pi}{c}\vec{J}_c(\vec{x},t) \tag{4.152}$$

Electrodynamics

The Hamiltonian for the electron in the electromagnetic field is

$$H = c\vec{\alpha}\cdot\left(\vec{P} - \frac{e}{c}\vec{A}\right) + \beta mc^2 + e\Phi \tag{4.153}$$

where m and e are the mass and charge of the electron ($m = 0.511\,\mathrm{MeV}/c^2$ and $e = -1.60\times10^{-19}$C).

The electromagnetic field exerts a force $\vec{F}$ and a torque $\vec{T}$ on the electron where

$$i\hbar\vec{F} = \left[\vec{P} - \frac{e}{c}\vec{A}, H\right] \tag{4.154}$$

$$i\hbar\vec{T} = \left[\vec{J}, H\right] \tag{4.155}$$

It follows using (4.153) that

$$\vec{F} = e\left(\vec{E} + \vec{\alpha} \times \vec{B}\right) \tag{4.156}$$

Lorentz force

(4.156) is the quantal version of the Lorentz force of classical electromagnetism. The Lorentz force takes its more customary form when $\vec{\alpha}$ is replaced by its positive energy projection $\Gamma_+ \vec{V}/c\Gamma_+$ as per (4.113).

Nonrelativistic Schrodinger equation

We assume that the potentials do not depend explicitly upon time.

The coordinate representation of the eigenvalue problem for the Hamiltonian (4.153) is

$$\begin{pmatrix} mc^2 + e\Phi & c\vec{\sigma} \cdot \vec{\Pi} \\ c\vec{\sigma} \cdot \vec{\Pi} & -mc^2 + e\Phi \end{pmatrix} \begin{pmatrix} \phi \\ \chi \end{pmatrix} = \epsilon \begin{pmatrix} \phi \\ \chi \end{pmatrix} \tag{4.157}$$

where

$$\vec{\Pi} = -i\hbar\vec{\nabla} - \frac{e}{c}\vec{A}(\vec{x}) \tag{4.158}$$

$$\begin{pmatrix} \phi \\ \chi \end{pmatrix} = \begin{pmatrix} < \vec{x}1 \mid \epsilon > \\ < \vec{x}2 \mid \epsilon > \\ < \vec{x}3 \mid \epsilon > \\ < \vec{x}4 \mid \epsilon > \end{pmatrix} \tag{4.159}$$

where $\mid \epsilon >$ is an eigenvector of (4.153) and ϵ is the corresponding eigenvalue.

We show in Section 4.18 that for motion in a homogeneous magnetic field the nonrelativistic limit of (4.157) is

$$\left(-\frac{\nabla^2}{2m} + e\Phi - \vec{\mu} \cdot \vec{B} \right) \phi = \epsilon' \phi \tag{4.160}$$

where $\epsilon' = \epsilon - mc^2$ and where

$$\vec{\mu} = \frac{e}{2mc} \left(\vec{L} + g\vec{S} \right) \tag{4.161}$$

$$g = 2 \tag{4.162}$$

Comments

1. **Dependence on the magnetic field**

Dirac's Hamiltonian (4.153) predicts the presence of a $\vec{\mu} \cdot \vec{B}$ term in the nonrelativistic equation (4.160)

The expression $-\vec{\mu} \cdot \vec{B}$ is used in *QLB: Introductory Topics* for the interaction for a magnetic moment with a magnetic field.

2. **Magnetic moment and g factor of the electron**

$\vec{\mu}$ (4.161) is the magnetic moment operator for a nonrelativistic electron.

g is the g factor of the electron.

Dirac's Hamiltonian (4.153) predicts that the g factor of the electron is 2.

3. **g factor: experiment and quantum electrodynamics**

The experimental value g_{exp} and value g_{qed} calculated using quantum electrodynamics for the g-factor of the electron are[1]

$$\frac{g_{exp}}{2} = 1.0011596521(93 \pm 10) \tag{4.163}$$

$$\frac{g_{qed}}{2} = 1.001159652190 \tag{4.164}$$

The Dirac prediction (4.162) is a remarkable result.

Magnetic moment: nonrelativistic projection

We give a second derivation of (4.161) and (4.162).

For motion in a homogeneous magnetic field it follows from (4.155) that the torque on the electron is

$$\vec{T} = \vec{M} \times \vec{B} \tag{4.165}$$

[1] The first calculation of g_{qed} by Schwinger in 1948 (see Schwinger (1958)) gave $g_{qed}/2 = 1 + \alpha/2\pi = 1.0011614$ where α is the fine-structure constant (4.169).

where

$$\vec{M} = \frac{e}{2}\left(\vec{X} \times \vec{\alpha}\right) \tag{4.166}$$

Comments

1. **Magnetic moment operator**

   $\vec{M}$ is the magnetic moment operator for a Dirac electron.

2. **Precession**

   It follows from (4.165) that the average value of the magnetic moment precesses about direction of the magnetic field.

3. **Nonrelativistic projection**

   We show in Section 4.18 that

$$\vec{M}_{nr} = \vec{\mu} \tag{4.167}$$

where $\vec{M}_{nr}$ is the nonrelativistic projection of $\vec{M}$ and $\vec{\mu}$ is given by (4.161).

Hydrogen atom

We assume that the electron is subjected to the attractive Coulomb force of a $+1.60 \times 10^{-19}$C charge fixed at the origin. The Hamiltonian for the electron is

$$\boxed{H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2 - \frac{e^2}{\sqrt{\vec{X} \cdot \vec{X}}} \qquad (4.168)}$$

To fourth-order in the fine-structure constant

$$\boxed{\alpha = \frac{e^2}{\hbar c} = \frac{1}{137.04} \qquad (4.169)}$$

the eigenvalues of (4.168) are

$$\boxed{\epsilon_{nj} = mc^2 + \epsilon_n \left[1 + \frac{\alpha^2}{n} \left(\frac{1}{j + \frac{1}{2}} - \frac{3}{4n} \right) \right] \qquad (4.170)}$$

where

$$\boxed{\epsilon_n = -\frac{1}{n^2} \left(\frac{me^4}{2\hbar^2} \right) = -\frac{\alpha^2 mc^2}{2n^2} \qquad (4.171)}$$

$$n = 1, 2, \cdots \tag{4.172}$$

$$j = l \pm \frac{1}{2} \tag{4.173}$$

$$l = 0, 1, 2, \cdots, n-1 \tag{4.174}$$

Comments

1. **Comparison with the nonrelativistic case**

ϵ_n is the energy eigenvalue of a nonrelativistic electron subjected to the attractive Coulomb force of a $+1.60 \times 10^{-19}$C charge fixed at the origin.

The term

$$\frac{\alpha^2}{n}\left(\frac{1}{j+\frac{1}{2}} - \frac{3}{4n}\right) \tag{4.175}$$

in (4.170) gives a fine structure to the nonrelativistic energy levels.

2. **Comparison with experiment**

ϵ_{nj} does not depend explicitly upon l; the $2S_{1/2}$ and $2P_{1/2}$ states, therefore, have the same energy. The $2P_{3/2}$ state has a higher energy by 45.2 μeV.

This is close to what is observed experimentally for the hydrogen atom, but it is not exact.

High precision laser spectroscopy of the hydrogen atom shows that the $2S_{1/2}$

and $2P_{3/2}$ levels lie 4.38 μeV and 42.2 μeV, respectively, above the $2P_{1/2}$ level.

The hydrogen atom $2S_{1/2}$ and $2P_{1/2}$ states are not degenerate in energy. The splitting of these levels (the Lamb shift) was first observed by Lamb and Retherford (1947).

3. **Comparison with quantum electrodynamics**

The Hamiltonian (4.168) does not yield the Lamb shift. It should not be surprising that there are physical effects not given by (4.168) since it does not explicitly include particle creation and annihiliation.

A theoretical explanation of the Lamb shift is given by quantum electrodynamics (QED), the relativistic quantum field theory of electrons, positrons and photons (see, for example, Schwinger (1958).). QED explicitly includes particle creation and annihilation; a very brief introduction is given in *QLB: Relativistic Quantum Field Theory*.

The Lamb shift arises partly because the electron-positron system can have intermediate states consisting of photons and electron-positron pairs. That is, it arises partly because of vacuum polarization effects.

QED gives perfect agreement with all electon-positron-photon experiments performed to date.

4.17 Negative energy states and Dirac's hole theory

A free Dirac particle prepared in a state with positive energy will remain in a state with positive energy because the Dirac Hamiltonian (4.10) does not couple positive and negative energy states. That is, if

$$| \psi_{-}(0) >= 0 \tag{4.176}$$

then

$$\Gamma_{-} | \psi(t) >= 0 \tag{4.177}$$

for all time t where Γ_{-} is the projection operator (4.109) onto the negative energy states.

(4.177) is not true in general for a Dirac particle in interaction. That is, if

$$H = c\vec{\alpha} \cdot \vec{P} + \beta mc^2 + V \tag{4.178}$$

and if (4.176) holds, in general,

$$\Gamma_{-} | \psi(t) > \neq 0 \tag{4.179}$$

for $t > 0$ since V may couple positive and negative energy states.[1] A particle prepared in a state of positive energy and subjected to such an interaction potential V will give up an arbitrarily large amount of energy. This is not what is observed.

One way of overcoming the above difficulty if to restrict the class of potentials to those which do not couple positive and negative energy states. If

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1 An example is $V = V\left(\vec{X}\right)$.

$$V = \Gamma_+ V \Gamma_+ \tag{4.180}$$

where Γ_+ is the projection operator (4.108) onto the positive energy states, then

$$\Gamma_\pm V \Gamma_\mp = 0 \tag{4.181}$$

and so if (4.176) holds, then so will (4.177) for all time t. That is, the particle prepared in a state with positive energy will remain in a state of positive energy when it is subjected to a potential satisfying (4.180).

Dirac (1930) suggested a way of overcoming the difficulty of negative energy states which did not use the projection operator method. Indeed, his way of overcoming the difficulty is a measure of his genius. He suggested that a particle cannot make a transition to a negative energy state because

all negative energy states are occupied, with one particle in each state in accordance with the Pauli Principle.

This is a remarkable suggestion.

Comments

1. Dirac's hole theory

A bonus arising from Dirac's suggestion is that an unoccupied negative energy state, that is, a hole in the filled negative energy sea, is manifested as something with a positive energy, since to make it disappear, that is, to fill the hole, one must add to the system a particle with negative energy.

2. Positrons and prediction of the existence of antimatter

The unoccupied negative energy states were eventually assumed by Dirac to be antielectrons, that is, positrons.

Dirac's hole theory implies the existence of antimatter. The positron was discovered in 1932, two years after Dirac proposed the hole theory.

3. Conversion of mass to energy

Dirac's hole theory gives a picture of the conversion of mass to energy: electron-positron annihilation occurs when the electron fills the hole.

4. Comparison with relativistic quantum field theory

Dirac's hole theory is, of course, a many-particle theory since it requires the existence of an infinite number of particles filling the negative energy sea. The negative energy sea has infinite negative energy and infinite negative charge.

Dirac's hole theory requires an enormous jump in logic: the single-particle system we started with requires the existence of an invisible infinite-particle system. The hole theory, illogical as it is, nevertheless paved the way for the construction of relativistic quantum field theory only a few years after Dirac invented it.

In *QLB: Relativistic Quantum Field Theory* we construct a relativistic quantum field theory of electrons and positrons where particles and antiparticles appear on equal footing and with positive energies. No reference is made to a negative energy sea. The theory is finite. The Dirac equation (4.58) arises as the field equation for the fermion-antifermion field operator $\psi(\vec{x}, t)$ in the Heisenberg picture. This operator contains electron and positron variables.

We show how Dirac's hole theory results from an incorrect interpretation of this finite field theory.

4.18 Some derivations

Derivation of (4.32)

We derive (4.32) by solving a first-order differential equation. Using

$$i\hbar\frac{d\vec{X}(t)}{dt} = U^{\dagger}(t)\left[\vec{X}, H\right]U(t) \tag{4.182}$$

and (4.21) it follows that

$$\frac{d\vec{X}(t)}{dt} = c\vec{\alpha}(t) \tag{4.183}$$

where

$$\vec{\alpha}(t) = U^{\dagger}(t)\vec{\alpha}U(t) \tag{4.184}$$

Now

$$i\hbar\frac{d\vec{\alpha}(t)}{dt} = U^{\dagger}(t)[\vec{\alpha}, H]U(t) \tag{4.185}$$

and

$$[\vec{\alpha}, H] = \{\vec{\alpha}, H\} - 2H\vec{\alpha} = 2c\vec{P} - 2H\vec{\alpha} \tag{4.186}$$

so

$$i\hbar\frac{d\vec{\alpha}(t)}{dt} = 2c\vec{P} - 2H\vec{\alpha}(t) \tag{4.187}$$

(4.187) is a first-order differential equation whose solution is

$$\vec{\alpha}(t) = \frac{c\vec{P}}{H} + e^{2iHt/\hbar}\left(\vec{\alpha} - \frac{c\vec{P}}{H}\right) \tag{4.188}$$

Substituting (4.188) into (4.183) and integrating yields (4.32).

Derivation of the Dirac equation (4.58)

The coordinate-space representation of the Schrodinger equation

$$H \mid \psi(t) >= i\hbar \frac{d}{dt} \mid \psi(t) > \tag{4.189}$$

is

$$< \vec{x}d \mid \left(c\vec{\alpha} \cdot \vec{P} + \beta mc^2 \right) \mid \psi(t) >= i\hbar \frac{\partial \psi_d(\vec{x},t)}{\partial t} \tag{4.190}$$

Now

$$< \vec{x}d \mid P^j \mid \psi(t) >= -i\hbar \frac{\partial \psi_d(\vec{x},t)}{\partial x^j} \tag{4.191}$$

so (4.190) is

$$\sum_{j=1}^{3} \sum_{d'=1}^{4} \left\{ -i\hbar c \alpha^j_{dd'} \frac{\partial}{\partial x^j} + mc^2 \beta_{dd'} \right\} \psi_{d'}(\vec{x},t) = i\hbar \frac{\partial \psi_d(\vec{x},t)}{\partial t} \tag{4.192}$$

or, more compactly,

$$\left(-i\hbar c\vec{\alpha} \cdot \vec{\nabla} + \beta mc^2 \right) \psi(\vec{x},t) = i\hbar \frac{\partial \psi(\vec{x},t)}{\partial t} \tag{4.193}$$

where α^j and β are the Dirac matrices (4.41) and (4.42). Finally, (4.193) can be written as (4.58).

Verification of (4.80)

(1.29) to (1.32) hold for the Dirac generators (4.10) to (4.13) if the fundamental dynamical variables (4.1) transform under space inversion as follows:

$$\mathrm{P}\vec{X}\mathrm{P}^\dagger = -\vec{X} \tag{4.194}$$

$$\mathrm{P}\vec{P}\mathrm{P}^\dagger = -\vec{P} \tag{4.195}$$

$$\mathrm{P}\vec{\alpha}\mathrm{P}^\dagger = -\vec{\alpha} \tag{4.196}$$

$$\mathrm{P}\beta\mathrm{P}^{\dagger} = \beta \tag{4.197}$$

It follows from (4.80) and (4.81) that for any linear operator A

$$\mathrm{P}A\mathrm{P}^{\dagger} = \beta\tilde{A}\beta \tag{4.198}$$

where

$$\begin{aligned}\hat{A} &= \sum_{d,d'=1}^{4} \int d^3x d^3x' \mid \vec{x}d >< -\vec{x}d \mid A \mid -\vec{x'}d' >< \vec{x'}d' \mid \\ &= \sum_{d,d'=1}^{4} \int d^3p d^3p' \mid \vec{p}d >< -\vec{p}d \mid A \mid -\vec{p'}d' >< \vec{p'}d' \mid\end{aligned} \tag{4.199}$$

In particular,

$$\tilde{\vec{X}} = -\vec{X} \tag{4.200}$$

$$\tilde{\vec{P}} = -\vec{P} \tag{4.201}$$

$$\tilde{\vec{\alpha}} = \vec{\alpha} \tag{4.202}$$

$$\tilde{\beta} = \beta \tag{4.203}$$

(4.194) to (4.197) follow from (4.198) and (4.200) to (4.203).

Verification of (4.82)

(1.33) to (1.36) hold for the Dirac generators (4.10) to (4.13) if the fundamental dynamical variables (4.1) transform under time reversal as follows:

$$\mathrm{T}\vec{X}\mathrm{T}^\dagger = \vec{X} \tag{4.204}$$

$$\mathrm{T}\vec{P}\mathrm{T}^\dagger = -\vec{P} \tag{4.205}$$

$$\mathrm{T}\vec{\alpha}\mathrm{T}^\dagger = -\vec{\alpha} \tag{4.206}$$

$$\mathrm{T}\beta\mathrm{T}^\dagger = \beta \tag{4.207}$$

It follows from (4.82) and (4.83) that for any linear operator A

$$\mathrm{T}A\mathrm{T}^\dagger = \Sigma^2 \breve{A} \Sigma^2 \tag{4.208}$$

where

$$\begin{aligned}\breve{A} &= \sum_{d,d'=1}^{4} \int d^3x d^3x' \mid \vec{x}d >< \vec{x}d \mid A \mid \vec{x}'d' >^* < \vec{x}'d' \mid \\ &= \sum_{d,d'=1}^{4} \int d^3p d^3p' \mid \vec{p}d >< -\vec{p}d \mid A \mid -\vec{p}'d' >^* < \vec{p}'d' \mid\end{aligned} \tag{4.209}$$

In particular,

$$\breve{\vec{X}} = \vec{X} \tag{4.210}$$

$$\breve{\vec{P}} = -\vec{P} \tag{4.211}$$

$$\breve{\alpha^j} = (-)^{j+1}\alpha^j \tag{4.212}$$

$$\breve{\beta} = \beta \tag{4.213}$$

(4.204) to (4.207) follow from (4.208) and (4.210) to (4.213).

Derivation of (4.116) to (4.119)

The functions $u_d^r(p)$ are determined by solving

$$\Lambda \mid u^r(\vec{p}) >= \hbar \lambda_r \mid u^r(\vec{p}) > \tag{4.214}$$

$$H \mid u^r(\vec{p}) >= \epsilon_{pr} \mid u^r(\vec{p}) > \tag{4.215}$$

in the $\mid \vec{p} d >$ representation. In this representation, (4.214) and (4.215) yield

$$\begin{pmatrix} \frac{1}{2}\vec{\sigma} \cdot \vec{\hat{p}} - \lambda_r & 0 \\ 0 & \frac{1}{2}\vec{\sigma} \cdot \vec{\hat{p}} - \lambda_r \end{pmatrix} \begin{pmatrix} \phi^r \\ \chi^r \end{pmatrix} = 0 \tag{4.216}$$

$$\begin{pmatrix} mc^2 - \epsilon_{pr} & c\vec{\sigma} \cdot \vec{p} \\ c\vec{\sigma} \cdot \vec{p} & -mc^2 - \epsilon_{pr} \end{pmatrix} \begin{pmatrix} \phi^r \\ \chi^r \end{pmatrix} = 0 \tag{4.217}$$

where

$$\phi^r = \begin{pmatrix} u_1^r(\vec{p}) \\ u_2^r(\vec{p}) \end{pmatrix} \tag{4.218}$$

$$\chi^r = \begin{pmatrix} u_3^r(\vec{p}) \\ u_4^r(\vec{p}) \end{pmatrix} \tag{4.219}$$

It follows from (4.217) that

$$u^r(\vec{p}) = \begin{pmatrix} \phi^r \\ \left(\frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon_p + mc^2}\right)\phi^r \end{pmatrix} \qquad \text{when} \quad r = 1, 2 \tag{4.220}$$

and

$$u^r(\vec{p}) = \begin{pmatrix} -\left(\frac{c\vec{\sigma}\cdot\vec{p}}{\epsilon_p + mc^2}\right)\chi^r \\ \chi^r \end{pmatrix} \qquad \text{when} \quad r = 3, 4 \tag{4.221}$$

Solving (4.216) completes the derivation.

Derivation of (4.160)

When $\epsilon + mc^2 - e\Phi > 0$, (4.157) yields

$$\chi = \frac{c\vec{\sigma} \cdot \vec{\Pi}}{\epsilon + mc^2 - e\Phi}\phi \tag{4.222}$$

and

$$\left(\frac{\left(c\vec{\sigma} \cdot \vec{\Pi}\right)^2}{\epsilon + mc^2 - e\Phi} + e\Phi \right)\phi = \left(\epsilon - mc^2\right)\phi \tag{4.223}$$

In the nonrelativistic limit,

$$\frac{c^2}{\epsilon + mc^2 - e\Phi} \rightarrow \frac{1}{2m} \tag{4.224}$$

and (4.222) and (4.223) simplify to

$$\chi = \frac{\vec{\sigma} \cdot \vec{\Pi}}{2mc}\phi \tag{4.225}$$

and

$$\left(\frac{\Pi^2}{2m} + e\Phi - \frac{e\hbar}{2mc}\vec{\sigma} \cdot \vec{B} \right)\phi = \epsilon'\phi \tag{4.226}$$

where

$$\epsilon' = \epsilon - mc^2 \tag{4.227}$$

and where we have used

$$\left(\vec{\sigma} \cdot \vec{\Pi}\right)^2 = \Pi^2 + i\vec{\sigma} \cdot \left(\vec{\Pi} \times \vec{\Pi}\right) \tag{4.228}$$

and

$$\vec{\Pi} \times \vec{\Pi} = \frac{i\hbar e}{c}\vec{B} \tag{4.229}$$

For motion in a constant magnetic field, the vector potential is

$$\vec{A} = \frac{1}{2}\left(\vec{B} \times \vec{X}\right) \tag{4.230}$$

from which

$$\vec{A} \cdot \vec{P} = \vec{P} \cdot \vec{A} = \frac{1}{2}\vec{B} \cdot \vec{L} \tag{4.231}$$

where

$$\vec{L} = \vec{X} \times \vec{P} \tag{4.232}$$

(4.160) then follows when terms proportional to e^2 are neglected.

Derivation of (4.167)

Nonrelativistic projection of an odd operator

A general state of the Dirac particle may be written as

$$| \psi >=| \psi_{large} > + | \psi_{small} > \tag{4.233}$$

where

$$| \psi_{large} >= \mathrm{B}_+ | \psi > \tag{4.234}$$

$$| \psi_{small} >= \mathrm{B}_- | \psi > \tag{4.235}$$

where

$$\mathrm{B}_+ = \frac{1}{2}(1+\beta) = \sum_{d=1,2} | \vec{x}d >< \vec{x}d | \tag{4.236}$$

$$\mathrm{B}_- = \frac{1}{2}(1-\beta) = \sum_{d=3,4} | \vec{x}d >< \vec{x}d | \tag{4.237}$$

and a general observable O of the Dirac particle may be written as

$$O = O_{even} + O_{odd} \tag{4.238}$$

where

$$O_{even} = \mathrm{B}_{+}O\mathrm{B}_{+} + \mathrm{B}_{-}O\mathrm{B}_{-} \tag{4.239}$$

$$O_{odd} = \mathrm{B}_{+}O\mathrm{B}_{-} + \mathrm{B}_{-}O\mathrm{B}_{+} \tag{4.240}$$

$\mathrm{B}_{\pm}$ are a complete set of orthogonal projection operators. O is said to be an even (odd) operator if O_{odd} (O_{even}) vanishes. An odd operator is linear in $\vec{\alpha}$.

Following (4.225), we write

$$\mid \psi_{small} >= \frac{\vec{\alpha} \cdot \left(\vec{P} - \frac{e}{c}\vec{A}\right)}{2mc} \mid \psi_{large} > \tag{4.241}$$

for a nonrelativistic Dirac particle in an electromagnetic field.

Approximation (4.241) yields

$$< \psi \mid O_{odd} \mid \psi >=< \psi_{large} \mid O_{nr} \mid \psi_{large} > \tag{4.242}$$

where

$$O_{nr} = \frac{1}{2mc}\left\{O_{odd}, \vec{\alpha} \cdot \left(\vec{P} - \frac{e}{c}\vec{A}\right)\right\} \tag{4.243}$$

(4.243) defines the nonrelativistic projection O_{nr} of an odd operator O_{odd}.

(4.167) follows from (4.243) when terms proportional to e^2 are neglected.

Chapter 5 SYSTEM OF n PARTICLES

In this chapter we consider a Lorentz invariant system consisting of n distinguishable particles with spin where n is any positive integer.

We follow the prescription given in Chapter 1 to describe a Lorentz invariant system: we specify the physical system by a set of fundamental dynamical variables and we construct Poincare generators in terms of these variables.

Fundamental dynamical variables for the system are given in Section 5.1 and a system of noninteracting particles is considered in Section 5.2. A system of interacting particles is considered in Section 5.3 and scattering equations for a system of interacting particles are given in Section 5.4. Some derivations are given in Section 5.5.

We assume that the particles are distinguishable for convenience. The methods of *QLB: Quantum Mechanics in Fock Space* can be used to describe a system of indistinguishable particles.

5.1 Fundamental dynamical variables

We consider the physical system to be a system of n particles with rest masses $m_1, m_2, ..., m_n$ and spins $s_1, s_2, ..., s_n$.

The Hilbert space $\mathcal{H}_n^{s_1 s_2 \cdots s_n}$ for the system is the direct product of n one-particle Hilbert spaces:

$$\boxed{\mathcal{H}_n^{s_1 s_2 \cdots s_n} = \mathcal{H}_1^{s_1} \otimes \mathcal{H}_1^{s_2} \otimes \cdots \otimes \mathcal{H}_1^{s_n} \qquad (5.1)}$$

$\mathcal{H}_1^{s_\alpha}$ denotes the Hilbert space for particle α and $\otimes$ denotes direct product.

The fundamental dynamical variables of the system are the Cartesian coordinates, momenta and spin of the individual particles

$$\boxed{X_1^j, P_1^j, S_1^j, X_2^j, P_2^j, S_2^j, ...X_n^j, P_n^j, S_n^j \qquad (5.2)}$$

where $j = 1, 2, 3$. These operators satisfy

$$\left[X_\alpha^j, X_\beta^k\right] = 0 \qquad (5.3)$$

$$\left[P_\alpha^j, P_\beta^k\right] = 0 \qquad (5.4)$$

$$\left[X_\alpha^j, P_\beta^k\right] = i\hbar\delta_{\alpha\beta}\delta_{jk} \qquad (5.5)$$

$$\left[S_\alpha^j, S_\beta^k\right] = i\hbar\delta_{\alpha\beta}\epsilon_{jkl}S_\alpha^l \qquad (5.6)$$

$$\vec{S}_\alpha \cdot \vec{S}_\alpha = s_\alpha(s_\alpha + 1)\hbar^2 \qquad (5.7)$$

$$\left[X_\alpha^j, S_\beta^k\right] = \left[P_\alpha^j, S_\beta^k\right] = 0 \qquad (5.8)$$

5.2 Noninteracting particles

In this section we consider a Lorentz invariant system of n noninteracting particles. It follows from (3.19) to (3.22) that the Poincare generators $H_0, \vec{P}_0, \vec{J}_0, \vec{K}_0$ for the system are

$$H_0 = \sum_{\alpha=1}^{n} H_\alpha \tag{5.9}$$

$$\vec{P}_0 = \sum_{\alpha=1}^{n} \vec{P}_\alpha \tag{5.10}$$

$$\vec{J}_0 = \sum_{\alpha=1}^{n} \vec{J}_\alpha \tag{5.11}$$

$$\vec{K}_0 = \sum_{\alpha=1}^{n} \vec{K}_\alpha \tag{5.12}$$

where

$$H_\alpha = \sqrt{P_\alpha^2 c^2 + m_\alpha^2 c^4} \tag{5.13}$$

$$\vec{J}_\alpha = \vec{X}_\alpha \times \vec{P}_\alpha + \vec{S}_\alpha \tag{5.14}$$

$$\vec{K}_\alpha = -\frac{1}{2c^2}\left(\vec{X}_\alpha H_\alpha + H_\alpha \vec{X}_\alpha\right) + \frac{\vec{S}_\alpha \times \vec{P}_\alpha}{H_\alpha + m_\alpha c^2} \tag{5.15}$$

Comments

1. **Notation**

We append the subscript 0 to noninteracting system operators to distinguish them from interacting system operators given in Section 5.3.

2. **Invariant mass**

The invariant mass M_0 (1.44) of the system is

$$\boxed{M_0 c^2 = \sqrt{H_0^2 - P_0^2 c^2} \qquad (5.16)}$$

That is

$$\boxed{H_0 = \sqrt{P_0^2 c^2 + M_0^2 c^4} \qquad (5.17)}$$

M_0 is not equal to the sum of the rest masses of the particles in the system.

3. **Centre of mass position and internal angular momentum**

The centre of mass position $\vec{\hat{X}}_0$ and internal angular momentum $\vec{\hat{S}}_0$ of the system are defined by (1.48) and (1.52), respectively. That is,

$$\vec{X}_0 = -\frac{c^2}{2}\left(\frac{1}{H_0}\vec{K}_0 + \vec{K}_0\frac{1}{H_0}\right) - \frac{c^2}{(H_0 + M_0c^2)H_0}\vec{P}_0 \times \vec{W}_0 \tag{5.18}$$

$$\vec{S}_0 = \frac{1}{M_0c}\left(\vec{W}_0 - \frac{c}{H_0 + Mc^2}W_0^0\vec{P}_0\right) \tag{5.19}$$

where W_0^μ is the Pauli-Lubanski four-vector (1.45):

$$W_0^0 = \vec{J}_0 \cdot \vec{P}_0 \tag{5.20}$$

$$\vec{W}_0 = \frac{1}{c}H_0\vec{J}_0 + c\vec{K}_0 \times \vec{P}_0 \tag{5.21}$$

$\vec{S}_0$ includes the spins of the individual particles and the orbital angular momenta of the particles about the centre of mass position of the system.

5.3 Interacting particles

In this section we consider a Lorentz invariant system of n interacting particles. The formalism developed in this section can be used, for example, to describe a system on interacting nucleons below the pion production threshold.

We must construct the Poincare generators $H, \vec{P}, \vec{J}, \vec{K}$ for the system in terms of (5.2). The coupling in the Poincare Algebra (1.1) to (1.9) requires that at least two of $H, \vec{P}, \vec{J}, \vec{K}$ not be equal to the noninteracting forms (5.9) to (5.12). Dirac (1949) discusses various possibilities for modifying the noninteracting generators; we consider the instant form of dynamics in which the Hamiltonian and booster change and the total momentum and total angular momentum remain unchanged:

$$H = \sqrt{P_0^2 c^2 + M^2 c^4} \tag{5.22}$$

$$\vec{P} = \vec{P}_0 \tag{5.23}$$

$$\vec{J} = \vec{J}_0 \tag{5.24}$$

$$\vec{K} = -\frac{1}{2c^2}\left(\vec{\tilde{X}}_0 H + H\vec{\tilde{X}}_0\right) + \frac{\vec{\tilde{S}}_0 \times \vec{P}_0}{H + Mc^2} \tag{5.25}$$

where

$$M = M_0 + V \tag{5.26}$$

where V satisfies

$$\left[V, \vec{P}_0\right] = \left[V, \vec{\tilde{X}}_0\right] = \left[V, \vec{\tilde{S}}_0\right] = 0 \tag{5.27}$$

Comments

1. **Invariant mass**

M is the invariant mass of the system; M is not equal to the sum of the rest masses of the particles in the system.

2. **Interaction potential; internal variables**

Interactions among the particles are specified entirely by the interaction potential V.

If $V = 0$, the system of particles is noninteracting.

(5.27) states that V must be a function of the internal variables for the system, that is, variables which commute with $\vec{P}_0, \vec{\hat{X}}_0, \vec{\hat{S}}_0$.

3. **Bakamjian-Thomas construction**

Bakamjian and Thomas (1953) give a method for construction of internal variables for a general n-particle system.

A readable account of the Bakamjian-Thomas construction is Kalyniak (1978).

4. **Nonrelativistic Hamiltonian**

It follows from the Galilei Algebra that the Hamiltonian H for a Galilei invariant system of n interacting particles has the form

$$\boxed{H = H_0 + V \qquad (5.28)}$$

where

$$\boxed{H_0 = \sum_{\alpha=1}^{n} \frac{P_\alpha^2}{2m_\alpha} = \frac{P_0^2}{2m} + \sum_{\alpha=1}^{n} \frac{P_\alpha'^2}{2m_\alpha} \qquad (5.29)}$$

where the nonrelativistic interaction potential V has the form

$$V = V\left(\vec{\xi}_1, \vec{\xi}_2, \cdots, \vec{\xi}_n\right) \tag{5.30}$$

where

$$\vec{\xi}_\alpha = \left\{\vec{X'}_\alpha, \vec{P'}_\alpha, \vec{S}_\alpha\right\} \tag{5.31}$$

where

$$\vec{X'}_\alpha = \vec{X}_\alpha - \vec{\hat{X}}_{nr} \tag{5.32}$$

$$\vec{P'}_\alpha = \vec{P}_\alpha - \frac{m_\alpha}{m}\vec{P}_0 \tag{5.33}$$

where

$$\vec{\hat{X}}_{nr} = \frac{1}{m}\sum_{\alpha=1}^{n} m_\alpha \vec{X}_\alpha \tag{5.34}$$

$$m = \sum_{\alpha=1}^{n} m_\alpha \tag{5.35}$$

$\vec{\hat{X}}_{nr}$ is the nonrelativistic centre of mass position of the system. $\vec{X'}_\alpha$ is the position of particle α with respect to the centre of mass position.

5. **Nonrelativistic internal variables**

The variables specified by (5.31) are internal variables for a nonrelativistic system. That is,

$$\left[X'^{j}_{\alpha}, P^{k}\right] = \left[X'^{j}_{\alpha}, \widehat{X}^{k}_{nr}\right] = 0 \qquad (5.36)$$

$$\left[P'^{j}_{\alpha}, P^{k}\right] = \left[P'^{j}_{\alpha}, \widehat{X}^{k}_{nr}\right] = 0 \qquad (5.37)$$

The nonrelativistic interaction potential (5.30) is a function of the internal variables for a nonrelativistic system.

6. **Cluster separability**

A requirement for a system of n interacting particles is that if the system is partitioned into two clusters that are separated by an infinite distance then the system must behave as two distinct systems which do not interact with each other. This is known as cluster separability (or the cluster decomposition principle).

For a nonrelativistic system this is accomplished by imposing suitable restrictions on the nonrelativistic interaction potential (5.30).

For a relativistic system this accomplished in the context of scattering theory by imposing suitable restrictions on the relativistic interaction potential V defined in (5.26).

5.4 Scattering theory

We give a very brief overview of scattering equations for a Lorentz invariant system of n interacting particles. Further details are in Monahan (1995); an introduction to the basic ideas of quantum scattering theory is given in *QLB: Scattering Theory*.

We assume that Poincare generators for the system are given by (5.22) to (5.25), that is, we assume in particular that the invariant mass M is of the form (5.26) and that the interaction potential V has been constructed subject to the restrictions (5.27). We assume further that requirements of cluster separability are satisfied.

Potentials and Green's operators

We assume that the interaction potential V has the form

$$V = \sum_{a=1}^{n_V} V_a \tag{5.38}$$

for some n_V and we define the mass operator M_a and potential V^a by

$$M_a = M_0 + V_a \tag{5.39}$$

$$M = M_a + V^a \tag{5.40}$$

Accordingly,

$$M = M_0 + V_a + V^a \tag{5.41}$$

and

$$V^a = \sum_{b \neq a} V_b \tag{5.42}$$

Comments

1. **Meaning of V_a, M_a and V^a**

   V_a is the interaction between particles in a subsystem (subsystem a) of the system.

   M_a is the invariant mass of the n-particle system when the only interactions in the system are those between the particles in subsystem a.

   V^a is the interaction between the particles in subsystem a and the particles in the rest of the n-particle system.

   Cluster separability means that $V^a \to 0$ when subsystem a is removed to infinite separation from the rest of the particles in the system.

2. **Extension to include $a = 0$**

   It is convenient to extend the values of a in (5.38), (5.40) and (5.41) to include $a = 0$ by defining

$$V_0 = 0 \tag{5.43}$$

in which case

$$V^0 = V \tag{5.44}$$

Accordingly, the decompositions (5.40) and (5.41) include (5.26) as a special case.

The decompositions (5.40) and (5.41) are the key to deriving the generalized Faddeev equations discussed in the next topic.

3. **Green's operators**

We define Green's operators

$$G_a(z) = \frac{1}{z - M_a} \tag{5.45}$$

$$G(z) = \frac{1}{z - M} \tag{5.46}$$

where z is a complex number.

$G_0(z)$ is the free-particle Green's operator; $G_a(z)$ is the Green's operator for the system when the only interactions in the system are those between the particles in subsystem a; $G(z)$ is the Green's operator for the system.

4. **Lippmann-Schwinger equations**

It follows from

$$G_a^{-1}(z) = G_0^{-1}(z) - V_a \tag{5.47}$$

and

$$G^{-1}(z) = G_a^{-1}(z) - V^a \tag{5.48}$$

that $G_a(z)$ and $G(z)$ satisfy the Lippmann-Schwinger equations

$$\boxed{\begin{aligned} G_a(z) &= G_0(z) + G_0(z)V_aG_a(z) = G_0(z) + G_a(z)V_aG_0(z) && (5.49) \\ G(z) &= G_a(z) + G_a(z)V^aG(z) = G_a(z) + G(z)V^aG_a(z) && (5.50) \end{aligned}}$$

Generalized Faddeev equations

We define T operators

$$\boxed{\begin{aligned} T_a(z) &= V_a + V_aG_a(z)V_a && (5.51) \\ T^{ba}(z) &= V^a + V^bG(z)V^a && (5.52) \end{aligned}}$$

Comments

1. **Physical significance of $T_a(z)$**

   The on-shell matrix elements of $T_a(z)$ are related to scattering cross sections for subsystem a.

2. **Physical significance of $T^{ba}(z)$**

   The on-shell matrix elements of $T^{ba}(z)$ are related to scattering cross sections for the n-particle system when subsystem a is in the initial state and subsystem b is in the final state.

3. **Kato-Birman invariance principle**

T operators for nonrelativistic scattering theory are defined in terms of the nonrelativistic Hamiltonian (5.28). The nonrelativistic T operator depends only on internal variables; there is no dependence on the total momentum $\vec{P}_0$ because the centre of mass motion may be separated out of the free Hamiltonian as per (5.29) and the nonrelativistic interaction potential depends only on internal variables.

For a relativistic system the centre of mass motion cannot be separated out of the free Hamiltonian (5.9) as per (5.29). The Kato-Birman invariance principle, however, states that T operators for relativistic systems can be expressed in terms of the mass operator as in (5.52); as in the nonrelativistic case, the relativistic T operators depend only on internal variables.

4. **Lippmann-Schwinger equation for $T_a(z)$**

We show in Section 5.5 that

$$\boxed{T_a(z) = V_a + V_a G_0(z) T_a(z) = V_a + V_a T_a(z) G_0(z) \qquad (5.53)}$$

$T_a(z)$ can be determined by solving the Lippmann-Schwinger equations (5.53); accordingly, $T_a(z)$ is a known operator.

5. **Lippmann-Schwinger equation for $T^{ba}(z)$**

We show in Section 5.5 that

$$T^{ba}(z) = V^a + V^b G_b(z) T^{ba}(z) \tag{5.54}$$

$T^{ba}(z)$ cannot in general be determined by solving the Lippmann-Schwinger equation (5.54) because there are δ-function singularities in the kernel which arise from intermediate states consisting of the interacting subsystem and free subsystems.

Indeed, it is clear that attempting to solve (5.54) is not the best strategy for determining $T^{ba}(z)$. Since $T_a(z)$ is known, it would appear advantageous to use $T_a(z)$ as input information in determining $T^{ba}(z)$. The generalized Faddeev equations of the next item employ this strategy.

6. **Generalized Faddeev equations**

We show in Section 5.5 that

$$T^{ba}(z) = V^a + \sum_{c \neq b} T_c(z) G_0(z) T^{ca}(z) \tag{5.55}$$

The generalized Faddeev equations (5.55) are a set of coupled integral equations for the operators $T^{ba}(z)$ whose input includes potentials and known T operators $T_c(z)$.

The mathematical attractiveness of (5.55) lies in the second term of the right side: there are no δ-function singularities in the kernel because of the restricted summation.

The Faddeev equations were originally derived to describe scattering in a nonrelativistic three-body system; (5.55), on the other hand, is appropriate for describing scattering in a relativistic system of n particles.

Three-body system

In this topic we illustrate the generalized Faddeev equations (5.55) by considering a relativistic three-body system.

We suppose that the three particles interact via two-body potentials, that is,

$$V = \frac{1}{2}\sum_{\alpha,\beta=1}^{3} V_{\alpha\beta} \tag{5.56}$$

where $V_{\alpha\alpha} = 0$ and $V_{\alpha\beta} = V_{\beta\alpha}$ and we define

$$V_a = V_{\beta\gamma} \tag{5.57}$$

for $a = 1, 2, 3$. V_a is the potential between particles β and γ and subsystem a consists of particles β and γ. Also,

$$V^a = V_{\alpha\beta} + V_{\alpha\gamma} \tag{5.58}$$

is the total potential for particle α.

We consider the special case of (5.55) when $a = b = 0$ and we write $T^{00}(z) = T(z)$. We show in Section 5.5 that

$$\boxed{T(z) = \sum_{a=1}^{3} T^a(z)} \tag{5.59}$$

where

$$\boxed{T^a(z) = T_a(z) + T_a(z)G_0(z)\sum_{b\neq a} T^b(z)} \tag{5.60}$$

Faddeev equations

(5.60) are a set of three coupled equations whose inputs are two-body T operators. The nonrelativistic version of (5.59) and (5.60) are the original Faddeev equations.

5.5 Some derivations

Derivation of (5.53)

It follows from (5.49) that

$$\begin{aligned} G_a(z)V_a &= [G_0(z) + G_0(z)V_aG_a(z)]V_a \\ &= G_0(z)[V_a + V_aG_a(z)V_a] = G_0(z)T_a(z) \end{aligned} \tag{5.61}$$

and

$$\begin{aligned} V_aG_a(z) &= V_a[G_0(z) + G_a(z)V_aG_0(z)] \\ &= [V_a + V_aG_a(z)V_a]G_0(z) = T_a(z)G_0(z) \end{aligned} \tag{5.62}$$

Derivation of (5.54)

It follows from (5.50) and (5.52) that

$$G(z)V^a = \Big[G_b(z) + G_b(z)V^b G(z)\Big]V^a$$

$$= G_b(z)\Big[V^a + V^b G(z)V^a\Big] = G_b(z)T^{ba}(z) \tag{5.63}$$

Derivation of (5.55)

It follows using (5.42), (5.63) and (5.62) that

$$V^b G(z)V^a = \sum_{c\neq b} V_c G(z)V^a = \sum_{c\neq b} V_c G_c(z)T^{ca}(z)$$

$$= \sum_{c\neq b} T_c(z)G_0(z)T^{ca}(z) \tag{5.64}$$

Derivation of (5.59) and (5.60)

It follows from (5.55) that

$$T(z) = V + \sum_{a=1}^{3} T_a(z)G_0(z)T^{a0}(z) \tag{5.65}$$

so, using (5.38), (5.59) holds where

$$T^a(z) = V_a + T_a(z)G_0(z)T^{a0}(z) \tag{5.66}$$

and

$$T^{a0}(z) = V + \sum_{b\neq a} T_b(z)G_0(z)T^{b0}(z)$$

$$= V_a + \sum_{b\neq a}\Big[V_b + T_b(z)G_0(z)T^{b0}(z)\Big] = V_a + \sum_{b\neq a} T^b(z) \tag{5.67}$$

Chapter 6 $\quad 2 \leftrightarrow 3$ PARTICLE SYSTEM

In this chapter we indicate how to extend the considerations of Chapter 5 to include particle creation and annihilation. The general mathematical formalism for handling particle creation and annihilation is given in *QLB: Quantum Mechanics in Fock Space*. In this chapter we consider the special case of a system of two particles where a third particle can be created and a system of three particles where one particle can be annihilated. We call this system the $2 \leftrightarrow 3$ particle system. A more complete description of the quantum mechanics of this system can be found in Monahan (1995).

The formalism given in this chapter can be used, for example, to describe nucleon-nucleon scattering and pion production and absorption reactions on two-nucleon systems above the threshold for single pion production and below the threshold for two-pion production. That is, it can be used to describe the reactions

$$N + N \leftrightarrow N + N \tag{6.1}$$

$$N + N \leftrightarrow N + N + \pi \tag{6.2}$$

$$N + N + \pi \leftrightarrow N + N + \pi \tag{6.3}$$

The formalism can be extended to describe a $2 \leftrightarrow 2$ particle system where the two two-particle systems are different. For example, it can be extended to describe the $\pi^- + p \rightarrow K^0 + \Lambda$ reaction.

The Hilbert space for the system is described in Section 6.1. an uncoupled interacting system is considered in Section 6.2; a coupled interacting system is considered in Section 6.3; and some derivations are given in Section 6.4.

6.1 Hilbert space, states and observables

We consider the physical system to be a system of two particles with rest masses m_1, m_2 and spins s_1, s_2 and of a third particle, with rest mass m_3 and spin s_3, which can be created and annihilated.

The Hilbert space $\mathcal{H}_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ for the system is the direct sum of a two-particle Hilbert space and a three-particle Hilbert space.

$$\boxed{\mathcal{H}_{2\leftrightarrow 3}^{s_1 s_2 s_3} = \mathcal{H}_{2}^{s_1 s_2} \oplus \mathcal{H}_{3}^{s_1 s_2 s_3}} \tag{6.4}$$

where $\mathcal{H}_{2}^{s_1 s_2}$ and $\mathcal{H}_{3}^{s_1 s_2 s_3}$ are given by (5.1) and $\oplus$ denotes the direct sum. Fundamental dynamical variables for the system are given by (5.2).

A state $\mid \psi >$ of the system is represented by the column matrix

$$\boxed{\mid \psi > = \begin{pmatrix} \mid \psi \underset{2}{>} \\ \mid \psi \underset{3}{>} \end{pmatrix}} \tag{6.5}$$

where $\mid \psi \underset{2}{>}$ is a vector in $\mathcal{H}_{2}^{s_1 s_2}$ and $\mid \psi \underset{3}{>}$ is a vector in $\mathcal{H}_{3}^{s_1 s_2 s_3}$. $\mid \psi \underset{2}{>}$ and $\mid \psi \underset{3}{>}$ are, respectively, the two- and three-particle components of $\mid \psi >$. $\mid \psi >$ has unit norm

$$\boxed{< \psi \mid \psi > = \underset{2}{<} \psi \mid \psi \underset{2}{>} + \underset{3}{<} \psi \mid \psi \underset{3}{>} = 1} \tag{6.6}$$

where $\underset{m}{<} \psi \mid \psi \underset{m}{>}$ is the probability that $\mid \psi >$ is an m-particle state.

An operator A in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ is represented by the 2×2 matrix

$$A = \begin{pmatrix} A^{\heartsuit} & A^{\flat} \\ A^{\sharp} & A^{\clubsuit} \end{pmatrix} \tag{6.7}$$

where $A^{\heartsuit}$ is an operator in $\maltese_2^{s_1 s_2}$ and $A^{\clubsuit}$ is an operator in $\maltese_3^{s_1 s_2 s_3}$. The operators $A^{\flat}$ and $A^{\sharp}$ link $\maltese_2^{s_1 s_2}$ and $\maltese_3^{s_1 s_2 s_3}$.

In particular, the number operator for the system is

$$N = \begin{pmatrix} 2 & 0 \\ 0 & 3 \end{pmatrix} \tag{6.8}$$

Classification of operators

An operator in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ can be expressed in terms of the four matrices

$$F^{\flat} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \tag{6.9}$$

$$F^{\sharp} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \tag{6.10}$$

$$F^{\flat} F^{\sharp} = \mathcal{P}^{\heartsuit} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \tag{6.11}$$

$$F^{\sharp} F^{\flat} = \mathcal{P}^{\clubsuit} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \tag{6.12}$$

which satisfy

$$\left(F^\flat\right)^2 = \left(F^\sharp\right)^2 = 0 \tag{6.13}$$

$$F^\flat F^\sharp + F^\sharp F^\flat = 1 \tag{6.14}$$

That is,

$$A = A^\heartsuit \mathcal{P}^\heartsuit + A^\flat F^\flat + A^\sharp F^\sharp + A^\clubsuit \mathcal{P}^\clubsuit \tag{6.15}$$

$\mathcal{P}^\heartsuit$ and $\mathcal{P}^\clubsuit$ are a complete set of orthogonal projection operators in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$; $\mathcal{P}^\heartsuit$ and $\mathcal{P}^\clubsuit$ project onto $\maltese_2^{s_1 s_2}$ and $\maltese_3^{s_1 s_2 s_3}$, respectively.

Comments

1. **Two-body operator**

An operator A in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ is a two-body operator if

$$A = \mathcal{P}^\heartsuit A \mathcal{P}^\heartsuit = A^\heartsuit \mathcal{P}^\heartsuit = \begin{pmatrix} A^\heartsuit & 0 \\ 0 & 0 \end{pmatrix} \tag{6.16}$$

2. **Three-body operator**

An operator A in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ is a three-body operator if

$$A = \mathcal{P}^{\clubsuit} A \mathcal{P}^{\clubsuit} = A^{\clubsuit} \mathcal{P}^{\clubsuit} = \begin{pmatrix} 0 & 0 \\ 0 & A^{\clubsuit} \end{pmatrix} \tag{6.17}$$

3. **Direct sum operator**

An operator A in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ is a direct sum operator if

$$A = \mathcal{P}^{\heartsuit} A \mathcal{P}^{\heartsuit} + \mathcal{P}^{\clubsuit} A \mathcal{P}^{\clubsuit} = A^{\heartsuit} \mathcal{P}^{\heartsuit} + A^{\clubsuit} \mathcal{P}^{\clubsuit} = \begin{pmatrix} A^{\heartsuit} & 0 \\ 0 & A^{\clubsuit} \end{pmatrix} \tag{6.18}$$

4. **Coupling operator**

An operator A in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ is a coupling operator if it has the form (6.7) or (6.15) where

$$A^{\flat} \neq 0 \qquad \text{or} \qquad A^{\sharp} \neq 0 \tag{6.19}$$

6.2 Uncoupled interacting system

In this section we describe a Lorentz invariant system of interacting particles in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$ when there is no particle creation or annihilation.

The Poincare generators $H, \vec{P}, \vec{J}, \vec{K}$ are given by (5.22) to (5.25) and the invariant mass M is given by (5.26) where $\vec{P}_0, \vec{J}_0, \vec{\vec{X}}_0, \vec{\vec{S}}_0, M_0, V$ are direct sum operators and where

$$\boxed{\left[V^{\spadesuit}, \vec{P}_0^{\spadesuit}\right] = \left[V^{\spadesuit}, \vec{\vec{X}}_0^{\spadesuit}\right] = \left[V^{\spadesuit}, \vec{\vec{S}}_0^{\spadesuit}\right] = 0 \qquad (6.20)}$$

where $\spadesuit = \heartsuit, \clubsuit$.

The interaction potential V has the form

$$\boxed{V = \sum_{a=0}^{4} V_a \qquad (6.21)}$$

where

$$V_0 = 0 \qquad (6.22)$$

$$V_4 = \begin{pmatrix} V_4^{\heartsuit} & 0 \\ 0 & 0 \end{pmatrix} \qquad (6.23)$$

$$V_a = \begin{pmatrix} 0 & 0 \\ 0 & V_a^{\clubsuit} \end{pmatrix} \qquad \text{if} \quad a = 1, 2, 3 \qquad (6.24)$$

That is,

$$
V = \begin{pmatrix} V_4^{\heartsuit} & 0 \\ 0 & \sum_{a=1}^{3} V_a^{\clubsuit} \end{pmatrix} \tag{6.25}
$$

$V_4^{\heartsuit}$ is the potential between particles 1 and 2 and, as in (5.57), $V_a^{\clubsuit}$ is the potential between particles β and γ.

The mass operator M_a and potential V^a are defined by

$$
M_a = M_0 + V_a \tag{6.26}
$$

$$
M = M_a + V^a \tag{6.27}
$$

Accordingly,

$$
M = M_0 + V_a + V^a \tag{6.28}
$$

and

$$
V^a = \sum_{b \neq a} V_b \tag{6.29}
$$

That is,

$$V^0 = V \tag{6.30}$$

$$V^4 = \begin{pmatrix} 0 & 0 \\ 0 & \sum_{\alpha=1}^{3} V_\alpha^{\clubsuit} \end{pmatrix} \tag{6.31}$$

$$V^a = \begin{pmatrix} V_4^{\heartsuit} & 0 \\ 0 & V_{\alpha\beta}^{\clubsuit} + V_{\alpha\gamma}^{\clubsuit} \end{pmatrix} \qquad \text{if} \quad a = 1, 2, 3 \tag{6.32}$$

Scattering equations

As in Chapter 5, scattering theory is expressed in terms of T operators

$$T_a(z) = V_a + V_a G_a(z) V_a \tag{6.33}$$

$$T^{ba}(z) = V^a + V^b G(z) V^a \tag{6.34}$$

where

$$G_a(z) = \frac{1}{z - M_a} \tag{6.35}$$

$$G(z) = \frac{1}{z - M} \tag{6.36}$$

Comments

1. **Direct sum operators**

$G_a(z)$ and $G(z)$ are direct sum operators.

Accordingly, $T_a(z)$ and $T^{ba}(z)$ are direct sum operators.

2. **Lippmann-Schwinger equations**

$T_4(z)$ is a two-body operator and $T_a(z)$ $(a = 1, 2, 3)$ are three-body operators.

$T_4(z)^\heartsuit$ and $T_a(z)^\clubsuit$ obey the Lippmann-Schwinger equations

$$\boxed{\begin{aligned} T_4(z)^\heartsuit &= V_4^\heartsuit + V_4^\heartsuit G_0(z)^\heartsuit T_4(z)^\heartsuit = V_4^\heartsuit + V_4^\heartsuit T_4(z)^\heartsuit G_0(z)^\heartsuit \qquad (6.37) \\ T_a(z)^\clubsuit &= V_a^\clubsuit + V_a^\clubsuit G_0(z)^\clubsuit T_a(z)^\clubsuit = V_a^\clubsuit + V_a^\clubsuit T_a(z)^\clubsuit G_0(z)^\clubsuit \qquad (6.38) \end{aligned}}$$

3. **Generalized Faddeev equations**

$T^{ba}(z)$ is a direct sum operator. When $a, b = 1, 2, 3$,

$$\boxed{T^{ba}(z)^\heartsuit = T_4(z)^\heartsuit \qquad (6.39)}$$

that is, $T^{ba}(z)^\heartsuit$ obeys the two-body Lippmann-Schwinger equation (6.37);

and $T^{ba}(z)^{\clubsuit}$ obeys the generalized three-body Faddeev equations

$$T^{ba}(z)^{\clubsuit} = V^{a\clubsuit} + \sum_{c \neq b=1}^{3} T_c(z)^{\clubsuit} G_0(z)^{\clubsuit} T^{ca}(z)^{\clubsuit} \tag{6.40}$$

6.3 Coupled interacting system

In this section we describe a Lorentz invariant system of interacting particles in $\maltese^{s_1 s_2 s_3}_{2-3}$ where particle 3 can be created or annihilated.

Particle creation and annihilation is accomplished and Lorentz invariance is maintained by modifying the interaction potential (6.21) to

$$V = \sum_{a=0}^{5} V_a \tag{6.41}$$

where V_0, V_1, V_2, V_3, V_4 are given by (6.22) to (6.24) and

$$V_5 = \begin{pmatrix} 0 & V_5^{\flat} \\ V_5^{\sharp} & 0 \end{pmatrix} \tag{6.42}$$

where

$$\left[V_5^{\natural}, \vec{P}_0^{\spadesuit}\right] = \left[V_5^{\natural}, \vec{\widehat{X}}_0^{\spadesuit}\right] = \left[V_5^{\natural}, \vec{\widehat{S}}_0^{\spadesuit}\right] = 0 \tag{6.43}$$

where $\natural = \flat, \sharp$ and $\spadesuit = \heartsuit, \clubsuit$. That is,

$$V = \begin{pmatrix} V_4^{\heartsuit} & V_5^{\flat} \\ V_5^{\sharp} & \sum_{a=1}^{3} V_a^{\clubsuit} \end{pmatrix} \tag{6.44}$$

It follows from (6.44) that

$$[H, N] \neq 0 \tag{6.45}$$

Comments

1. **Coupling operators**

V_5 is a coupling operator. More specifically,

$$< \phi \mid V_5 \mid \psi > = \underset{2}{<} \phi \mid V_5^{\flat} \mid \psi \underset{3}{>} + \underset{3}{<} \phi \mid V_5^{\sharp} \mid \psi \underset{2}{>} \tag{6.46}$$

Accordingly, V and H are coupling operators.

2. **Particle creation and annihilation**

It follows from (6.45) that a state of particles 1 and 2 can evolve in time to a state of particles 1, 2 and 3 and a state of particles 1, 2 and 3 can evolve in time to a state of particles 1 and 2. That is, particle 3 can be created or annihilated.

More specifically, if at time zero

$$\boxed{\mid \psi >= \begin{pmatrix} \mid \psi \underset{2}{>} \\ 0 \end{pmatrix} \qquad (6.47)}$$

then at time t

$$\boxed{\mid \psi(t) >= U(t) \mid \psi >= \begin{pmatrix} \mid \psi(t) \underset{2}{>} \\ \mid \psi(t) \underset{3}{>} \end{pmatrix} \qquad (6.48)}$$

Similarly, a three-particle state prepared at time zero can evolve to a two-particle state at a later time.

Generalized Faddeev equations

Scattering theory is expressed in terms of T operators (6.33) and (6.34) where $a, b = 0, 1, 2, 3, 4, 5$. $T^{ba}(z)$ involves the potentials

$$V^0 = V \tag{6.49}$$

$$V^4 = \begin{pmatrix} 0 & V_5^\flat \\ V_5^\sharp & \sum_{\alpha=1}^{3} V_\alpha^\clubsuit \end{pmatrix} \tag{6.50}$$

$$V^a = \begin{pmatrix} V_4^\heartsuit & V_5^\flat \\ V_5^\sharp & V_{\alpha\beta}^\clubsuit + V_{\alpha\gamma}^\clubsuit \end{pmatrix} \quad \text{if} \quad a = 1, 2, 3 \tag{6.51}$$

$$V^5 = \begin{pmatrix} V_4^\heartsuit & 0 \\ 0 & \sum_{\alpha=1}^{3} V_\alpha^\clubsuit \end{pmatrix} \tag{6.52}$$

Comments

1. **Direct sum and coupling operators**

$G_a(z)$ $(a = 0, 1, 2, 3, 4)$ are direct sum operators and $G_5(z)$ and $G(z)$ are coupling operators.

Accordingly, $T_a(z)$ $(a = 1, 2, 3, 4)$ are direct sum operators and $T_5(z)$ and $T^{ba}(z)$ are coupling operators.

2. **Lippmann-Schwinger equations for $T_a(z)$ $(a = 1, 2, 3, 4)$**

$T_4(z)$ is a two-body operator and $T_a(z)$ $(a = 1, 2, 3)$ are three-body operators.

$T_4(z)^\heartsuit$ and $T_a(z)^\clubsuit$ obey the Lippmann-Schwinger equations

$$T_4(z)^{\heartsuit} = V_4^{\heartsuit} + V_4^{\heartsuit} G_0(z)^{\heartsuit} T_4(z)^{\heartsuit} = V_4^{\heartsuit} + V_4^{\heartsuit} T_4(z)^{\heartsuit} G_0(z)^{\heartsuit} \tag{6.53}$$

$$T_a(z)^{\clubsuit} = V_a^{\clubsuit} + V_a^{\clubsuit} G_0(z)^{\clubsuit} T_a(z)^{\clubsuit} = V_a^{\clubsuit} + V_a^{\clubsuit} T_a(z)^{\clubsuit} G_0(z)^{\clubsuit} \tag{6.54}$$

3. **Lippmann-Schwinger equation for $T_5(z)$**

We show in Section 6.4 that

$$T_5(z) = \begin{pmatrix} T_5(z)^{\heartsuit} & T_5(z)^{\flat} \\ T_5(z)^{\sharp} & T_5(z)^{\clubsuit} \end{pmatrix} \tag{6.55}$$

where $T_5(z)^{\heartsuit}$ and $T_5(z)^{\clubsuit}$ obey the Lippmann-Schwinger equations

$$T_5(z)^{\heartsuit} = \mathcal{U}^{\heartsuit}(z) + \mathcal{U}^{\heartsuit}(z) G_0(z)^{\heartsuit} T_5(z)^{\heartsuit} \tag{6.56}$$

$$T_5(z)^{\clubsuit} = \mathcal{U}^{\clubsuit}(z) + \mathcal{U}^{\clubsuit}(z) G_0(z)^{\clubsuit} T_5(z)^{\clubsuit} \tag{6.57}$$

where

$$\begin{aligned} \mathcal{U}(z) &= V_5 G_0(z) V_5 \\ &= \begin{pmatrix} V_5^{\flat} G_0(z)^{\clubsuit} V_5^{\sharp} & 0 \\ 0 & V_5^{\sharp} G_0(z)^{\heartsuit} V_5^{\flat} \end{pmatrix} = \begin{pmatrix} \mathcal{U}(z)^{\heartsuit} & 0 \\ 0 & \mathcal{U}(z)^{\clubsuit} \end{pmatrix} \end{aligned} \tag{6.58}$$

and where

$$T_5(z)^\sharp = V_5^\sharp + V_5^\sharp G_0(z)^\heartsuit T_5(z)^\heartsuit \tag{6.59}$$

$$T_5(z)^\flat = V_5^\flat + V_5^\flat G_0(z)^\clubsuit T_5(z)^\clubsuit \tag{6.60}$$

4. **Potential $\mathcal{U}(z)$**

The direct sum operator $\mathcal{U}(z)$ is an effective potential in $\maltese_{2\leftrightarrow 3}^{s_1 s_2 s_3}$.

$\mathcal{U}^\heartsuit(z)$ is an effective two-body potential which maps $\maltese_2^{s_1 s_2}$ to itself via $\maltese_3^{s_1 s_2 s_3}$; it describes a process where a two-body system emits and reabsorbs a third particle.

$\mathcal{U}^\clubsuit(z)$ is an effective three-body potential which maps $\maltese_3^{s_1 s_2 s_3}$ to itself via $\maltese_2^{s_1 s_2}$; it describes a process where a three-body system absorbs and re-emits a particle.

5. **Generalized Faddeev equations**

It follows as in Chapter 5 that

$$T^{ba}(z) = V^a + \sum_{c \neq b} T_c(z) G_0(z) T^{ca}(z) \tag{6.61}$$

The generalized Faddeev equations (6.61) are a set of coupled integral equations for the operators $T^{ba}(z)$ which are appropriate for describing scattering in the relativistic $2 \leftrightarrow 3$ system.

The equations for $T^{ba}(z)$ $(a, b = 0, 1, 2, 3, 4)$ involve V_5 and $T_5(z)$ and therefore include a production contribution $2 \rightarrow 3 \rightarrow 2$ to the two-body process $2 \rightarrow 2$ and an annihilation contribution $3 \rightarrow 2 \rightarrow 3$ to the three-body process $3 \rightarrow 3$.

6.4 Some derivations

Derivation of (6.56) to (6.60)

It follows from the Lippmann-Schwinger equation

$$T_5(z) = V_5 + V_5 G_0(z) T_5(z) \tag{6.62}$$

that (6.55) holds where

$$T_5(z)^{\heartsuit} = V_5^{\flat} G_0(z)^{\clubsuit} T_5(z)^{\sharp} \tag{6.63}$$

$$T_5(z)^{\sharp} = V_5^{\sharp} + V_5^{\sharp} G_0(z)^{\heartsuit} T_5(z)^{\heartsuit} \tag{6.64}$$

$$T_5(z)^{\flat} = V_5^{\flat} + V_5^{\flat} G_0(z)^{\clubsuit} T_5(z)^{\clubsuit} \tag{6.65}$$

$$T_5(z)^{\clubsuit} = V_5^{\sharp} G_0(z)^{\heartsuit} T_5(z)^{\flat} \tag{6.66}$$

Substitution of (6.64) into (6.63) and (6.65) into (6.66) yields (6.56) and (6.57), respectively.

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