QUANTUM LEAPS AND BOUNDS

Relativistic Quantum Mechanics

Malcolm McMillan

Department of Physics

University of British Columbia

Contents

Preface

The six volumes of notes *Quantum Leaps and Bounds (QLB)* form the basis of the introductory graduate quantum mechanics course I have given in the Department of Physics at the University of British Columbia at various times since 1973.

The six volumes of *QLB* are

- *Introductory Topics*: a collection of miscellaneous topics in introductory quantum mechanics
- *Scattering Theory*: an introduction to the basic ideas of quantum scattering theory by considering the scattering of a relativistic spinless particle from a fixed target
- *Quantum Mechanics in Fock Space*: an introduction to the second-quantization description of nonrelativistic many-body systems
- *Relativistic Quantum Mechanics*: an introduction to incorporating special relativity in quantum mechanics
- *Some Lorentz Invariant Systems*: some examples of systems incorporating special relativity in quantum mechanics
- *Relativistic Quantum Field Theory*: an elementary introduction to the relativistic quantum field theory of spinless bosons, spin $\frac{1}{2}$ fermions and antifermions and to quantum electrodynamics, the relativistic quantum field theory of electrons, positrons and photons

QLB assumes no familiarity with relativistic quantum mechanics. It does assume that students have taken undergraduate courses in nonrelativistic quantum mechanics which include discussion of the nonrelativistic Schrodinger equation and the solutions of some standard problems (e.g., the one-dimensional harmonic oscillator and the hydrogen atom) and perturbation theory and other approximation methods.

QLB assumes also that students will take other graduate courses in condensed matter physics, nuclear and particle physics and relativistic quantum field theory. Accordingly, our purpose in *QLB* is to introduce some basic ideas and formalism and thereby give students sufficient background to read the many excellent texts on these subjects.

I am happy to have this opportunity to thank my friends and colleagues R. Barrie, B. Bergersen, M. Bloom, J. Feldman, D.H. Hearn, W.W. Hsieh, R.I.G. Hughes, F.A. Kaempffer, P.A. Kalyniak, R.H. Landau, E.L. Lomon, A.H. Monahan, W. Opechowski, M.H.L. Pryce, A. Raskin, P. Rastall, L. Rosen, L. Sobrino, F. Tabakin, A.W. Thomas, E.W. Vogt and G.M. Volkoff for sharing their knowledge of quantum mechanics with me.

I also thank my wife, Henrietta, for suggesting the title for these volumes of notes. Quite correctly, she found my working title *Elements of Intermediate Quantum Mechanics* a bore.

RELATIVISTIC QUANTUM MECHANICS

Chapter 1 INTRODUCTORY REMARKS

This volume of *QLB* gives a introduction to incorporating special relativity in quantum mechanics. That is, described here is a theory which includes Einstein's

$$E = mc^2 \tag{1.1}$$

with Heisenberg's

$$\Delta x \Delta p \geq \hbar/2 \tag{1.2}$$

How does relativistic quantum mechanics differ from nonrelativistic quantum mechanics? That is, what new feature does special relativity bring to quantum mechanics? The answer lies in (1.1), the possibility of converting energy to mass and *vice versa.*

In nonrelativistic quantum mechanics one deals with physical systems where the total mass is fixed and invariate for all time. One solves the one-body problem, the two-body problem, and so on. And one says, for example, that the hydrogen atom consists of a proton and an electron and the deuteron consists of a proton and a neutron. These are nonrelativistic statements. These composite particles (and others) consist of these two-body configurations to be sure, but they also consist of other multiparticle configurations. That is, there is a nonzero probability that the state of the hydrogen atom has components corresponding to a proton and an electron and also, for example, to a proton and an electron and any number of photons, electron-positron pairs and pions. The state of the deuteron has similar components as do the states of all composite particles. Relativistic quantum mechanics incorporates these effects.

Relativistic effects are numerically small at the atomic and molecular levels. One can understand all of biology and chemistry, and much of physics, without incorporating special relativity into quantum mechanics.

For example, nonrelativistic quantum mechanics predicts that the lowest energy state of the free hydrogen atom is the $1S_{1/2}$ state with an energy of —13.58 eV, and that the $2S_{1/2}$, $2P_{1/2}$, $2P_{3/2}$ states are degenerate with an energy of —3.40 eV. This is experimentally true at that level of accuracy. Measurement of transition energies at the μeV level using high precision laser spectroscopy, however, shows that these states are not degenerate in energy: the $2P_{3/2}$ and $2S_{1/2}$ levels lie 42.2μeV and 4.38μeV, respectively, above the $2P_{1/2}$ level. The splitting of the $2P_{1/2}$ and $2S_{1/2}$ levels is called the Lamb Shift. It was discovered in 1947 by W.E. Lamb and R.C. Retherford.

The Dirac equation for the hydrogen atom, which was invented by P.A.M. Dirac in 1928 (and is discussed in *QLB: Some Lorentz Invariant Systems*) incorporates special relativity into quantum mechanics without particle creation and annihilation. It gives an improvement over nonrelativistic quantum mechanics for the fine structure of the energy levels but it does not predict the Lamb Shift. It predicts that the $2S_{1/2}$ and $2P_{1/2}$ states are degenerate and that the $2P_{3/2}$ state has a higher energy by 45.2μeV.

Quantum electrodynamics, the relativistic quantum field theory of electrons and photons, which was invented in the 1930's and refined by R.P Feynman, J. Schwinger and S. Tomonaga in the 1940's (and is discussed briefly in *QLB: Relativistic Quantum Field Theory*), incorporates special relativity into quantum mechanics including particle creation and annihilation. It yields the Lamb Shift and gives perfect agreement with all electron-positron-photon experiments performed to date.

Relativistic effects are not small at the subatomic level. For example, it is clearly essential to include particle creation and annihilation effects when one tries to interpret experimental data for the reactions

$$p + p \rightarrow d + \pi^+ \tag{1.3}$$

and

$$p + p \rightarrow p + p + K^+ + K^- \tag{1.4}$$

These are the basic reactions for the production of positive pions at the TRIUMF accelator and for the production of positive and negative kaon beams in proton-proton collisions.

Quite apart from the largeness or smallness of relativistic effects, the study of relativistic quantum mechanics forces one to consider what are the fundamental entities in terms of which one describes the physical world. Searching for these fundamental entities remains a strong human quest. The reason perhaps lies in the statement[1]

> "Once we figure out how the dice are made, we may be able to figure out who is throwing them."

The basic equations of relativistic quantum mechanics are a set of commutation relations called the Poincare Algebra. We present the Poincare Algebra in Chapter 2.

The basic equations of nonrelativistic quantum mechanics are a set of commutation relations called the Galilei Algebra. We present the Galilei Algebra in Chapter 2. The Galilei Algebra is a special case of the Poincare Algebra.

Lorentz invariance of a physical system is defined in Chapter 3; the Poincare Algebra is derived in Chapter 4; space inversion and time reversal are discussed in Chapter 5; and the centre of mass position, centre of mass velocity and internal angular momentum of a Lorentz invariant system are discussed in Chapter 6.

The Appendix gives some matrices which arise in previous chapters. The volume concludes with lists of selected reference books, journal articles and theses.

---

[1] Attributed to Graeme Ross, Fredericton, New Brunswick, by Stephen Strauss, *Globe and Mail*, 1990.

Chapter 2 INTRODUCTION TO THE POINCARE ALGEBRA

In order to describe a Lorentz invariant physical system using quantum mechanics it is necessary to determine the Poincare generators of the system in terms of the fundamental dynamical variables of the system. In this chapter we present and comment on the the Poincare generators and the Poincare Algebra. Derivations and some definitions are given later: a Lorentz invariant physical system and Poincare transformations are defined and discussed in detail in Chapter 3 and the Poincare Algebra is derived in Chapter 4.

The Poincare generators are a set of ten Hermitian operators: the Hamiltonian and the three components of the total momentum, total angular momentum and Lorentz booster for the system. As discussed in Chapter 3 these operators generate time translations, spatial displacements, rotations and Lorentz boosts. The Poincare Algebra is a set of commutation relations satisfied by the Poincare generators; these commutation relations are the basic equations of relativistic quantum mechanics.

We also present and comment on the Galilei generators and Galilei Algebra which are appropriate for describing a Galilei invariant physical system. The Galilei generators are the Hamiltonian and the three components of the total momentum, total angular momentum and Galilei booster for the system. These operators generate time translations, spatial displacements, rotations and Galilei boosts. The Galilei Algebra is a set of commutation relations satisfied by the Galilei generators; these commutation relations are the nonrelativistic approximation of the Poincare Algebra and are the basic equations of nonrelativistic quantum mechanics. Nonrelativistic quantum mechanics is arrived at as a nonrelativistic approximation to relativistic quantum mechanics.

2.1 Poincare Algebra

In order to describe a physical system which is Lorentz invariant one must construct from the fundamental dynamical variables for the system ten Hermitian operators H, P^j, J^j, K^j (where $j = 1, 2, 3$) satisfying

$$\left[P^j, P^k\right] = 0 \tag{2.1}$$

$$\left[P^j, H\right] = 0 \tag{2.2}$$

$$\left[J^j, P^k\right] = i\hbar\epsilon_{jkl}P^l \tag{2.3}$$

$$\left[J^j, H\right] = 0 \tag{2.4}$$

$$\left[J^j, J^k\right] = i\hbar\epsilon_{jkl}J^l \tag{2.5}$$

$$\left[K^j, P^k\right] = -i\hbar\delta_{jk}H/c^2 \tag{2.6}$$

$$\left[K^j, H\right] = -i\hbar P^j \tag{2.7}$$

$$\left[K^j, J^k\right] = i\hbar\epsilon_{jkl}K^l \tag{2.8}$$

$$\left[K^j, K^k\right] = -i\hbar\epsilon_{jkl}J^l/c^2 \tag{2.9}$$

where $\hbar = h/2\pi$, h is Planck's constant, c is the speed of light, δ_{jk} is the Kronecker delta symbol and ϵ_{jkl} is the Levi-Civita permutation symbol.

2.2 Comments on the Poincare Algebra

1. **Lorentz invariant physical system**

A Lorentz invariant physical system is defined and discussed in some detail in Chapter 3.

Examples of Lorentz invariant physical systems are given in *QLB: Some Lorentz Invariant Systems.*

2. **Poincare Algebra: the basic equations of physics**

(2.1) to (2.9) are the Poincare Algebra. The Poincare Algebra is the Lie Algebra for the Poincare group.

The appearance of Planck's constant h and the speed of light c in (2.6) and (2.9) indicates explicitly that the Poincare Algebra involves quantum mechanics and special relativity.

The Poincare Algebra was first derived in the 1930's. The central role it plays in relativistic quantum mechanics is emphasized in Dirac (1949).

The corresponding equations for a Galilei invariant (or nonrelativistic) physical system, the Galilei Algebra, are given in item 8.

(2.1) to (2.9) are the basic equations of relativistic quantum mechanics. Indeed, since nonrelativistic quantum mechanics and classical mechanics are approximations to relativistic quantum mechanics, (2.1) to (2.9) are the basic equations of physics.

3. **Poincare generators**

The ten Hermitian operators H, P^j, J^j, K^j are the Poincare generators for the physical system.

H is the Hamiltonian for the system.

P^j, J^j, K^j are the jth component of the total momentum, total angular momentum and Lorentz booster, respectively, for the system.

As discussed in Chapter 3, these operators generate time translations, spatial displacements, rotations and Lorentz boosts.

4. Symmetry in the Poincare Algebra

(2.1) to (2.9) are invariant under the replacement

$$H, K^j \rightarrow -H, -K^j \tag{2.10}$$

With one exception, the Hamiltonians in the examples in *QLB: Some Lorentz Invariant Systems* are chosen to have positive spectral values.

The Hamiltonian for the Dirac particle discussed in *QLB: Some Lorentz Invariant Systems* is the one exception: it has both positive and negative spectral values. The negative energy states of the Dirac particle have no physical interpretation in a one-particle theory. We outline in *QLB: Some Lorentz Invariant Systems* how Dirac's brilliant interpretation of these states in 1930 predicted the existence of antiparticles and led to the invention of relativistic quantum field theory.

5. Constants of the motion

(2.2) and (2.4) show that all components of the total momentum and the total angular momentum are constants of the motion. It is often convenient to describe a system using eigenkets of the total momentum and eigenvectors of the total angular momentum.

(2.7) shows that none of the components of the Lorentz booster are constants of the motion. It is generally not convenient to describe a system using the eigenkets of the Lorentz boosters.

6. **Equations (2.1) to (2.5): the equations not involving K^j**

These equations are derived without involving Lorentz boosts. They are the same in nonrelativistic and relativistic quantum mechanics. (See item 8.)

7. **Equations (2.6) to (2.9): the equations involving K^j**

These equations involve the speed of light c.

The equations are in two pairs: one pair couples H and P^j and the other couples J^j and K^j.

We show in Chapter 3 that the coupling of H and P^j yields a mixing of energy and momentum under a Lorentz boost. This mixing is familiar from classical mechanics.

We show in Chapter 6 that the coupling of J^j and K^j yields a Wigner rotation of internal angular momentum under a Lorentz boost.

8. **The nonrelativistic limit: the Galilei Algebra**

The Galilei Algebra is a set of commution relations appropriate for describing a Galilei invariant physical system. The Galilei Algebra involves the Galilei generators which are the Hamiltonian and the three components of the total momentum, total angular momentum and Galilei booster for the system. As discussed in Chapter 3 these operators generate time translations, spatial displacements, rotations and Galilei boosts.

Since nonrelativistic quantum mechanics is the special case of relativistic quantum mechanics corresponding to taking the speed of light c to be infinite, the Galilei Algebra is identical to the Poincare Algebra except for (2.6) and (2.9). These are the only equations in the Poincare Algebra which involve c.

We show in Chapter 3 that the Galilei Algebra differs from the Poincare Algebra only in having (2.6) and (2.9) replaced by

$$\left[K^j, P^k\right] = -i\hbar m \delta_{jk} \quad (2.11)$$

$$\left[K^j, K^k\right] = 0 \quad (2.12)$$

(2.1) to (2.9) with (2.6) and (2.9) replaced by (2.11) and (2.12), respectively, are the basic equations of nonrelativistic quantum mechanics.

9. Comparison of the Galilei and Poincare Algebras

a. Notation

We use the same symbols for the Galilei and Poincare generators for notational convenience. It will be clear from the context when we are dealing with nonrelativistic or relativistic quantum mechanics.

b. Mass

The parameter m in (2.11) is the mass of the system. It follows from the Galilei Algebra that every Galilei invariant system is characterized by its mass. This implies that mass is conserved for Galilei invariant systems.

There is no mass parameter in the Poincare Algebra and consequently no requirement that Lorentz invariant systems conserve mass. Indeed, the Poincare Algebra allows the conversion of mass to energy and *vice versa.*

The nonappearance of mass in the Poincare Algebra raises the question of why each fundamental particle in Nature (the electron, the photon, *etc.*) is labelled by its rest mass. We show in Chapter 4 that the Poincare Algebra allows construction of a Lorentz invariant operator which for a single particle is the rest mass of the particle.

c. **Relative simplicity of the Galilei Algebra**

The equations in the Galilei Algebra involving K^j do not couple H and P^j or J^j and K^j. There is no mixing of energy and momentum or rotation of angular momentum under a Galilei boost.

The relative simplicity of the Galilei Algebra makes nonrelativistic quantum mechanics simpler then relativistic quantum mechanics.

10. **Determining the Poincare and Galilei generators**

The mathematical problem to be solved in order to describe a Lorentz invariant or Galilei invariant physical system is to construct the Poincare or Galilei generators in terms of fundamental dynamical variables for the system. These variables satisfy a fundamental algebra which is usually another set of commutation relations or anticommutation relations. That is, the mathematical problem is solve a coupled set of commutation relations in terms of given operators which satisfy given algebraic relations. There are, unfortunately, no well-developed mathematical methods for doing this.

For nonrelativistic quantum mechanics, the problem can be reduced to solving partial differential equations for which there are well-defined mathematical methods. For relativistic quantum mechanics, however, one must generally work directly with the Poincare Algebra. The methods for doing this are less systematic and often involve trial and error and intelligent guessing.

Chapter 3 POINCARE TRANSFORMATIONS

The basic equations of relativistic quantum mechanics, the Poincare Algebra, are presented and discussed in Chapter 2 and derived in Chapter 4. In this chapter we set the stage for the derivation in Chapter 4 by discussing Poincare transformations and defining what is meant by Lorentz invariance of a physical system.

Poincare transformations of preparation and measurement apparatuses are defined in Section 3.1, Lorentz invariance of a physical system is defined in Section 3.2, Poincare transformations in Hilbert space are given in Section 3.3, three- and four-vecotr operators are discussed in Sections 3.4 and 3.5, Poincare transformations of some operators are given in Section 3.6 and derivations of some results are given in Section 3.7.

3.1 Poincare transformations of apparatuses

An apparatus used to prepare a state $\mid \psi >$ of a physical system is placed in a fixed inertial frame S. The placement of the apparatus in S is specified by ten real numbers which may be chosen to be

- the time it was switched on (one number)
- the position of a fixed point on the apparatus (three numbers)
- the angles axes fixed to the apparatus make with the axes in S (three numbers)
- the velocity of a fixed point on the apparatus as measured in S (three numbers)

The placement of the apparatus in S is now changed without changing its intrinsic structure. The changed apparatus prepares a state $\mid \psi' >$. The change in placement of the apparatus involves changing one or more of the above ten numbers; it can be characterized by

$$x^\mu \to x'^\mu \tag{3.1}$$

$$x'^\mu = \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{3.2}$$

where $\mu = 0, 1, 2, 3$ and where

$$\left(x^0, x^1, x^2, x^3\right) = (ct, x, y, z) = (ct, \vec{x}) \tag{3.3}$$

are the space-time coordinates of a point on the apparatus before the change (t being the time the apparatus was switched on) and

$$\left(x'^0, x'^1, x'^2, x'^3\right) = \left(ct', x', y', z'\right) = \left(ct', \vec{x'}\right) \tag{3.4}$$

are the space-time coordinates of the same point on the apparatus after the change.

The quantities $\Lambda^\mu{}_\nu$ and a^μ in (3.2) are independent of x^μ and x'^μ and characterize the change in placement of the apparatus. The $\Lambda^\mu{}_\nu$ correspond to rotations and Lorentz boosts; the a^μ correspond to time translations and displacements.

Similar considerations can be given to a second apparatus which measures the value of some observable of the system. We denote the observable measured in the first placement by A and the observable measured in the second placement by A'. Then, for example,

$$< \psi \mid A \mid \psi > \tag{3.5}$$

is the average value of results of measurements of the observable A for the system when it has been prepared in the state $\mid \psi >$ and

$$< \psi' \mid A' \mid \psi' > \tag{3.6}$$

is the average value of results of measurements of the observable A' for the system when it has been prepared in the state $\mid \psi' >$.

Comments

1. **Active view**

We consider the active view of space-time transformations in the above. That is, we consider a single fixed intertial frame S and we imagine preparing states and measuring observables by moving apparatuses in this fixed frame.

x^μ and x'^μ in (3.2) are space-time coordinates in S of a point on an apparatus before and after a change of the placement of the apparatus in S.

$|\ \psi >$ and $|\ \psi' >$ are the states prepared by a preparation apparatus before and after a change of the placement of the preparation apparatus in S.

A and A' are the observables measured by a measurement apparatus before and after a change of the placement of the measurement apparatus in S.

2. **Passive view**

A second view of space-time transformations is the passive view. In this view one considers a second frame S' which is fixed to an apparatus and moves with it. In this view one does not move apparatuses around; instead, one considers states and observables in two frames S and S'.

3. **Passive view: interpretation of x^μ**

As with the active view, x'^μ in (3.2) are space-time coordinates in S of a point on an apparatus after a change of the placement of the apparatus in S.

In the passive view, x^μ in (3.2) are space-time coordinates of the same point on the apparatus in S'.

That is, in the passive view, (3.2) relates the space-time coordinates of the same point in two different frames: x^μ are the coordinates in S' and x'^μ are the coordinates in S.

4. **Passive view: interpretation of $|\psi'>$ and A'**

As with the active view, $|\psi>$ and A refer to a state and observable in S.

In the passive view, $|\psi'>$ is the state which the state-preparation apparatus prepares in S' and A' is the observable which the A—measuring apparatus measures in S'.

5. **Another notation for (3.1)**

The change (3.1) will also be written as

$$(\vec{x}, t) \to \left(\vec{x'}, t'\right) \tag{3.7}$$

It will be clear from the context which notation is being used.

6. **Notation**

Transformation (3.2) will be denoted as (Λ, a).

7. **Homogeneous and inhomogeneous transformations**

(Λ, a) is homogeneous if $a^\mu = 0$ and inhomogeneous if $a^\mu \neq 0$.

8. **Time translation**

The change

$$(\vec{x}, t) \to \left(\vec{x'}, t'\right) = (\vec{x}, t - \tau) \tag{3.8}$$

describes a time translation. It corresponds to an apparatus in S being switched on τ seconds earlier.

9. **Displacement**

The change

$$(\vec{x}, t) \to \left(\vec{x'}, t'\right) = (x + a, y, z, t) \tag{3.9}$$

describes a displacement of an apparatus in S along the x-axis by a distance a.

10. **Rotation**

The change

$$(\vec{x}, t) \to \left(\vec{x'}, t'\right) = (x, y\cos\theta - z\sin\theta, y\sin\theta + z\cos\theta, t) \tag{3.10}$$

describes a rotation of an apparatus in S about the x-axis by an angle θ.

11. **Lorentz boost**

The change

$$(\vec{x}, t) \to \left(\vec{x'}, t'\right) = \left(\gamma(x + vt), y, z, \gamma\left(t + \frac{vx}{c^2}\right)\right) \tag{3.11}$$

where

$$\gamma = \left(1 - \frac{v^2}{c^2}\right)^{-\frac{1}{2}} \tag{3.12}$$

describes a Lorentz boost of an apparatus in S along the x-axis by speed v; c is the speed of light and γ is the Lorentz factor of the boost.

(3.11) may also be expressed as

$$(x^\mu) \to (x'^\mu) = (x^0 \cosh u + x^1 \sinh u, x^0 \sinh u + x^1 \cosh u, x^2, x^3) \tag{3.13}$$

where

$$\tanh u = \frac{v}{c} \tag{3.14}$$

u is the rapidity of the boost.

12. **General transformation**

Every transformation (3.2) can be expressed as a product of one or more one-parameter transformations of the form (3.8), (3.9), (3.10), (3.13).

13. **Homogeneous Lorentz transformation**

The x'^μ and x^μ satisfy

$$\left(x'^0\right)^2 - \left(x'^1\right)^2 - \left(x'^2\right)^2 - \left(x'^3\right)^2 = \\ = \left(x^0\right)^2 - \left(x^1\right)^2 - \left(x^2\right)^2 - \left(x^3\right)^2 \tag{3.15}$$

when (3.2) is homogeneous. $(\Lambda, 0)$ is a homogeneous Lorentz transformation.

14. **Lorentz group**

The $(\Lambda, 0)$ form the Lorentz group. A subgroup of the Lorentz group is the rotation group.

15. **Restricted Lorentz transformation**

$(\Lambda, 0)$ excludes space inversion and time reversal. The determinant of the $\Lambda^{\mu}{}_{\nu}$ equals unity and $\Lambda^0{}_0 \geq 1$. That is, $(\Lambda, 0)$ is both proper and orthochronous. $(\Lambda, 0)$ is a restricted Lorentz transformation.

Space inversion and time reversal are considered in Chapter 5.

16. **Poincare transformation**

(Λ, a) is a restricted Lorentz transformation combined with a space-time translation. (Λ, a) is a Poincare transformation. The (Λ, a) form the Poincare group. The Poincare group is a Lie group.

3.2 Lorentz invariance of a physical system

A physical system is Lorentz invariant if the same average is obtained for the results of measurements of every observable of the system in every state of the system when the same Poincare transformation is carried out on both the preparation apparatus and the measuring apparatus.

That is, a physical system is Lorentz invariant if

$$< \psi' \mid A' \mid \psi' > = < \psi \mid A \mid \psi > \tag{3.16}$$

for every observable A and every state $\mid \psi >$ of the system and for every Poincare transformation (3.2) of both preparation and measuring apparatuses.

(3.16) holds if the states $\mid \psi' >$ and $\mid \psi >$ and the observables A' and A are related according to

$$\mid \psi' > = O \mid \psi > \tag{3.17}$$

$$A' = OAO^{\dagger} \tag{3.18}$$

where O is either a linear unitary operator or an antilinear antiunitary operator.

It follows from item 12 of Section 3.1 that each operator O corresponding to a Poincare transformation (Λ, a) can be written as the product of one or more of ten one-parameter operators

$$O_1(\eta_1), O_2(\eta_2), \cdots, O_{10}(\eta_{10}) \tag{3.19}$$

where η labels the real parameter for a one-parameter Poincare transformation.

These operators satisfy

$$O_\alpha(0) = 1 \tag{3.20}$$

$$O_\alpha(\eta_1)O_\alpha(\eta_2) = O_\alpha(\eta_1 + \eta_2) \tag{3.21}$$

Since the square of a unitary operator or an antiunitary operator is a unitary operator, it follows from (3.21) that each $O_\alpha(\eta)$ is a unitary operator. It follows from (3.20) and (3.21) that $O_\alpha(\eta)$ can be written in the form

$$O_\alpha(\eta) = e^{-iC_\alpha \eta} \tag{3.22}$$

where C_α is a Hermitian operator. The ten unitary operators (3.19) are thus determined by the ten Hermitian operators

$$C_1, C_2, \cdots, C_{10} \tag{3.23}$$

Comments

1. **Poincare generators**

(3.23) are the generators of the Poincare group or the Poincare generators.

2. **Labels for the Poincare generators**

The Poincare generators (3.23) are labelled as follows:

$H/\hbar$ generates a time translation.
$P^j/\hbar$ generates a displacement along the j–axis.
$J^j/\hbar$ generates a rotation about the j–axis.
$cK^j/\hbar$ generates a boost along the j–axis.

3. **Labels for the unitary Poincare operators**

The unitary Poincare operators (3.19) are labelled as follows:

$$U(t) = e^{-iHt/\hbar} \tag{3.24}$$

$$D^j(a) = e^{-iP^j a/\hbar} \tag{3.25}$$

$$R^j(\theta) = e^{-iJ^j\theta/\hbar} \tag{3.26}$$

$$L^j(u) = e^{-icK^j u/\hbar} \tag{3.27}$$

4. **Form invariance of the Poincare Algebra**

In the passive view of space-time transformations (3.18) relates observables measured in two different inertial frames. That is, A' given by (3.18) is the observable which the A-measuring apparatus measures in S'.

In view of the unitarity of O the Poincare Algebra (2.1) to (2.9) has the same form when written in terms of the unprimed or primed generators. That is, the basic equations of relativistic quantum mechanics are form invariant under a general Poincare transformation. The speed of light c and Planck's constant $\hbar$ have the same value in all reference frames.

3.3 Unitary Poincare operators

The unitary Poincare operators (3.24) to (3.27) are the operators in Hilbert space which correspond to the Poincare transformations given Section 3.1. That is, (3.24) to (3.27) give Poincare transformations in Hilbert space. We comment on these operators in this section. We generally consider the active view of space-time transformations; the passive view is considered when convenient.

Comments

1. **Evolution operator**

$U(t)$ defined by (3.24) is the evolution operator for the system.

$U(t)$ corresponds to the time translation

$$(\vec{x}, t_0) \rightarrow (\vec{x}, t_0 - t) \tag{3.28}$$

The time labels in (3.28) refer to the times an apparatus was switched on; we label the evolution operator by the difference in these times.

Correspondingly, we label a state by the time it has evolved since the time the preparation apparatus was switched on. If $| \psi >$ is the state of the system in S at time zero (that is, which has evolved for zero time) then

$$| \psi(t) >= U(t) | \psi > \tag{3.29}$$

is the state of the system S at time t (that is, which has evolved for time t). A state prepared earlier has evolved for a longer time.

2. **Schrodinger equation**

It follows on differentiating (3.29) with respect to t that

$$H | \psi(t) >= i\hbar \frac{d}{dt} | \psi(t) > \tag{3.30}$$

(3.30) is the Schrodinger equation for the system.

3. **Displacement operator**

$D^j(a)$ defined by (3.25) is the displacement operator along the j-axis of S for the system.

$D^1(a)$, for example, corresponds to the displacement (3.9).

If $| \psi(t) >$ is the state prepared by a preparation apparatus in S at time t then

$$D^j(a) | \psi(t) > \tag{3.31}$$

is the state prepared at time t by the same apparatus displaced in S by a along the j-axis for the system.

4. **Displacement operator for a general displacement**

The operator

$$D(\vec{x}) = D^1(x)D^2(y)D^3(z) \tag{3.32}$$

is the displacement operator for a general displacement of the system in S. It follows using (2.1) that

$$D(\vec{x}) = e^{-i\vec{P}\cdot\vec{x}/\hbar} \tag{3.33}$$

$D(\vec{x})$ corresponds to the Poincare transformation

$$(0, 0, 0, t) \to (x, y, z, t) \tag{3.34}$$

If $\mid \psi(t) >$ is the state prepared in S by a preparation apparatus at the origin at time t then

$$D(\vec{x}) \mid \psi(t) > \tag{3.35}$$

is the state prepared in S by the same apparatus at (x, y, z) at time t.

5. **Space-time displacement operator**

The operator

$$D(\vec{x}, t) = D(\vec{x})U(t) \quad (3.36)$$

is the space-time displacement operator for the system in$S$. It follows using (2.1) and (2.2) that

$$D(\vec{x}, t) = e^{-i(\vec{P}\cdot\vec{x}+Ht)/\hbar} \quad (3.37)$$

$D(\vec{x}, t)$ corresponds to the Poincare transformation

$$(0, 0, 0, t_0) \to (x, y, z, t_0 - t) \quad (3.38)$$

If $\mid \psi >$ is the state prepared in S by a preparation apparatus at the origin at time zero then

$$D(\vec{x}, t) \mid \psi > \quad (3.39)$$

is the state prepared in S by the same apparatus at (x, y, z) at time t.

6. Rotation operator

$R^j(\theta)$ defined by (3.26) is the rotation operator about the j-axis in S for the system.

$R^1(\theta)$, for example, corresponds to the rotation (3.10).

If $\mid \psi(t) >$ is the state prepared in S by a preparation apparatus at time t then

$$R^j(\theta) \mid \psi(t) > \tag{3.40}$$

is the state prepared in S at time t by the same apparatus rotated by θ about the j-axis for the system.

7. Rotation operator for a general rotation

The operator

$$R(\alpha, \beta, \gamma) = R^3(\alpha) R^2(\beta) R^3(\gamma) \tag{3.41}$$

is the rotation operator for a general rotation of the system in S.[1]

$R(\alpha, \beta, \gamma)$ corresponds to the Poincare transformation

$$(\vec{x}, t) \rightarrow (\vec{x}_R, t) \tag{3.42}$$

where α, β, γ are the Euler angles corresponding to the rotation of $\vec{x}$ to $\vec{x}_R$.

If $\mid \psi(t) >$ is the state prepared in S by a preparation apparatus at time t then

$$R(\alpha, \beta, \gamma) \mid \psi(t) > \tag{3.43}$$

is the state prepared in S at time t by the same apparatus rotated through Euler angles α, β, γ.

[1] We follow the convention for the sequence of rotations used by Rose (1956), equation (4.7), page 51.

8. Lorentz boost operator

$L^j(u)$ defined by (3.27) is the Lorentz boost operator along the j-axis in S for the system.

$L^1(u)$, for example, corresponds to the Lorentz boost (3.13).

If $| \psi(t) >$ is the state prepared in S by a preparation apparatus at time t then

$$L^j(u) \mid \psi(t) > \tag{3.44}$$

is the state prepared in S at time t by the same apparatus boosted with rapidity u along the j-axis for the system.

9. Einstein addition of velocities

It follows from (3.27) that

$$L^j(u_1)L^j(u_2) = L^j(u_1 + u_2) \tag{3.45}$$

When written in terms of velocities (3.45) is

$$L^j(v_1)L^j(v_2) = L^j\left(\frac{v_1 + v_2}{1 + v_1 v_2/c^2}\right) \tag{3.46}$$

(3.46) expresses the Einstein addition of velocities.

3.4 Rotations and 3–vector operators

In this section we define and discuss operators which transform under a rotation like the components of a 3-vector.

Definition and comments

1. 3-vector operators

If $\left(A^1, A^2, A^3\right)$ satisfy

$$\left[J^j, A^k\right] = i\hbar\epsilon_{jkl}A^l \tag{3.47}$$

then

$$R^j(\theta)A^a R^{j\dagger}(\theta) = r^j_{ab}(\theta)A^b \tag{3.48}$$

where $R^j(\theta)$ is the rotation operator (3.26) and $r^j_{ab}(\theta)$ are the matrix elements of the rotation matrix $r^j(\theta)$ (A.4).

That is, $\left(A^1, A^2, A^3\right)$ transform under a rotation like the components of a 3–vector. Accordingly, we write

$$\vec{A} = A^1\vec{i} + A^2\vec{j} + A^3\vec{k} \tag{3.49}$$

where $\vec{i}, \vec{j}, \vec{k}$ are unit vectors along the axes of S. $\vec{A}$ is a vector operator.

2. Examples

It follows from (2.3), (2.5), (2.8) and (3.47) that

$$(P^1, P^2, P^3) \quad (J^1, J^2, J^3) \quad (K^1, K^2, K^3) \tag{3.50}$$

transform under a rotation like the components of a 3–vector.

3. Rotation of $\vec{k}$ to $\vec{m}$

It follows from (3.48) that

$$R(\theta, \varphi)\vec{A} \cdot \vec{k} R^{\dagger}(\theta, \varphi) = \vec{A} \cdot \vec{m} \tag{3.51}$$

where

$$R(\theta, \varphi) = R^3(\varphi) R^2(\theta) \tag{3.52}$$

and

$$\vec{m} = \sin\theta \cos\varphi\, \vec{\imath} + \sin\theta \sin\varphi\, \vec{\jmath} + \cos\theta\, \vec{k} \tag{3.53}$$

(3.52) corresponds in S to the rotation of $\vec{k}$ to $\vec{m}$.

It follows from (3.51) that an apparatus in S which measures $\vec{A} \cdot \vec{k}$ will measure $\vec{A} \cdot \vec{m}$ when rotated by angles corresponding to (3.52).

Application of the above to Stern-Gerlach experiments for spin $\frac{1}{2}$ and spin 1 particles is given in *QLB: Introductory Topics.* (A Stern-Gerlach apparatus measures the component of spin of a particle along the direction of the gradient of the inhomogeneous magnetic field of the apparatus.)

4. **Invariant operators**

If (A^1, A^2, A^3) and (B^1, B^2, B^3) each satisfy (3.47) it follows that

$$\left[J^j, \vec{A} \cdot \vec{B}\right] = 0 \tag{3.54}$$

and therefore

$$R^j(\theta)\, \vec{A} \cdot \vec{B}\, R^{j\dagger}(\theta) = 0 \tag{3.55}$$

That is, $\vec{A} \cdot \vec{B}$ is invariant under a rotation.

3.5 Lorentz boosts and 4-vector operators

In this section we define and discuss operators which transform under a Lorentz boost like components of a 4–vector.

Definition and comments

1. **4–vector operators**

If (A^0, A^1, A^2, A^3) satisfy

$$[K^j, A^0] = -\frac{i\hbar}{c} A^j \tag{3.56}$$

$$\left[K^j, A^k\right] = -\frac{i\hbar}{c} \delta_{jk} A^0 \tag{3.57}$$

that is, if

$$[K^j, A^\mu] = \frac{i\hbar}{c} (g^{j\mu} A^0 - g^{0\mu} A^j) \tag{3.58}$$

then

$$L^j(u) A^\mu L^{j\dagger}(u) = l^{j\mu}{}_\nu(u) A^\nu \tag{3.59}$$

where $L^j(u)$ is the Lorentz boost operator (3.27) and $l^{j\mu}{}_\nu(u)$ are the matrix elements of the Lorentz transformation matrix $l^j(u)$ (A.8).

Accordingly, (A^0, A^1, A^2, A^3) transform under a Lorentz boost like the components of a 4–vector.

2. **Example**

It follows from (2.6), (2.7), (3.56) and (3.57) that

$$\left(\frac{H}{c}, P^1, P^2, P^3\right) \tag{3.60}$$

transform under a Lorentz boost like the components of a 4–vector.

3. **Invariant operators**

If $\left(A^0, A^1, A^2, A^3\right)$ and $\left(B^0, B^1, B^2, B^3\right)$ each satisfy (3.58) it follows that[1]

$$\left[K^j, A.B\right] = 0 \tag{3.61}$$

and therefore

$$L^j(u)\, A.B\, L^{j\dagger}(u) = 0 \tag{3.62}$$

That is, $A.B$ is invariant under a Lorentz boost.

In particular, it follows from from (2.6) and (2.7) that $P.P$ is invariant under a Lorentz boost.

3.6 Transformations of the Poincare generators

In this section we present some Poincare transformations of Poincare generators and we interpret these transformations in the passive view of space-time transformations.

1 We recall that $A.B = A_\mu B^\mu = A^\mu B_\mu = A^0 B^0 - A \cdot B$

Time translation

$$U(t)HU^\dagger(t) = H \tag{3.63}$$

$$U(t)\vec{P}U^\dagger(t) = \vec{P} \tag{3.64}$$

$$U(t)\vec{J}U^\dagger(t) = \vec{J} \tag{3.65}$$

$$U(t)\vec{K}U^\dagger(t) = \vec{K} + \vec{P}t \tag{3.66}$$

Displacement

$$D(\vec{x})HD^\dagger(\vec{x}) = H \tag{3.67}$$

$$D(\vec{x})\vec{P}D^\dagger(\vec{x}) = \vec{P} \tag{3.68}$$

$$D(\vec{x})\vec{J}D^\dagger(\vec{x}) = \vec{J} - \vec{x} \times \vec{P} \tag{3.69}$$

$$D(\vec{x})\vec{K}D^\dagger(\vec{x}) = \vec{K} + \vec{x}H/c^2 \tag{3.70}$$

Rotation

$$R^j(\theta)HR^{j\dagger}(\theta) = H \tag{3.71}$$

$$R^1(\theta)P^1R^{1\dagger}(\theta) = P^1 \qquad (3.72)$$

$$R^1(\theta)P^2R^{1\dagger}(\theta) = P^2\cos\theta + P^3\sin\theta \qquad (3.73)$$

$$R^1(\theta)P^3R^{1\dagger}(\theta) = P^3\cos\theta - P^2\sin\theta \qquad (3.74)$$

and similarly for (J^1, J^2, J^3) and (K^1, K^2, K^3).

Lorentz boost

$$L^1(u)HL^{1\dagger}(u) = H\cosh u - cP^1\sinh u \qquad (3.75)$$

$$L^1(u)P^1L^{1\dagger}(u) = P^1\cosh u - \frac{H}{c}\sinh u \qquad (3.76)$$

$$L^1(u)P^2L^{1\dagger}(u) = P^2 \qquad (3.77)$$

$$L^1(u)P^3L^{1\dagger}(u) = P^3 \qquad (3.78)$$

$$L^1(u)J^1L^{1\dagger}(u) = J^1 \qquad (3.79)$$

$$L^1(u)J^2L^{1\dagger}(u) = J^2\cosh u + cK^3\sinh u \qquad (3.80)$$

$$L^1(u)J^3L^{1\dagger}(u) = J^3\cosh u - cK^2\sinh u \qquad (3.81)$$

$$L^1(u)K^1L^{1\dagger}(u) = K^1 \tag{3.82}$$

$$L^1(u)K^2L^{1\dagger}(u) = K^2\cosh u - \frac{1}{c}J^3\sinh u \tag{3.83}$$

$$L^1(u)K^3L^{1\dagger}(u) = K^3\cosh u + \frac{1}{c}J^2\sinh u \tag{3.84}$$

Lorentz boost of the space-time displacement operator

$$L^1(u)D(x)L^{1\dagger}(u) = D(x_L) \tag{3.85}$$

where

$$D(x) = D(\vec{x}, t) \tag{3.86}$$

is the space-time displacement operator and where

$$(x_L^\mu) = (x^0\cosh u - x^1\sinh u, x^0\sinh u - x^1\cosh u, x^2, x^3) \tag{3.87}$$

Lorentz boost of an observable

$$L^1(u)A(x)L^{1\dagger}(u) = A_L(x_L) \tag{3.88}$$

where

$$A(x) = D(x)AD^\dagger(x) \tag{3.89}$$

and

$$A_L(x_L) = D(x_L)A_LD^\dagger(x_L) \tag{3.90}$$

where

$$A_L = L^1(u)AL^{1\dagger}(u) \tag{3.91}$$

Comments

1. **Interpretation in the passive view**

In the following items we interpret (3.63) to (3.91) in the passive view of space-time transformations.

In this view, for example,

$$H \quad \text{and} \quad \vec{P} \tag{3.92}$$

which appear on the right sides of (3.75) to (3.78) are the energy and momentum of a system as measured by a stationary observer in S and

$$L^1(u)HL^{1\dagger}(u) \quad \text{and} \quad L^1(u)\vec{P}L^{1\dagger}(u) \tag{3.93}$$

which appear on the left sides of (3.75) to (3.78) are the energy and momentum of the same system as measured by a stationary observer in a frame S_L which is Lorentz boosted along the x-axis of S with rapidity u.

2. **Invariance of the Hamiltonian**

(3.63) follows because H commutes with itself and (3.67) and (3.71) follow from (2.2) and (2.4): H is invariant under time translations, space displacements and space rotations.

3. **Hamiltonian in the boosted frame**

(3.75) shows that H is not invariant under Lorentz boosts; there is a mixing of energy and momentum under a Lorentz boost.

(3.75) to (3.78) correspond to the classical expressions.

4. **Poincare generators as a rank 2 contravariant tensor**

We show in Section that (3.79) to (3.84) imply that (J^1, J^2, J^3) and (cK^1, cK^2, cK^3) transform under a Lorentz boost as components of an antisymmetric rank 2 contravariant tensor.

5. **Lorentz boost of the space-time displacement operator**

We denote the space-time coordinates of the same event measured in S and S_L by

$$\left(x^0, x^1, x^2, x^3\right) \quad \text{and} \quad \left(x_L^0, x_L^1, x_L^2, x_L^3\right) \tag{3.94}$$

respectively. Then, (3.87) holds as per (3.13).

The left side of (3.85) is the space-time displacement operator in S_L.

(3.85) states that the space-time displacement operator in S_L has the same functional form as the space-time displacement operator in S but evaluated at the boosted space-time point $\left(x_L^\mu\right)$.

6. **Lorentz boost of an observable**

(3.88) to (3.91) are interpreted as follows:

A is an observable measured by a stationary measurement apparatus in S.

A_L is the observable measured by a stationary measurement apparatus in S_L.

$A(x)$ is the observable measured by a measurement apparatus in S which has undergone a space-time displacement in S by x^μ.

$A_L(x_L)$ is the observable measured by a measurement apparatus in S_L which has undergone a space-time displacement in S_L by x_L^μ.

It follows from (3.88) that the space-time properties of a system and of a Lorentz boosted system are the same when the boosted system is viewed in a boosted coordinate system.

3.7 Some derivations

Derivation of (3.48)

The derivation involves calculating

$$\frac{d}{d\theta} R^j(\theta) A^a R^{j\dagger}(\theta) \tag{3.95}$$

and using

$$\frac{dr^j_{ab}(\theta)}{d\theta} = \epsilon_{jac} r^j_{cb}(\theta) \tag{3.96}$$

Derivation of (3.59)

The derivation involves calculating

$$\frac{d}{du} L^j(u) A^\mu L^{j\dagger}(u) \tag{3.97}$$

and using

$$\frac{dl^{j\mu}{}_\nu(u)}{du} = g^{j\mu} l^{j0}{}_\nu(u) - g^{0\mu} l^{jj}{}_\nu(u) \tag{3.98}$$

Derivation of (3.85)

The derivation uses

$$\begin{aligned} L^1(u)\Big(\vec{P} \cdot \vec{x} + Ht\Big) L^{1\dagger}(u) &= L^1(u) \vec{P} L^{1\dagger}(u) \cdot \vec{x} + L^1(u) H L^{1\dagger}(u) t \\ &= \vec{P} \cdot \vec{x}_L + H t_L \end{aligned} \tag{3.99}$$

Chapter 4 MORE ABOUT THE POINCARE ALGEBRA

We give a further discussion of the Poincare Algebra (2.1) to (2.9) in this chapter. The derivation of the Poincare Algebra is discussed in Section 4.1 and the Poincare Algebra is written in a manifestly covariant form in Section 4.2. Lorentz invariants, operators invariant under all Poincare transformations, are constructed in Section 4.3. Section 4.4 contains some matrix representations of the Lorentz Algebra, which is that part of the Poincare Algebra corresponding to homogeneous Poincare transformations (rotations and boosts). Also included is a demonstration of the relationship between γ–matrices and Lorentz boosts and a 4×4 matrix representation of the Poincare generators. Finally, some derivations are given in Section 4.5.

4.1 Derivation of the Poincare Algebra

Kalyniak (1978) contains a complete derivation of the Poincare Algebra. We illustrate the method of derivation in this section and we give some derivations in Section 4.5.

It follows from the group properties of Poincare transformations that two Poincare transformations in succession yield another Poincare transformation. We consider two successive Poincare transformations followed by their inverses in the infinitesimal limit of the Poincare transformation parameters; the Poincare Algebra is derived by considering the corresponding unitary operators in the infinitesimal limit. It is sufficient to consider the infinitesimal limit as this yields relationships among the Poincare generators (the Poincare Algebra) which, because the unitary Poincare operators are determined in terms of the Poincare generators by (3.22), determine relationships among the unitary Poincare operators for all values of the Poincare transformation parameters.

We consider Poincare transformations (3.22) generated by Poincare generators C_α and C_β with parameters σ and τ, respectively, followed by their inverses. The corresponding unitary operator on Hilbert space is

$$e^{iC_\beta \tau} e^{iC_\alpha \sigma} e^{-iC_\beta \tau} e^{-iC_\alpha \sigma} \tag{4.1}$$

and expanding (4.1) in the parameters σ and τ yields

$$e^{iC_\beta \tau} e^{iC_\alpha \sigma} e^{-iC_\beta \tau} e^{-iC_\alpha \sigma} = 1 + \sigma\tau\big[C_\alpha, C_\beta\big] + \cdots \tag{4.2}$$

The result of the above transformations is another Poincare transformation. That is, up to a phase factor, the corresponding unitary operator is the product of one or more of the ten unitary operators (3.24) to (3.27). Expanding the resultant unitary operator in the parameters σ and τ yields

$$1 + \sigma\tau\big(C_{\alpha\beta} + b_{\alpha\beta} I\big) + \ldots \tag{4.3}$$

where $C_{\alpha\beta}$ is the generator of the net transformation and $b_{\alpha\beta}$ is a constant corresponding to the phase factor. It follows from (4.2) and (4.3) that

$$\big[C_\alpha, C_\beta\big] = C_{\alpha\beta} + b_{\alpha\beta} I \tag{4.4}$$

The Poincare Algebra (2.1) to (2.9) is derived by considering (4.4) for all Poincare generators. The complete derivation shows that all constants $b_{\alpha\beta}$ may be eliminated from (2.1) to (2.9): the Jacobi identity is used to show that a number of the phase constants vanish and those which are nonzero may be absorbed into the Poincare generators.

We derive some of the equations in the Poincare Algebra in Section 4.5; we assume there that all phase constants have been been eliminated by the above procedure.

4.2 Manifest covariance

In view of (3.75) to (3.84) we define operators P^0 and $M^{\mu\nu}$ by

$$P^0 = \frac{1}{c} H \tag{4.5}$$

$$\left(M^{23}, M^{31}, M^{12}\right) = \left(J^1, J^2, J^3\right) \tag{4.6}$$

$$\left(M^{01}, M^{02}, M^{03}\right) = \left(cK^1, cK^2, cK^3\right) \tag{4.7}$$

$$M^{\mu\nu} = -M^{\nu\mu} \tag{4.8}$$

It follows from (3.75) to (3.84) that

$$L^1(u) P^\mu L^{1\dagger}(u) = l^{1\mu}{}_\alpha P^\alpha \tag{4.9}$$

$$L^1(u) M^{\mu\nu} L^{1\dagger}(u) = l^{1\mu}{}_\alpha \; l^{1\nu}{}_\beta \; M^{\alpha\beta} \tag{4.10}$$

where $l^{1\mu}{}_\nu$ are the matrix elements (A.8) of the Lorentz matrix (A.5).

The Poincare Algebra (2.1) to (2.9) takes the form

$$[P^{\mu}, P^{\nu}] = 0 \tag{4.11}$$

$$[M^{\mu\nu}, P^{\sigma}] = i\hbar(g^{\nu\sigma}P^{\mu} - g^{\mu\sigma}P^{\nu}) \tag{4.12}$$

$$[M^{\mu\nu}, M^{\sigma\tau}] = i\hbar(g^{\nu\sigma}M^{\mu\tau} - g^{\mu\sigma}M^{\nu\tau} + g^{\nu\tau}M^{\sigma\mu} - g^{\mu\tau}M^{\sigma\nu}) \tag{4.13}$$

where $g^{\mu\nu}$ is the metric tensor.[1]

Comments

1. **Transformation under Lorentz boosts**

   In view of the appearance of the matrix elements of the Lorentz matrix (A.5) in (4.9) and (4.10), P^{μ} and $M^{\mu\nu}$ are said to transform under Lorentz boosts as a contravariant vector and a rank 2 contravariant tensor, respectively.

2. **Manifestly covariant form of the Poincare Algebra**

   (4.11) to (4.13) are said to be a manifestly covariant form of the Poincare Algebra since they are expressed entirely in terms of four-vectors and four-tensors.

   The manifestly covariant form of the Poincare Algebra is convenient when one considers Poincare invariant physical systems with quantum fields as fundamental dynamical variables.

[1] $g^{\mu\nu}$ is diagonal; $g^{00} = -g^{11} = -g^{22} = -g^{33} = 1$

4.3 Lorentz invariants

We construct two independent operators M and $W.W$ which commute with all Poincare generators. These operators are Lorentz invariants; they are the Casimir operators for the Poincare group.

There are only two independent Lorentz invariants for a general Lorentz invariant system. As shown in *QLB: Some Lorentz Invariant Systems* this implies that a single particle can be labelled by its rest mass and spin.

Invariant mass

We define the invariant mass M of a Lorentz invariant system by

$$M = \frac{1}{c}\sqrt{P.P} \tag{4.14}$$

That is,

$$Mc^2 = \sqrt{H^2 - P^2c^2} \tag{4.15}$$

Comments

1. **Lorentz invariant**

   M commutes with all Poincare generators; it is a Lorentz invariant.

2. **One-particle system**

We show in *QLB: Some Lorentz Invariant Systems* that M is the rest mass of a single Lorentz invariant particle with arbitrary spin.

3. **Nonconservation of mass for a many-particle system**

   The above comment does not imply that mass is conserved in all Lorentz invariant systems. We show in *QLB: Some Lorentz Invariant Systems* that M is not equal to the sum of the rest masses of a Lorentz invariant many-particle system with a fixed number of particles.

4. **Interaction in a many-particle system**

   We show in *QLB: Some Lorentz Invariant Systems* that the interactions among the particles in a Lorentz invariant many-particle system with a fixed number of particles may be specified entirely by the invariant mass M.

Pauli-Lubanski four-vector

The Pauli-Lubanski four-vector W^μ is defined as

$$W^\mu = -\frac{1}{2}\epsilon^{\mu\nu\sigma\tau} P_\nu M_{\sigma\tau} \tag{4.16}$$

where $\epsilon^{\mu\nu\sigma\tau}$ is the unit antisymmetric tensor. That is,

$$W^0 = \vec{J} \cdot \vec{P} \tag{4.17}$$

$$\vec{W} = \frac{H}{c}\vec{J} + c\vec{K} \times \vec{P} \tag{4.18}$$

It follows from (4.16) that

$$P_\mu W^\mu = 0 \tag{4.19}$$

$$[P^\mu, W^\nu] = 0 \tag{4.20}$$

$$[M^{\mu\nu}, W^\sigma] = i\hbar(g^{\nu\sigma}W^\mu - g^{\mu\sigma}W^\nu) \tag{4.21}$$

$$[W^\mu, W^\nu] = i\hbar\epsilon^{\mu\nu\sigma\tau}W_\sigma P_\tau \tag{4.22}$$

Comments

1. **Lorentz invariant**

W^μ transforms under Lorentz boosts as a contravariant four-vector (4.9).

$W.W$ commutes with all Poincare generators; $W.W$ is a Lorentz invariant.

2. **Pauli-Lubanski four-vector and internal angular momentum**

The Pauli-Lubanski four-vector was invented independently by W. Pauli and F. Lubanski in order to construct the internal angular momentum of a general Lorentz invariant system.

We define the internal angular momentum of a general Lorentz invariant system in terms of W^μ in Chapter 6.

4.4 Matrix representations of the Lorentz Algebra

In this section we consider matrix representations of the Lorentz Algebra which is that part of the Poincare Algebra corresponding to homogeneous Poincare transformations (rotations and boosts). The Lorentz Algebra is (2.5), (2.8) and (2.9) or alternatively (4.13).

In particular we consider 2×2 and 4×4 matrix representations of the Lorentz Algebra. The latter leads naturally to γ–matrices which were originally invented by Dirac for description of the Dirac particle.

Decoupled form of the Lorentz Algebra

On defining

$$A^j = \frac{1}{2}(J^j + icK^j) \tag{4.23}$$

$$B^j = \frac{1}{2}(J^j - icK^j) \tag{4.24}$$

which yields

$$J^j = A^j + B^j \tag{4.25}$$

$$K^j = i(B^j - A^j)/c \tag{4.26}$$

it follows from (2.5), (2.8) and (2.9) that

$$\left[A^j, A^k\right] = i\hbar\epsilon_{jkl}A^l \tag{4.27}$$

$$\left[B^j, B^k\right] = i\hbar\epsilon_{jkl}B^l \tag{4.28}$$

$$\left[A^j, B^k\right] = 0 \tag{4.29}$$

Comments

1. **Decoupled form of the Lorentz Algebra**

   (4.27) to (4.29) are the decoupled form of the Lorentz Algebra.

   A^j and B^j satisfy the same commutation relations (2.5) as J^j.

2. **Matrix representations**

   The irreducible representations of the rotation group are the $2j+1$ by $2j+1$ rotation matrices[2]

$$D^j_{m'm}(\alpha, \beta, \gamma) \tag{4.30}$$

where $j = 0, \frac{1}{2}, 1, \cdots$ and α, β, γ are Euler angles characterizing the rotation.

It follows from (4.27) to (4.29) that the finite-dimensional irreducible representations of the Lorentz group are direct products of two $2j+1$ by $2j+1$

[2] We follow the convention and notation used by Rose (1957).

matrices. That is, they are of the form

$$D^{jj'}(\Lambda) = D^j(\Lambda) \otimes D^{j'}(\Lambda) \tag{4.31}$$

These irreducible representations in general are not unitary because when A^j and B^j are Hermitian K^j isn't.

2×2 matrix representations

The Lorentz Algebra (4.27) to (4.29) is satisfied by following two sets of 2×2 matrices

$$A^j = \frac{1}{2}\hbar\sigma^j \tag{4.32}$$

$$B^j = 0 \tag{4.33}$$

and

$$A^j = 0 \tag{4.34}$$

$$B^j = \frac{1}{2}\hbar\sigma^j \tag{4.35}$$

where $\sigma^1, \sigma^2, \sigma^3$ are the Pauli matrices (A.9) to (A.11).

The Poincare generators corresponding to (4.32) to (4.35) are[1]

$$J^j = \frac{1}{2}\hbar\sigma^j \tag{4.36}$$

$$K^j = -\frac{i\hbar}{2c}\sigma^j \tag{4.37}$$

and

$$J^j = \frac{1}{2}\hbar\sigma^j \tag{4.38}$$

$$K^j = \frac{i\hbar}{2c}\sigma^j \tag{4.39}$$

Group $SL(2,c)$

For a restricted homogeneous Lorentz transformation, (3.2) can be written as the matrix equation

$$X' = OXO^\dagger \tag{4.40}$$

where

[1] We note that the LA is invariant under the replacement $(J^j, K^j) \rightarrow (J^j, -K^j)$.

$$X' = \begin{pmatrix} ct' + z' & x' - iy' \\ x' + iy' & ct' - z' \end{pmatrix} \tag{4.41}$$

$$X = \begin{pmatrix} ct + z & x - iy \\ x + iy & ct - z \end{pmatrix} \tag{4.42}$$

and

$$O = \begin{pmatrix} e & f \\ g & h \end{pmatrix} \tag{4.43}$$

where e, f, g, h are complex parameters such that

$$\det(O) = eh - fg = 1 \tag{4.44}$$

Comments

1. Another form

(4.42) may be expressed as

$$X = x^{\mu}\sigma^{\mu} \tag{4.45}$$

where σ^0 is the 2×2 identity matrix.

2. Determinants

The determinants of X' and X are

$$\det(X') = (ct')^2 - x'^2 - y'^2 - z'^2 \tag{4.46}$$

$$\det(X) = (ct)^2 - x^2 - y^2 - z^2 \tag{4.47}$$

(4.44) ensures that

$$(ct')^2 - x'^2 - y'^2 - z'^2 = (ct)^2 - x^2 - y^2 - z^2 \tag{4.48}$$

(4.48) characterizes a homogeneous Lorentz transformation.

3. Special case: unitary matrix

When (4.43) is unitary it follows that

$$t' = t \tag{4.49}$$

$$x'^2 + y'^2 + z'^2 = x^2 + y^2 + z^2 \tag{4.50}$$

(4.49) and (4.50) characterize a rotation.

4. Groups

The group of matrices (4.43) satisfying (4.44) is called $SL(2, c)$.

The group of unitary matrices (4.43) satisfying (4.44) is called $SU(2)$.

$SU(2)$ is a subgroup of $SL(2,c)$.

$SL(2,c)$ is the covering group for the restricted homogeneous Lorentz group.

$SU(2)$ is the covering group for the rotation group.

5. **Notation for matrices**

Matrices (4.43) generated by (4.36) and (4.37) are denoted by $D^{\frac{1}{2}0}(\Lambda)$.

Matrices (4.43) generated by (4.38) and (4.39) are denoted by $D^{0\frac{1}{2}}(\Lambda)$.

6. **Example: rotation about the z-axis**

For a rotation by θ about the z-axis,

$$D^{0\frac{1}{2}}(\Lambda) = R^3(\theta) = e^{-iJ^3\theta/\hbar} = e^{-i\sigma^3\theta/2} = \begin{pmatrix} e^{-i\theta/2} & 0 \\ 0 & e^{i\theta/2} \end{pmatrix} \tag{4.51}$$

so (4.40) is

$$X' = R^3(\theta) X R^{3\dagger}(\theta) \tag{4.52}$$

that is,

$$\left(\vec{x'}, t'\right) = (x\cos\theta - y\sin\theta, x\sin\theta + y\cos\theta, z, t) \tag{4.53}$$

as per (3.10).

7. Example: boost along the z-axis

For a boost with rapidity u along the z-axis,

$$D^{0\frac{1}{2}}(\Lambda) = L^3(u) = e^{-icK^3u/\hbar} = e^{\sigma^3 u/2} = \begin{pmatrix} e^{u/2} & 0 \\ 0 & e^{-u/2} \end{pmatrix} \tag{4.54}$$

so (4.40) is

$$X' = L^3(u) X L^{3\dagger}(u) \tag{4.55}$$

that is,

$$(x'^\mu) = \left(x^0 \cosh u + x^3 \sinh u, x^1, x^2, x^0 \sinh u + x^3 \cosh u\right) \tag{4.56}$$

as per (3.13).

4×4 matrix representation; γ–matrices

The 4×4 matrices

$$J^j = \begin{pmatrix} \frac{1}{2}\hbar\sigma^j & 0 \\ 0 & \frac{1}{2}\hbar\sigma^j \end{pmatrix} \tag{4.57}$$

$$K^j = \begin{pmatrix} -\frac{i\hbar}{2c}\sigma^j & 0 \\ 0 & \frac{i\hbar}{2c}\sigma^j \end{pmatrix} \tag{4.58}$$

satisfy the Lorentz Algebra (2.5), (2.8) and (2.9).

Comments

1. **Direct sum**

(4.57) is the direct sum of the matrices (4.36) and (4.38).

(4.58) is the direct sum of the matrices (4.37) and (4.39).

2. **Reducible representation**

(4.57) and (4.58) generate a reducible representation of $SL(2,c)$.

Transformation matrices generated by (4.57) and (4.58) are denoted by

$$D^{\frac{1}{2}0}(\Lambda) \oplus D^{0\frac{1}{2}}(\Lambda) \tag{4.59}$$

where $\oplus$ denotes direct sum.

3. Spin

It follows from (4.57) that

$$\vec{J} \cdot \vec{J} = s(s+1)\hbar^2 \qquad \text{where} \qquad s = \frac{1}{2} \tag{4.60}$$

4. Generators and γ-matrices

It follows using (4.6) and (4.7) that (4.57) and (4.58) may be written as

$$M^{\mu\nu} = \frac{i\hbar}{4}[\gamma^\mu, \gamma^\nu] \tag{4.61}$$

where the γ-matrices are in the Weyl representation (A.34) to (A.36).

Since (4.61) satisfies the Lorentz Algebra (4.13) using only (A.24) it follows that (4.61) holds for every representation of the γ-matrices.

5. Transformation properties of γ-matrices

It follows on calculation that

$$L^3(u)\gamma^5\left[L^3(u)\right]^{-1} = \gamma^5 \tag{4.62}$$

and

$$L^3(u)\gamma^\mu\left[L^3(u)\right]^{-1} = l^{3\mu}{}_\nu\gamma^\nu \tag{4.63}$$

where

$$L^3(u) = \begin{pmatrix} e^{-u/2} & 0 & 0 & 0 \\ 0 & e^{u/2} & 0 & 0 \\ 0 & 0 & e^{u/2} & 0 \\ 0 & 0 & 0 & e^{-u/2} \end{pmatrix} \tag{4.64}$$

as per (4.54) and (4.59) and where $l^{3\mu}{}_\nu$ are the matrix elements of the Lorentz transformation matrix (A.7)

That is, the γ^μ transform under Lorentz boosts generated by (4.58) as a contravariant vector.

In general,

$$S(\Lambda)\gamma^\mu S(\Lambda)^{-1} = \Lambda^\mu{}_\nu\gamma^\nu \tag{4.65}$$

where $S(\Lambda)$ is a 4×4 representation of $SL(2,c)$ and the $\Lambda^\mu{}_\nu$ characterize rotations and Lorentz boosts as in (3.2).

(4.65) shows the importance of the γ-matrices in a description of spin $\frac{1}{2}$ systems.

6. Dirac particle

γ-matrices arise again in *QLB: Some Lorentz Invariant Systems* when we discuss the quantum mechanics of a Dirac particle.

γ-matrices were first considered by Dirac in his discussion of the Dirac particle. We see from the above, however, that they arise more generally via the 4×4 matrix representation (4.59) of $SL(2,c)$.

4×4 matrix representation of the Poincare Algebra

It follows using (A.24) and (A.25) that the matrices (4.61) along with the four matrices

$$P^{\mu} = \left(1 \pm \gamma^5\right)\gamma^{\mu} \tag{4.66}$$

satisfy the Poincare Algebra (4.11) to (4.13).

Comments

1. **Invariance of the Poincare generators**

   Since (4.61) and (4.66) satisfy the Poincare Algebra using only (A.24) and (A.25), it follows that (4.61) and (4.66) hold for every representation of the γ-matrices.

2. **Invariant mass**

   It follows from (4.14) that

$$M = 0 \tag{4.67}$$

3. **Helicity, chirality**

It follows on calculation that (6.36) holds with

$$\Lambda = \mp\frac{3}{2}\hbar \tag{4.68}$$

where Λ is the helicity.

The coefficient of γ^5 in (4.68) is related to the sign of the helicity; in this context γ^5 is called the chirality operator. The magnitude of the helicity is unexplained.

4.5 Some derivations

Derivation of $[J^2, P^1] = -i\hbar\ P^3$

The operator

$$e^{iP^1a/\hbar}e^{iJ^2\theta/\hbar}e^{-iP^1a/\hbar}e^{-iJ^2\theta/\hbar} \tag{4.69}$$

corresponds to the sequence of transformations

$$(x,y,z,t) \to (x',y,z',t) \to (x'',y,z',t) \to (x''',y,z'',t) \to (x'''',y,z'',t) \tag{4.70}$$

where

$$x' = x\cos\theta + z\sin\theta \tag{4.71}$$

$$z' = -x\sin\theta + z\cos\theta \tag{4.72}$$

$$x'' = x' + a \tag{4.73}$$

$$x''' = x''\cos\theta - z'\sin\theta \tag{4.74}$$

$$z'' = x'' \sin\theta + z' \cos\theta \tag{4.75}$$

$$x'''' = x''' - a \tag{4.76}$$

In the limit when a and θ are infinitesimal, the net transformation is

$$(x, y, z, t) \rightarrow (x, y, z + a\theta, t) \tag{4.77}$$

which is a displacement by $a\theta$ along the z-axis. Expanding the corresponding unitary operators

$$e^{iP^1a/\hbar}e^{iJ^2\theta/\hbar}e^{-iP^1a/\hbar}e^{-iJ^2\theta/\hbar} = 1 + \frac{a\theta}{\hbar^2}[J^2, P^1] + \cdots \tag{4.78}$$

$$e^{-iP^3a\theta/\hbar} = 1 - \frac{ia\theta}{\hbar}P^3 + \cdots \tag{4.79}$$

yields

$$[J^2, P^1] = -i\hbar P^3 \tag{4.80}$$

Derivation of $[K^1, P^1] = -i\hbar\, H/c^2$

The operator

$$e^{iP^1a/\hbar}e^{icK^1u/\hbar}e^{-iP^1a/\hbar}e^{-icK^1u/\hbar} \tag{4.81}$$

corresponds to the sequence of transformations

$$(x, y, z, t) \rightarrow (x', y, z, t') \rightarrow (x'', y, z, t') \rightarrow (x''', y, z, t'') \rightarrow (x'''', y, z, t'') \tag{4.82}$$

where

$$x' = \gamma(x + vt) \tag{4.83}$$

$$t' = \gamma\left(t + \frac{vx}{c^2}\right) \tag{4.84}$$

$$x'' = x' + a \tag{4.85}$$

$$x''' = \gamma(x'' - vt') \tag{4.86}$$

$$t'' = \gamma\left(t' - \frac{vx''}{c^2}\right) \tag{4.87}$$

$$x'''' = x''' - a \tag{4.88}$$

In the limit when a and v are infinitesimal, the net transformation is

$$(x, y, z, t) \to \left(x, y, z, t - \frac{va}{c^2}\right) \tag{4.89}$$

which is a time translation by va/c^2. Expanding the corresponding unitary operators

$$e^{iP^1a/\hbar}e^{icK^1u/\hbar}e^{-iP^1a/\hbar}e^{-icK^1u/\hbar} = 1 + \frac{va}{\hbar^2}[K^1, P^1] + \cdots \tag{4.90}$$

$$e^{-iHva/c^2\hbar} = 1 - \frac{iva}{c^2\hbar}H + \cdots \tag{4.91}$$

yields

$$[K^1, P^1] = -i\hbar H/c^2 \tag{4.92}$$

Comment

1. Comparison with the Galilei Algebra

Regarding the corresponding expression (2.11) in the Galilei Algebra, in the limit when c is infinite the net transformation (4.89) is

$$(x, y, z, t) \to (x, y, z, t) \tag{4.93}$$

which is the identity transformation. Expanding the corresponding unitary operators yields

$$\left[K^1, P^1\right] = -i\hbar m \tag{4.94}$$

where m is a phase constant.

It is not possible to eliminate the phase constant m in the Galilei Algebra using the Jacobi identity or by redefining the Galilei generators to include m.

Derivation of $[K^1, H] = -i\hbar\ P^1$

The operator

$$e^{iH\tau/\hbar}e^{icK^1u/\hbar}e^{-iH\tau/\hbar}e^{-icK^1u/\hbar} \tag{4.95}$$

corresponds to the sequence of transformations

$$(x, y, z, t) \rightarrow (x', y, z, t') \rightarrow (x', y, z, t'') \rightarrow (x'', y, z, t''') \rightarrow (x'', y, z, t'''') \tag{4.96}$$

where

$$x' = \gamma(x + vt) \tag{4.97}$$

$$t' = \gamma\left(t + \frac{vx}{c^2}\right) \tag{4.98}$$

$$t'' = t' - \tau \tag{4.99}$$

$$x'' = \gamma(x' - vt'') \tag{4.100}$$

$$t''' = \gamma\left(t'' - \frac{vx'}{c^2}\right) \tag{4.101}$$

$$t'''' = t''' + \tau \tag{4.102}$$

In the limit when v and τ are infinitesimal, the net transformation is

$$(x, y, z, t) \to (x + v\tau, y, z, t) \tag{4.103}$$

which is a displacement by $v\tau$ along the x–axis. Expanding the corresponding unitary operators

$$e^{iH\tau/\hbar}e^{icK^1u/\hbar}e^{-iH\tau/\hbar}e^{-icK^1u/\hbar} = 1 + \frac{v\tau}{\hbar^2}\left[K^1, H\right] + \cdots \tag{4.104}$$

$$e^{-iP^1v\tau/\hbar} = 1 - \frac{iv\tau}{\hbar}P^1 + \cdots \tag{4.105}$$

yields

$$\left[K^1, H\right] = -i\hbar P^1 \tag{4.106}$$

Derivation of $\left[\,K^1, K^2\right]\ = -i\hbar\ J^3/c^2$

The operator

$$e^{icK^2u_2/\hbar}e^{icK^1u_1/\hbar}e^{-icK^2u_2/\hbar}e^{-icK^1u_1/\hbar} \tag{4.107}$$

corresponds to the sequence of transformations

$$(x, y, z, t) \to \left(x', y, z, t'\right) \to \left(x', y', z, t''\right) \to \left(x'', y', z, t'''\right) \to \left(x'', y'', z, t''''\right) \tag{4.108}$$

where

$$x' = \gamma_1(x + v_1 t) \tag{4.109}$$

$$t' = \gamma_1\left(t + \frac{v_1 x}{c^2}\right) \tag{4.110}$$

$$y' = \gamma_2\left(y + v_2 t'\right) \tag{4.111}$$

$$t'' = \gamma_2\left(t' + \frac{v_2 y}{c^2}\right) \tag{4.112}$$

$$x'' = \gamma_1\left(x' - v_1 t''\right) \tag{4.113}$$

$$t''' = \gamma_1\left(t'' - \frac{v_1 x'}{c^2}\right) \tag{4.114}$$

$$y'' = \gamma_2(y' - v_2 t''') \tag{4.115}$$

$$t'''' = \gamma_2\left(t''' - \frac{v_2 y'}{c^2}\right) \tag{4.116}$$

In the limit when v_1 and v_2 infinitesimal, the net transformation is

$$(x, y, z, t) \rightarrow \left(x - \frac{v_1 v_2 y}{c^2}, y + \frac{v_1 v_2 x}{c^2}, z, t\right) \tag{4.117}$$

which is a rotation by

$$v_1 v_2 / c^2 \tag{4.118}$$

about the 3–axis. Expanding the corresponding unitary operators

$$e^{icK^2 u_2/\hbar} e^{icK^1 u_1/\hbar} e^{-icK^2 u_2/\hbar} e^{-icK^1 u_1/\hbar} = 1 + \frac{v_1 v_2}{\hbar^2}\left[K^1, K^2\right] + \cdots \tag{4.119}$$

$$e^{-iJ^3 v_1 v_2/c^2\hbar} = 1 - \frac{i v_1 v_2}{c^2\hbar} J^3 + \cdots \tag{4.120}$$

yields

$$\left[K^1, K^2\right] = -i\hbar J^3/c^2 \tag{4.121}$$

Comment

1. Comparison with the Galilei Algebra

Regarding the corresponding expression (2.12) in the Galilei Algebra, in the limit when c is infinite the net transformation (4.117) is

$$(x, y, z, t) \rightarrow (x, y, z, t) \tag{4.122}$$

which is the identity transformation. Expanding the corresponding unitary operators yields

$$[K^1, K^2] = b_{12} \tag{4.123}$$

where b_{12} is a phase constant. This phase constant may be shown to vanish using the Jacobi identity.

Chapter 5 SPACE INVERSION AND TIME REVERSAL

In this chapter we extend the set of transformations of apparatuses considered in Chapter 3 to include space inversion and time reversal.

The space inversion transformation corresponds to viewing the preparation and measurement processes in a mirror. The time reversal transformation corresponds to viewing a motion picture of the preparation and measurement process in reverse. In this chapter we explore consequences of assuming that Lorentz invariant systems are also space inversion invariant and time reversal invariant.

Unlike the Poincare transformations (3.24) to (3.27), the space inversion and time reversal transformations cannot be characterized by continuous transformations from the identity. A consequence is that the time reversal operator is an antilinear antiunitary operator.

Space inversion is discussed in Section 5.1 and time reversal is discussed in Section 5.2. Some derivations are given in Section 5.3.

5.1 Space inversion

The space inversion transformation is the following transformation of space-time points

$$(\vec{x}, t) \rightarrow \left(\vec{x'}, t'\right) = (-\vec{x}, t) \tag{5.1}$$

on a preparation or measurement apparatus.

The original preparation apparatus prepares $\mid \psi >$ and space-inverted preparation apparatus prepares $\mid \psi_{inv} >$. The original measuring apparatus measures A and the space-inverted measuring apparatus measures A_{inv}.

A physical system is space-inversion invariant if the same average is obtained for the results of measurements of every observable of the system in every state of the system when the space-inversion transformation is carried out on both the preparation apparatus and the measuring apparatus.

That is, a physical system is space-inversion invariant if

$$< \psi_{inv} \mid A_{inv} \mid \psi_{inv} > = < \psi \mid A \mid \psi > \qquad (5.2)$$

for every observable A and every state $\mid \psi >$ of the system.

(5.2) holds if

$$\mid \psi_{inv} > = \mathrm{P} \mid \psi > \qquad (5.3)$$

$$A_{inv} = \mathrm{P} A \mathrm{P}^\dagger \qquad (5.4)$$

$$\mathrm{P}\mathrm{P}^\dagger = \mathrm{P}^\dagger \mathrm{P} = 1 \qquad (5.5)$$

where P is the operator on the Hilbert space which corresponds to (5.1). Since the product of successive space inversions leaves apparatuses unchanged

$$\mathrm{P}^2 = 1 \qquad (5.6)$$

so

$$P^\dagger = P \tag{5.7}$$

P is either a linear unitary operator or an antilinear antiunitary operator.

Parity

The eigenvalues of P are ± 1. Eigenvectors of P belonging to eigenvalue $+1$ are said to have positive parity; eigenvectors of P belonging to eigenvalue -1 are said to have negative parity.

Parity projectors

We define parity projectors $P_\pm$ by

$$P_\pm = \frac{1}{2}(1 \pm P) \tag{5.8}$$

Then

$$P_+ + P_- = 1 \tag{5.9}$$

$$(P_\pm)^2 = P_\pm \tag{5.10}$$

$$P_\pm P_\mp = 0 \tag{5.11}$$

Comments

1. **Positive and negative parity components of a state**

It follows from (5.9) to (5.11) that a state $| \psi >$ may be written as

$$| \psi >=| \psi_+ > + | \psi_- > \tag{5.12}$$

where

$$| \psi_\pm >= \mathrm{P}_\pm | \psi > \tag{5.13}$$

$| \psi_\pm >$ are eigenvectors of P belonging to eigenvalues ± 1.

$| \psi_+ >$ and $| \psi_- >$ are, repectively, the positive and negative parity components of $| \psi >$.

2. **Observable invariant under space inversion**

An observable A is invariant under space inversion if

$$\mathrm{P}A\mathrm{P}^\dagger = A \tag{5.14}$$

It follows from (5.12) and (5.14) that

$$< \phi_+ | A | \psi_- >=< \phi_- | A | \psi_+ >= 0 \tag{5.15}$$

for any states $| \phi >$ and $| \psi >$.

That is, a space-inversion invariant observable does not couple states of opposite parity.

Transformation of the unitary Poincare operators

We show in Section 5.3 that the evolution operator (3.24), displacement operator (3.25), rotation operator (3.26) and Lorentz boost operator (3.27) transform under space inversion as follows:

$$\mathrm{P}U(t)\mathrm{P}^{\dagger} = U(t) \tag{5.16}$$

$$\mathrm{P}D^{j}(a)\mathrm{P}^{\dagger} = D^{j}(-a) \tag{5.17}$$

$$\mathrm{P}R^{j}(\theta)\mathrm{P}^{\dagger} = R^{j}(\theta) \tag{5.18}$$

$$\mathrm{P}L^{j}(u)\mathrm{P}^{\dagger} = L^{j}(-u) \tag{5.19}$$

Transformation of the Poincare generators

We show in Section 5.3 that P is linear and that the Poincare generators transform under space inversion as follows

$$\mathrm{P}H\mathrm{P}^{\dagger} = H \tag{5.20}$$

$$\mathrm{P}\vec{P}\mathrm{P}^{\dagger} = -\vec{P} \tag{5.21}$$

$$\mathrm{P}\vec{J}\mathrm{P}^{\dagger} = \vec{J} \tag{5.22}$$

$$\mathrm{P}\vec{K}\mathrm{P}^{\dagger} = -\vec{K} \tag{5.23}$$

provided H and $-H$ have different spectra. (Most Hamiltonians of interest satisify this proviso; it holds if the spectrum of H is bounded below and but above.)

Comments

1. **Conservation of parity**

   (5.20) states that the Hamiltonian is invariant under space inversion; parity is conserved.

2. **Vectors and pseudovectors**

   In view of the signs of the right sides of (5.21) to (5.23), $\vec{P}$ and $\vec{K}$ transform under space inversion like vectors and $\vec{J}$ transforms like a pseudovector.

Space-inverted state $\mid \psi_{inv}(t) >$

We recall that the original preparation apparatus prepares $\mid \psi >$ and the space-inverted apparatus prepares $\mid \psi_{inv} >$. These are two different states of the system. They are related according to (5.3).

The states evolve under the influence of the Hamiltonian H and at time t are

$$| \psi(t) >= U(t) | \psi > \tag{5.24}$$

$$| \psi_{inv}(t) >= U(t) | \psi_{inv} > \tag{5.25}$$

where $U(t)$ is the evolution operator (3.24).

It follows using (5.16) that

$$| \psi_{inv}(t) >= \mathrm{P} | \psi(t) > \tag{5.26}$$

(5.26) says that the space-inverted state at time t is obtained by performing the space inversion operation on the originally prepared state at time t.

5.2 Time reversal

The time reversal transformation is the following transformation

$$(\vec{x}, t) \rightarrow \left(\vec{x'}, t'\right) = (\vec{x}, -t) \tag{5.27}$$

of space-time points on a preparation or measurement apparatus.

The original preparation apparatus prepares $| \psi >$ and the time-reversed preparation apparatus prepares $| \psi_{rev} >$. The original measuring apparatus measures A and the time-reversed measuring apparatus measures A_{rev}.

A physical system is time-reversal invariant if the same average is obtained for the results of measurements of every observable of the system in every state of the system when the time—reversal transformation is carried out on both the preparation apparatus and the measuring apparatus.

That is, a physical system is time-reversal invariant if

$$< \psi_{rev} \mid A_{rev} \mid \psi_{rev} >=< \psi \mid A \mid \psi > \tag{5.28}$$

for every observable A and every state $\mid \psi >$ of the system.

(5.28) holds if

$$\mid \psi_{rev} >= \mathrm{T} \mid \psi > \tag{5.29}$$

$$A_{rev} = \mathrm{T} A \mathrm{T}^{\dagger} \tag{5.30}$$

$$\mathrm{T}\mathrm{T}^{\dagger} = \mathrm{T}^{\dagger}\mathrm{T} = 1 \tag{5.31}$$

where T is the operator on the Hilbert space which corresponds to (5.27). Since the product of successive time reversals leaves apparatuses unchanged

$$\mathrm{T}^2 = 1 \tag{5.32}$$

so

$$T^\dagger = T \qquad (5.33)$$

T is either a linear unitary operator or an antilinear antiunitary operator.

Transformation of the unitary Poincare operators

We show in Section 5.3 that the evolution operator (3.24), displacement operator (3.25), rotation operator (3.26) and Lorentz boost operator (3.27) transform under time reversal as follows

$$\mathrm{T}U(t)\mathrm{T}^\dagger = U(-t) \qquad (5.34)$$

$$\mathrm{T}D^j(a)\mathrm{T}^\dagger = D^j(a) \qquad (5.35)$$

$$\mathrm{T}R^j(\theta)\mathrm{T}^\dagger = R^j(\theta) \qquad (5.36)$$

$$\mathrm{T}L^j(u)\mathrm{T}^\dagger = L^j(-u) \qquad (5.37)$$

Transformation of the Poincare generators

We show in Section 5.3 that T is antilinear and that the Poincare generators transform under time reversal as follows

$$\mathrm{T} H \mathrm{T}^{\dagger} = H \tag{5.38}$$

$$\mathrm{T} \vec{P} \mathrm{T}^{\dagger} = -\vec{P} \tag{5.39}$$

$$\mathrm{T} \vec{J} \mathrm{T}^{\dagger} = -\vec{J} \tag{5.40}$$

$$\mathrm{T} \vec{K} \mathrm{T}^{\dagger} = \vec{K} \tag{5.41}$$

provided H and $-H$ have different spectra.

Comments

1. **Invariance under time reversal**

   (5.38) states that the Hamiltonian is invariant under time reversal.

2. **Motion reversal operator**

   In view of (5.39) T is also called the motion reversal operator.

Time-reversed state $\mid \psi_{rev}(t) >$

We recall that the original preparation apparatus prepares $\mid \psi >$ and the time-reversed preparation apparatus prepares $\mid \psi_{rev} >$. These are two different states of the system. They are related according to (5.29).

The states evolve under the influence of the Hamiltonian H and at time t are

$$| \psi(t) >= U(t) | \psi > \tag{5.42}$$

$$| \psi_{rev}(t) >= U(t) | \psi_{rev} > \tag{5.43}$$

where $U(t)$ is the evolution operator (3.24).

It follows using (5.16) and (5.34) that

$$| \psi_{rev}(t) >= \mathrm{T} | \psi(-t) > \tag{5.44}$$

Comments

1. **Time-reversed state at time t**

   (5.44) says that the time-reversed state at time t by obtained by performing the time reversal operation on the originally prepared state at time $-t$.

2. **Meaning of $| \psi(-t) >$**

   It is necessary to clarify the meaning of $| \psi(-t) >$. $| \psi(t) >$ has evolved from the state $| \psi >$ prepared at time zero. It can also be regarded, however, as having evolved under the influence of H from a state prepared at an earlier time, and, in particular, from $| \psi(-t) >$.

5.3 Some derivations

Derivation of (5.16)

The left side of (5.16) corresponds to a space inversion followed by a time translation by t followed by another space inversion. That is, it corresponds to the following sequence of spacetime transformations:

$$(\vec{x},\tau) \to (-\vec{x},\tau) \to (-\vec{x},\tau - t) \to (\vec{x},\tau - t) \tag{5.45}$$

The resultant transformation is a time translation by t, as given by the right side of (5.16).

Derivation of (5.17)

The left side of (5.17) corresponds to a space inversion followed by a space displacement by a along the j-axis followed by another space inversion. That is, it corresponds to the following sequence of spacetime transformations:

$$(\vec{x},t) \to (-\vec{x},t) \to (-\vec{x} + a\vec{u}_j,t) \to (\vec{x} - a\vec{u}_j,t) \tag{5.46}$$

The resultant transformation is a space displacement by $-a$ along the j-axis, as given by the right side of (5.17).

Derivation of (5.18)

The left side of (5.18) corresponds to a space inversion followed by a rotation by θ about the j-axis followed by another space inversion. That is, for $j = 1$ it corresponds to the following sequence of spacetime transformations:

$$(x,y,z,t) \to (x',y',z',t') \to (x'',y'',z'',t'') \to (x''',y''',z''',t''') \tag{5.47}$$

where

$$(x',y',z',t') = (-x,-y,-z,t) \tag{5.48}$$

$$(x'',y'',z'',t'') = (x',y'\cos\theta + z'\sin\theta, -y'\sin\theta + z'\cos\theta, t') \tag{5.49}$$

$$(x''', y''', z''', t''') = (-x'', -y'', -z'', t'') =$$

$$= (x, y\cos\theta + z\sin\theta, -y\sin\theta + z\cos\theta, t) \tag{5.50}$$

The resultant transformation is a rotation by θ about the 1-axis, as given by the right side of (5.18).

Derivation of (5.19)

The left side of (5.19) corresponds to a space inversion followed by a boost by v along the j-axis followed by another space inversion. That is, for $j = 1$ it corresponds to the following sequence of spacetime transformations:

$$(x, y, z, t) \rightarrow (x', y', z', t') \rightarrow (x'', y'', z'', t'') \rightarrow (x''', y''', z''', t''') \tag{5.51}$$

where

$$(x', y', z', t') = (-x, -y, -z, t) \tag{5.52}$$

$$(x'', y'', z'', t'') = \left(\gamma(x' + vt'), y', z', \gamma\left(t' + \frac{vx'}{c^2}\right)\right) \tag{5.53}$$

$$(x''', y''', z''', t''') = (-x'', -y'', -z'', t'') =$$

$$= \left(\gamma(x - vt), y, z, \gamma\left(t - \frac{vx}{c^2}\right)\right) \tag{5.54}$$

The resultant transformation is a boost by $-v$ along the 1-axis, as given by the right side of (5.19).

Proof that P is linear

Since

$$\mathrm{P}U(t)\mathrm{P}^\dagger = \mathrm{P}e^{-iHt/\hbar}\mathrm{P}^\dagger = e^{-\mathrm{P}iH\mathrm{P}^\dagger t/\hbar} \tag{5.55}$$

it follows that (5.16) can be written as

$$e^{-\mathrm{P}iH\mathrm{P}^\dagger t/\hbar} = e^{-iHt/\hbar} \tag{5.56}$$

Therefore

$$\mathrm{P}H\mathrm{P}^\dagger = H \tag{5.57}$$

if P is linear and

$$\mathrm{P}H\mathrm{P}^\dagger = -H \tag{5.58}$$

if P is antilinear.

Now $\mathrm{P}H\mathrm{P}^\dagger$ and H have the same spectra so (5.58) cannot hold if H and $-H$ have different spectra. P must therefore be linear.

Derivation of (5.21) to (5.23)

(5.21) to (5.23) follow from (5.17) to (5.19).

Applying $\mathrm{P}\cdots\mathrm{P}^\dagger$ to both sides of the Poincare Algebra (2.1) to (2.9) yields consistent results, of course, when (5.20) to (5.23) are used.

Derivation of (5.34)

The left side of (5.34) corresponds to a time reversal followed by a time translation by t followed by another time reversal. That is, it corresponds to the following sequence of spacetime transformations:

$$(\vec{x},\tau) \to (\vec{x},-\tau) \to (\vec{x},-\tau-t) \to (\vec{x},\tau+t) \tag{5.59}$$

The resultant transformation is a time translation by $-t$ as given by the right side of (5.34).

Derivation of (5.35) and (5.36)

(5.35) and (5.36) follow immediately since they involve space displacements and rotations and these transformations are unaffected by (5.27).

Derivation of (5.37)

The left side (5.37) corresponds to a time reversal followed by a Lorentz boost by v followed by another time reversal. That is, for $j = 1$ it corresponds to the following sequence of spacetime transformations:

$$(x, y, z, t) \to (x', y', z', t') \to (x'', y'', z'', t'') \to (x''', y''', z''', t''') \tag{5.60}$$

where

$$(x', y', z', t') = (x, y, z, -t) \tag{5.61}$$

$$(x'', y'', z'', t'') = \left(\gamma(x' + vt'), y', z', \gamma\left(t' + \frac{vx'}{c^2}\right)\right) \tag{5.62}$$

$$\begin{aligned}(x''', y''', z''', t''') &= (x'', y'', z'', -t'') = \\ &= \left(\gamma(x - vt), y, z, \gamma\left(t - \frac{vx}{c^2}\right)\right)\end{aligned} \tag{5.63}$$

The resultant transformation is a Lorentz boost by $-v$, as given by the right side of (5.37).

Proof that T is antilinear

Since

$$\mathrm{T}U(t)\mathrm{T}^\dagger = \mathrm{T}e^{-iHt/\hbar}\mathrm{T}^\dagger = e^{-\mathrm{T}iH\mathrm{T}^\dagger t/\hbar} \tag{5.64}$$

it follows that (5.34) can be written as

$$e^{-\mathrm{T}iH\mathrm{T}^\dagger t/\hbar} = e^{iHt/\hbar} \tag{5.65}$$

Therefore

$$\mathrm{T}H\mathrm{T}^\dagger = -H \tag{5.66}$$

if T is linear and

$$\mathrm{T}H\mathrm{T}^\dagger = H \tag{5.67}$$

if T is antilinear.

Now $\mathrm{T}H\mathrm{T}^\dagger$ and H have the same spectra so (5.66) cannot hold if H and $-H$ have different spectra. T must therefore be antilinear.

Derivation of (5.39) to (5.41)

(5.39) to (5.41) follow from (5.35) to (5.37).

Applying $\mathrm{T}\cdots\mathrm{T}^\dagger$ to both sides of the Poincare Algebra (2.1) to (2.9) yields consistent results, of course, when (5.38) to (5.41) are used.

Chapter 6 CENTRE OF MASS POSITION AND INTERNAL ANGULAR MOMENTUM

In this chapter we define operators for the centre of mass position and internal angular momentum of a Lorentz invariant physical system. These operators are defined in terms of the Poincare generators for the system. The motivation for defining centre of mass position and internal angular momentum comes from experience in classical mechanics and in nonrelativistic quantim mechanics where it has been found useful to separate the motion of a system into motion of the centre of mass and motion of the system relative to the centre of mass. We carry out the same separation for relativistic quantum mechanics.

Conditions on the centre of mass position and internal angular momentum are given in Section 6.1 and definitions are given in Section 6.2. The helicity of a system is defined in Section 6.3 and some Poincare transformations are given in Section 6.4. Some derivations are given in Section 6.5.

6.1 Conditions and commutation relations

Centre of mass position $\vec{\widehat{X}}$ and internal angular momentum $\vec{\widehat{S}}$ of a system are defined in terms of the Poincare generators for the system in order that the requirements listed below are satisfied.

1. **Heisenberg's Uncertainty Relation**

It is required that Heisenberg's Uncertainty Relation holds for centre of mass motion:

$$\Delta \widehat{X}^j \Delta P^k \geq \frac{1}{2}\hbar\delta_{jk} \tag{6.1}$$

2. **Centre of mass velocity**

We show in Section 6.5 that the centre of mass velocity $\vec{V}$ of the system is defined as

$$i\hbar\vec{V} = \left[\vec{\hat{X}}, H\right] \tag{6.2}$$

It is required that (6.2) yields

$$\vec{V} = \frac{c^2\vec{P}}{H} \tag{6.3}$$

It follows from (6.3) that

$$\vec{\hat{X}}(t) = \vec{\hat{X}} + \vec{V}t \tag{6.4}$$

where $\vec{\hat{X}}(t)$ is the centre of mass position in the Heisenberg picture.

3. **Orbital angular momentum**

It is required that the total angular momentum $\vec{J}$ of the system can be written as

$$\vec{J} = \vec{\hat{L}} + \vec{\hat{S}} \tag{6.5}$$

where

$$\vec{\hat{L}} = \vec{\hat{X}} \times \vec{P} \tag{6.6}$$

$\vec{\hat{L}}$ is the orbital angular momentum of the centre of mass of the system.

4. **Constant of the motion**

It is required that $\vec{\hat{S}}$ be a constant of the motion.

5. **Compatibility**

It is required that $\widehat{X}^j$ and $\widehat{X}^k$, $\widehat{S}^j$ and $\widehat{X}^k$, and $\widehat{S}^j$ and $\widehat{P}^k$ be compatible.

Commutation relations

(6.1) and the compatibility conditions hold if

$$\left[\widehat{X}^j, \widehat{X}^k\right] = 0 \tag{6.7}$$

$$\left[\widehat{X}^j, P^k\right] = i\hbar\delta_{jk} \tag{6.8}$$

$$\left[\widehat{S}^j, \widehat{X}^k\right] = 0 \tag{6.9}$$

$$\left[\widehat{S}^j, P^k\right] = 0 \tag{6.10}$$

It follows from (2.3) and (2.5) that

$$\left[J^j, \widehat{L}^k\right] = i\hbar\epsilon_{jkl}\widehat{L}^l \tag{6.11}$$

$$\left[J^j, \widehat{S}^k\right] = i\hbar\epsilon_{jkl}\widehat{S}^l \tag{6.12}$$

and

$$\left[\widehat{L}^j, \widehat{L}^k\right] = i\hbar\epsilon_{jkl}\widehat{L}^l \tag{6.13}$$

$$\left[\widehat{S}^j, \widehat{S}^k\right] = i\hbar\epsilon_{jkl}\widehat{S}^l \tag{6.14}$$

$$\left[\widehat{L}^j, \widehat{S}^k\right] = 0 \tag{6.15}$$

$\vec{\widehat{S}}$ is a constant of the motion if

$$\left[\vec{\widehat{S}}, H\right] = 0 \tag{6.16}$$

6.2 Definitions

Centre of mass position $\vec{\widehat{X}}$ and internal angular momentum $\vec{\widehat{S}}$ of a system are defined in terms of the Poincare generators for the system as

$$\vec{\hat{X}} = -\frac{c^2}{2}\left(\frac{1}{H}\vec{K} + \vec{K}\frac{1}{H}\right) - \frac{c}{(E + Mc^2)MH}\vec{P} \times \vec{W} \qquad (6.17)$$

$$\vec{\hat{S}} = \frac{1}{Mc}\left(\frac{H}{E}\vec{W} - \frac{c}{E + Mc^2}W^0\vec{P}\right) \qquad (6.18)$$

where

$$E = \sqrt{H^2} = \sqrt{P^2c^2 + M^2c^4} \qquad (6.19)$$

where M is the invariant mass (4.14) and W^μ is the Pauli-Lubanski four-vector (4.16).

Comments

1. **Satisfying the requirements**

   $\vec{\hat{X}}$ and $\vec{\hat{S}}$ given by (6.17) and (6.18) satisfy (6.3), (6.5) and (6.7) to (6.16).

2. **Pryce-Newton-Wigner position operator**

   (6.17) is the Pryce-Newton-Wigner position operator.

   (6.17) was derived by M.H.L. Pryce (1935) and re-invented by T.D. Newton and E.P. Wigner (1948).

3. **Lorentz booster**

(6.17) and (6.18) can be solved for $\vec{J}$ and $\vec{K}$ to yield (6.5) and

$$\vec{K} = -\frac{1}{2c^2}\left(\vec{\hat{X}}H + H\vec{\hat{X}}\right) + \frac{H}{[E(E+Mc^2)]}\vec{\hat{S}} \times \vec{P} \tag{6.20}$$

4. **Lorentz invariant**

It follows on calculation that

$$W.W = -(Mc)^2\,\vec{\hat{S}} \cdot \vec{\hat{S}} \tag{6.21}$$

That is, $\vec{\hat{S}} \cdot \vec{\hat{S}}$ is a Lorentz invariant.

5. **Single-particle system**

We show in *QLB: Some Lorentz Invariant Systems* that for a single relativistic particle with spin s

$$\vec{\hat{X}} = \vec{X} \tag{6.22}$$

$$\vec{\hat{S}} = \vec{S} \tag{6.23}$$

where $\vec{X}$ and $\vec{S}$ are the position and spin of the particle.

6. **Galilei booster**

The Galilei booster $\vec{K}_G$ for a nonrelativistic system of n particles with rest masses $m_1, m_2, ..., m_n$ is

$$\vec{K}_G = -\sum_{\alpha=1}^{n} m_\alpha \vec{X}_\alpha \tag{6.24}$$

where $\vec{X}_1, \vec{X}_2, ..., \vec{X}_n$ are the position operators for the individual particles.

7. **Nonrelativistic limit**

We show in *QLB: Some Lorentz Invariant Systems* that for a nonrelativistic system of n particles with rest masses $m_1, m_2, ..., m_n$

$$\vec{\hat{X}} = \frac{1}{m} \sum_{\alpha=1}^{n} m_\alpha \vec{X}_\alpha \tag{6.25}$$

$$\vec{\hat{S}} = \sum_{\alpha=1}^{n} \vec{S}_\alpha \tag{6.26}$$

where

$$m = \sum_{\alpha=1}^{n} m_\alpha \tag{6.27}$$

where $\vec{X}_1, \vec{X}_2, ..., \vec{X}_n$ and $\vec{S}_1, \vec{S}_2, ..., \vec{S}_n$ are the position and spin operators for the individual particles.

6.3 Helicity

Helicity Λ of a physical system is defined as

$$\Lambda = \frac{\vec{S} \cdot \vec{P}}{\sqrt{\vec{P} \cdot \vec{P}}} \tag{6.28}$$

It follows on calculation that

$$[\Lambda, H] = 0 \tag{6.29}$$

$$\left[\Lambda, \vec{P}\right] = 0 \tag{6.30}$$

$$\left[\Lambda, \vec{J}\right] = 0 \tag{6.31}$$

$$\left[\Lambda, \vec{K}\right] = i\hbar Mc\left(P\vec{S} - \Lambda\vec{P}\right) \tag{6.32}$$

where $P = \sqrt{\vec{P} \cdot \vec{P}}$.

Comments

1. **Invariances**

It follows from (6.29) to (6.32) that Λ is invariant under time translations, displacements and rotations and is not invariant under Lorentz boosts except for systems with $M = 0$. Such systems are discussed in the following topic.

2. **Eigenvalues for a single-particle system**

For a single-particle system with spin s, Λ is the projection of the intrinsic spin of the particle along the direction of the momentum of the particle.

The eigenvalues of Λ are $\lambda\hbar$ where $\lambda = s, s-1, \cdots, -s$.

3. **Values of helicity and the general state of a single-particle system**

The general state of a single-particle system with spin s and nonzero rest mass is a linear combination of states with helicity $\lambda\hbar$ where $\lambda = s, s-1, \cdots, -s$.

The above statement is not true for a single-particle system with spin s and zero rest mass as shown in the following topic.

Lorentz invariance for a system with $M = 0$

It follows from (6.29) to (6.32) that Λ commutes with all Poincare generators for a system with $M = 0$. That is, Λ is a Lorentz invariant for a system with $M = 0$. A system with $M = 0$ may be labelled by a unique value of helicity.

Second proof

The Lorentz invariance of Λ for a system with $M = 0$ follows also from consideration of the Pauli-Lubanski four-vector (4.16). It follows using (6.28) that

$$W^0 = \vec{\hat{S}} \cdot \vec{P} = \Lambda P \tag{6.33}$$

and, when $M = 0$, (6.18) yields

$$\vec{W} = \frac{W^0}{P}\vec{P} = \Lambda\vec{P} \tag{6.34}$$

where we have taken

$$H = E = \sqrt{H^2} = Pc \tag{6.35}$$

It therefore follows that

$$W^\mu = \Lambda P^\mu \tag{6.36}$$

(6.36) shows explicitly that Λ is a Lorentz invariant.

Comments

1. **Pseudoscalar**

   We show in Section 6.4 that Λ changes sign under space inversion.

   Λ is a pseudoscalar.

2. **System which does not conserve parity**

   The general state of a single-particle system with spin s and zero rest mass which is described by a Hamiltonian which is not invariant under space inversion is a state with helicity $\lambda\hbar$ where either $\lambda = s$ or $\lambda = -s$.

   Neutrinos and antineutrinos are examples of such particles $\left(s = \frac{1}{2}\right)$. Every neutrino state is an eigenstate of helicity with $\lambda = -\frac{1}{2}$; every antineutrino state is an eigenstate of helicity with $\lambda = \frac{1}{2}$. This point is discussed again in *QLB: Relativistic Quantum Field Theory* when a quantum field description of a Lorentz invariant system of free neutrinos and antineutrinos is described.

3. **System which conserves parity**

   The general state of a single-particle system with spin s and zero rest mass which is described by a Hamiltonian which is invariant under space inversion is a linear combination of states with helicity $\lambda\hbar$ where $\lambda = s$ or $\lambda = -s$.

A photon is such a particle ($s = 1$). This point is discussed again in *QLB: Relativistic Quantum Field Theory* when a quantum field description of a Lorentz invariant system of free photons is described.

6.4 Space-time transformations

In this section we record some space-time transformations of $\vec{\hat{X}}$, $\vec{V}$, $\vec{\hat{S}}$ and Λ and we give further properties of the space-inverted state $| \psi_{inv}(t) >$ and the time-reversed state $| \psi_{rev}(t) >$ defined in Chapter 5.

Einstein addition of velocities

It follows from (6.3) and from (3.75) to (3.78) that

$$L^j(u)V^jL^{j\dagger}(u) = \frac{V^j - v}{1 - V^jv/c^2} \tag{6.37}$$

$$L^j(u)V^kL^{j\dagger}(u) = \frac{V^k/\gamma}{1 - V^jv/c^2} \qquad (j \neq k) \tag{6.38}$$

Interpretation

The left sides of (6.37) and (6.38) are the components of the centre of mass velocity as measured in an inertial frame boosted with speed v along the j-axis.

(6.37) and (6.38) are the quantal versions of the Einstein addition of velocities formulae.

Lorentz boost of position

It follows from (6.17) that

$$L^1(u)\hat{X}^1 L^{1\dagger}(u) = \frac{1}{\gamma}\hat{X}^1 + \frac{v}{2c\gamma}\left\{\hat{X}^1, \frac{V^1/c}{1 - vV^1/c^2}\right\} \qquad (6.39)$$

$$L^1(u)\hat{X}^j L^{1\dagger}(u) = \hat{X}^j + \frac{v}{2c}\left\{\hat{X}^1, \frac{V^j/c}{1 - vV^1/c^2}\right\} \qquad (j = 2, 3) \qquad (6.40)$$

when $\vec{\hat{S}} = 0$.

Interpretation

(6.39) and (6.40) appear at first sight to be quite unfamiliar. Monahan (1995) shows how in the Heisenberg picture (6.39) and (6.40) correspond to the familiar classical results.

Wigner rotation

The internal angular momentum $\vec{\hat{S}}$ changes under a Lorentz boost because $\hat{S}^j$ depends on K^j. It follows from (6.18) that

$$L^j(u)\vec{\hat{S}}L^{j\dagger}(u) = \vec{\hat{S}}\cos\theta_W + (1 - \cos\theta_W)\left(\vec{N}\cdot\vec{\hat{S}}\right)\vec{N} + \sin\theta_W \vec{N}\times\vec{\hat{S}} \quad (6.41)$$

where

$$\vec{N} = \frac{\vec{\imath} \times \vec{P}}{|\vec{\imath} \times \vec{P}|} \tag{6.42}$$

$$\theta_W = 2\tan^{-1}\left(\frac{|\vec{\imath} \times \vec{P}|}{(H + Mc^2)\coth\frac{u}{2} - c\vec{\imath}\cdot\vec{P}}\right) \tag{6.43}$$

Comments

1. **Interpretation**

(6.41) describes a rotation of $\vec{\hat{S}}$ by angle θ_W about an axis $\vec{N}$ which is perpendicular to $\vec{\hat{S}}$ and the boost direction $\vec{\imath}$. This rotation is called a Wigner rotation.

2. **Galilei invariant systems**

The Wigner rotation is a purely relativistic phenomenon. There is no Wigner rotation in Galilei invariant systems because the Galilei booster $\vec{K}_G$ (6.24) commutes with $\vec{\hat{S}}$.

Space inversion and time reversal

The transformations of $\vec{\hat{X}}$, $\vec{V}$, $\vec{\hat{S}}$ and Λ under space inversion P and time reversal T follow from the definitions (6.3), (6.17) and (6.18) and from the transformation equations (5.20) to (5.23) and (5.38) to (5.41) for the Poincare generators.

For space inversion,

$$\mathrm{P}\vec{\vec{X}}\mathrm{P}^\dagger = -\vec{\vec{X}} \tag{6.44}$$

$$\mathrm{P}\vec{V}\mathrm{P}^\dagger = -\vec{V} \tag{6.45}$$

$$\mathrm{P}\vec{\vec{S}}\mathrm{P}^\dagger = \vec{\vec{S}} \tag{6.46}$$

$$\mathrm{P}\Lambda\mathrm{P}^\dagger = -\Lambda \tag{6.47}$$

For time reversal,

$$\mathrm{T}\vec{\vec{X}}\mathrm{T}^\dagger = \vec{\vec{X}} \tag{6.48}$$

$$\mathrm{T}\vec{V}\mathrm{T}^\dagger = -\vec{V} \tag{6.49}$$

$$\mathrm{T}\vec{\vec{S}}\mathrm{T}^\dagger = -\vec{\vec{S}} \tag{6.50}$$

$$\mathrm{T}\Lambda\mathrm{T}^\dagger = \Lambda \tag{6.51}$$

Pseudoscalar; pseudovector

Λ changes sign under space inversion and $\vec{\vec{S}}$ does not; Λ is a pseudoscalar and $\vec{\vec{S}}$ is a pseudovector.

Averages in the states $| \psi_{inv}(t) >$ and $| \psi_{rev}(t) >$

We recall from Chapter 5 that $| \psi_{inv}(t) >$ and $| \psi_{rev}(t) >$ are states which have evolved under the Hamiltonian H from states prepared at time zero by space-inverted and time-reversed preparation apparatuses, respectively. $| \psi(t) >$ has evolved under H from the state prepared at time zero by the original preparation apparatus.

It follows using (5.26) that

$$< \psi_{inv}(t) \mid \vec{\hat{X}} \mid \psi_{inv}(t) >= - < \psi(t) \mid \vec{\hat{X}} \mid \psi(t) > \tag{6.52}$$

$$< \psi_{inv}(t) \mid \vec{V} \mid \psi_{inv}(t) >= - < \psi(t) \mid \vec{V} \mid \psi(t) > \tag{6.53}$$

$$< \psi_{inv}(t) \mid \vec{\hat{S}} \mid \psi_{inv}(t) >=< \psi(t) \mid \vec{\hat{S}} \mid \psi(t) > \tag{6.54}$$

$$< \psi_{inv}(t) \mid \Lambda \mid \psi_{inv}(t) >= - < \psi(t) \mid \Lambda \mid \psi(t) > \tag{6.55}$$

It follows using (5.44) that

$$< \psi_{rev}(t) \mid \vec{\hat{X}} \mid \psi_{rev}(t) >=< \psi(-t) \mid \vec{\hat{X}} \mid \psi(-t) > \tag{6.56}$$

$$< \psi_{rev}(t) \mid \vec{V} \mid \psi_{rev}(t) >= - < \psi(-t) \mid \vec{V} \mid \psi(-t) > \tag{6.57}$$

$$< \psi_{rev}(t) \mid \vec{\hat{S}} \mid \psi_{rev}(t) >= - < \psi(-t) \mid \vec{\hat{S}} \mid \psi(-t) > \tag{6.58}$$

$$< \psi_{rev}(t) \mid \Lambda \mid \psi_{rev}(t) >=< \psi(-t) \mid \Lambda \mid \psi(-t) > \tag{6.59}$$

Comments

1. **Averages in the state $\mid \psi_{inv}(t) >$**

   The average centre of mass position and velocity both have opposite signs for the states $\mid \psi_{inv}(t) >$ and $\mid \psi(t) >$.

   The average internal angular momentum has the same sign for the states $\mid \psi_{inv}(t) >$ and $\mid \psi(t) >$.

2. **Averages in the state $\mid \psi_{rev}(t) >$**

   The average centre of mass velocity and internal angular momentum both have opposite signs for the states $\mid \psi_{rev}(t) >$ and $\mid \psi(-t) >$.

   The average centre of mass position has the same sign for the states $\mid \psi_{rev}(t) >$ and $\mid \psi(-t) >$.

6.5 Some derivations

Derivation of (6.2) and (6.3)

The centre of mass velocity is the time rate of change of the centre of mass position. That is,

$$\vec{V}(t) = \frac{d}{dt}\vec{\hat{X}}(t) \tag{6.60}$$

The operators in (6.60) are in the Heisenberg picture. They are related to Schrodinger picture operators $\vec{\hat{X}}$ and $\vec{V}$ according to

$$\vec{\hat{X}}(t) = U^{\dagger}(t)\vec{\hat{X}}U(t) = e^{iHt/\hbar}\vec{\hat{X}}e^{-iHt/\hbar} \tag{6.61}$$

$$\vec{V}(t) = U^{\dagger}(t)\vec{V}U(t) = e^{iHt/\hbar}\vec{V}e^{-iHt/\hbar} \tag{6.62}$$

Carrying out the differentiation on the right side of (6.60) yields

$$\frac{d}{dt}\vec{\hat{X}}(t) = \frac{i}{\hbar}e^{iHt/\hbar}\left[H, \vec{\hat{X}}\right]e^{-iHt/\hbar} \tag{6.63}$$

and comparison with (6.62) yields (6.2).

Evaluating the right side of (6.2) using (6.17) yields (6.3).

Appendix: Some Matrices

Rotation matrices

Rotation matrices $r^1(\theta), r^2(\theta), r^3(\theta)$ are defined as

$$r^1(\theta) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\theta & \sin\theta \\ 0 & -\sin\theta & \cos\theta \end{pmatrix} \tag{A.1}$$

$$r^2(\theta) = \begin{pmatrix} \cos\theta & 0 & -\sin\theta \\ 0 & 1 & 0 \\ \sin\theta & 0 & \cos\theta \end{pmatrix} \tag{A.2}$$

$$r^3(\theta) = \begin{pmatrix} \cos\theta & \sin\theta & 0 \\ -\sin\theta & \cos\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \tag{A.3}$$

That is,

$$r^j_{ab}(\theta) = \delta_{ab}\cos\theta + \delta_{ja}\delta_{jb}(1-\cos\theta) + \epsilon_{jab}\sin\theta \tag{A.4}$$

(A.1) to (A.3) are involved in coordinate transformations under rotations.

$r^2(\beta)$ and $r^3(\gamma)$ are identical to $M(\beta)$ and $M(\gamma)$, respectively, on page 65, Rose (1957).

Lorentz transformation matrices

Lorentz transformation matrices $l^1(u), l^2(u), l^3(u)$ are defined as

$$l^1(u) = \begin{pmatrix} \cosh u & -\sinh u & 0 & 0 \\ -\sinh u & \cosh u & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.5}$$

$$l^2(u) = \begin{pmatrix} \cosh u & 0 & -\sinh u & 0 \\ 0 & 1 & 0 & 0 \\ -\sinh u & 0 & \cosh u & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \tag{A.6}$$

$$l^3(u) = \begin{pmatrix} \cosh u & 0 & 0 & -\sinh u \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ -\sinh u & 0 & 0 & \cosh u \end{pmatrix} \tag{A.7}$$

That is,

$$l^{j\mu}{}_{\nu}(u) = \delta^{\mu}{}_{\nu} + \left(\delta^{\mu}{}_{0}\delta^{\nu}{}_{0} + \delta^{\mu}{}_{j}\delta^{\mu}{}_{j}\right)(\cosh u - 1) - \left(\delta^{\mu}{}_{0}\delta^{\nu}{}_{j} + \delta^{\mu}{}_{j}\delta^{\mu}{}_{j0}\right)\sinh u \tag{A.8}$$

μ labels the rows and ν labels the columns of $l^j(u)$.

(A.5) to (A.7) are involved in coordinate transformations under Lorentz boosts.

Pauli matrices

Pauli matrices $\sigma_x, \sigma_y, \sigma_z$ are defined as

$$\sigma_x = \sigma^1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.9}$$

$$\sigma_y = \sigma^2 = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{A.10}$$

$$\sigma_z = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.11}$$

It follows that

$$\sigma^j \sigma^k = \delta_{jk} + i\epsilon_{jkl}\sigma^l \tag{A.12}$$

$$\left(\vec{\sigma} \cdot \vec{A}\right)\left(\vec{\sigma} \cdot \vec{B}\right) = \vec{A} \cdot \vec{B} + i\vec{\sigma} \cdot \left(\vec{A} \times \vec{B}\right) \tag{A.13}$$

$$\left(\vec{\sigma} \cdot \vec{A}\right)^2 = A^2 \qquad \text{if} \qquad \left[A^j, A^k\right] = 0 \tag{A.14}$$

$$e^{a\sigma^j} = \cosh a + \sigma^j \sinh a \tag{A.15}$$

Dirac matrices

Dirac matrices $\alpha^1, \alpha^2, \alpha^3, \beta$ satisfy

$$\left\{\alpha^j, \alpha^k\right\} = 2\delta_{jk} \tag{A.16}$$

$$\left\{\alpha^j, \beta\right\} = 0 \tag{A.17}$$

$$\beta^2 = 1 \tag{A.18}$$

Dirac representation:

$$\alpha^j = \begin{pmatrix} 0 & \sigma^j \\ \sigma^j & 0 \end{pmatrix} \tag{A.19}$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.20}$$

Each element in the matrices on the right side of (A.19) and (A.20) is a 2×2 matrix.

γ-matrices

γ–matrices $\gamma^0, \gamma^1, \gamma^2, \gamma^3, \gamma^5$ are related to Dirac matrices $\alpha^1, \alpha^2, \alpha^3, \beta$ by

$$\gamma^0 = \beta \tag{A.21}$$

$$\gamma^j = \beta\alpha^j \tag{A.22}$$

$$\gamma^5 = \gamma_5 = i\gamma^0\gamma^1\gamma^2\gamma^3 \tag{A.23}$$

It follows that

$$\{\gamma^\mu, \gamma^\nu\} = 2g^{\mu\nu} \tag{A.24}$$

$$\{\gamma^5, \gamma^\mu\} = 0 \tag{A.25}$$

$$\left(\gamma^0\right)^2 = -\left(\gamma^1\right)^2 = -\left(\gamma^2\right)^2 = -\left(\gamma^3\right)^2 = \left(\gamma^5\right)^2 = 1 \tag{A.26}$$

$$\gamma^{\mu\dagger} = \gamma^0\gamma^\mu\gamma^0 \tag{A.27}$$

$$\gamma^{0\dagger} = \gamma^0 \tag{A.28}$$

$$\gamma^{j\dagger} = -\gamma^j \tag{A.29}$$

$$\gamma^{5\dagger} = \gamma^5 \tag{A.30}$$

Dirac representation

$$\gamma^0 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.31}$$

$$\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{A.32}$$

$$\gamma^5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.33}$$

Weyl representation

$$\gamma^0 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{A.34}$$

$$\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{A.35}$$

$$\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix} \tag{A.36}$$

Chiral representation

$$\gamma^0 = \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix} \tag{A.37}$$

$$\gamma^j = \begin{pmatrix} 0 & \sigma^j \\ -\sigma^j & 0 \end{pmatrix} \tag{A.38}$$

$$\gamma^5 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \tag{A.39}$$

Comments

1. **Notation**

   Each element in the matrices on the right side of (A.31) to (A.39) is a 2×2 matrix.

2. **Unitary transformations**

   Let

$$U_{\pm} = \frac{1}{\sqrt{2}}\left(1 \pm \gamma^5\gamma^0\right) \tag{A.40}$$

$$V_{\pm} = U_{\pm}^2 = \pm\gamma^5\gamma^0 \tag{A.41}$$

Then

$$U_\pm^\dagger = \frac{1}{\sqrt{2}}\left(1 \mp \gamma^5\gamma^0\right) \tag{A.42}$$

$$V_\pm^\dagger = \mp\gamma^5\gamma^0 = -V_\pm \tag{A.43}$$

$$U_\pm\gamma^0 U_\pm^\dagger = \pm\gamma^5 \tag{A.44}$$

$$U_\pm\gamma^j U_\pm^\dagger = \gamma^j \tag{A.45}$$

$$U_\pm\gamma^5 U_\pm^\dagger = \mp\gamma^0 \tag{A.46}$$

$$U_\pm\gamma^5\gamma^0 U_\pm^\dagger = \gamma^5\gamma^0 \tag{A.47}$$

$$V_\pm\gamma^0 V_\pm^\dagger = -\gamma^0 \tag{A.48}$$

$$V_\pm\gamma^j V_\pm^\dagger = \gamma^j \tag{A.49}$$

$$V_\pm\gamma^5 V_\pm^\dagger = -\gamma^5 \tag{A.50}$$

$$V_\pm\gamma^5\gamma^0 V_\pm^\dagger = \gamma^5\gamma^0 \tag{A.51}$$

3. Dirac, Weyl and chiral representations

The Dirac, Weyl and chiral representations are related as follows:

$$U_+\gamma^\mu_{\text{dirac}}U_+^\dagger = \gamma^\mu_{\text{weyl}} \tag{A.52}$$

$$U_-\gamma^\mu_{\text{dirac}}U_-^\dagger = \gamma^\mu_{\text{chiral}} \tag{A.53}$$

$$V_\pm\gamma^\mu_{\text{weyl}}V_\pm^\dagger = \gamma^\mu_{\text{chiral}} \tag{A.54}$$

SELECTED REFERENCE BOOKS

M. Abramowitz and I. A. Stegun, eds., *Handbook of Mathematical Functions*, U.S. National Bureau of Standards, 1964.

G. Arfken, *Mathematical Methods for Physicists*, Academic Press, Inc., 3rd. ed., 1985.

H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One- and Two-Electron Atoms*, Academic Press, Inc., 1957.

L. Biedenharn and H. van Dam, eds., *Quantum Theory of Angular Momentum*, Academic Press, Inc., 1965.

J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics*, McGraw-Hill, Inc., 1964.

——, *Relativistic Quantum Fields*, McGraw-Hill, Inc., 1965.

N. N. Bogliubov, A. A. Logunov, and I. T. Todorov, *Introduction to Axiomatic Quantum Field Theory*, W. A. Benjamin, Inc., 1975.

F. Calogero, *Variable Phase Approach to Potential Scattering*, Academic Press, Inc., 1967.

P. A. M. Dirac, *The Principles of Quantum Mechanics*, Oxford University Press, 4th. ed., 1958.

M. L. Goldberger and K. M. Watson, *Collision Theory*, John Wiley & Sons, Inc., 1964.

R. I. G. Hughes, *The Stucture and Interpretation of Quantum Mechanics*, Harvard University Press, 1989.

C. Itzykson and J. B. Zuber, *Quantum Field Theory*, McGraw-Hill, Inc., 1980.

J. M. Jauch, *Foundations of Quantum Mechanics*, Addison-Wesley Publishing Company, Inc., 1965.

J. M. Jauch and F. Rohrlich, *The Theory of Electrons and Positrons: The Relativistic Quantum Field Theory of Charged Particles with Spin One-Half*, Addison-Wesley Publishing Company, Inc., 1955.

T. F. Jordan, *Linear Operators for Quantum Mechanics*, John Wiley & Sons, Inc., 1969.

F. A. Kaempffer, *Concepts in Quantum Mechanics*, Academic Press, Inc., 1965.

Y. S. Kim and M. E. Noz, *Theory and Applications of the Poincare Group*, D. Reidel Publishing Company, 1986.

R. H. Landau, *Quantum Mechanics II*, John Wiley & Sons, Inc., 2nd. ed., 1996.

R. D. Mattuck, *A Guide to Feynman Diagrams in the Many-Body Problem*, Dover Publications, Inc., 2nd. ed., 1976.

A. Messiah, *Quantum Mechanics I & II*, John Wiley & Sons, Inc., 1958.

P. L. Morse and H. Feshbach, *Methods for Theoretical Physics*, McGraw-Hill, Inc., 1953.

R. G. Newton, *Scattering Theory of Waves and Particles*, McGraw-Hill, Inc., 1966.

R. Omnes, *The Interpretation of Quantum Mechanics*, Princeton University Press, 1994.

D. Pines, ed., *The Many-Body Problem*, W. A. Benjamin, Inc., 1961.

F. Riesz and B. Sz.-Nagy, *Functional Analysis*, Frederick Ungar Publishing Company, Inc., 1955.

M. E. Rose, *Elementary Theory of Angular Momentum*, John Wiley & Sons, Inc., 1957.

S. S. Schweber, *An Introduction to Relativistic Quantum Field Theory*, Row, Peterson and Company, 1961.

J. Schwinger, ed., *Selected Papers on Quantum Electrodynamics*, Dover Publications, Inc., 1958.

——, *Particles, Sources and Fields*, Addison-Wesley Publishing Company, Inc., 1970.

A. E. Sommerfeld, *Partial Differential Equations in Physics*, Academic Press, Inc., 1949.

R. F. Streater and A. S. Wightman, *PCT, Spin and Statistics, and All That*, Benjamin/Cummings Publishing Company, Inc., 1964.

J. R. Taylor, *Scattering Theory: The Quantum Theory on Nonrelativistic Collisions*, John Wiley & Sons, Inc., 1972.

E. C. Titchmarsh, *The Theory of Functions*, Oxford University Press, 1939.

W.-K. Tung, *Group Theory in Physics*, World Scientific Publishing Company, Inc., 1994.

D. A. Varshalovich, A. N. Moskalev, and V. K. Khersonskii, *Quantum Theory of Angular Momentum*, World Scientific Publishing Company, Inc., 1988.

J. von Neumann, *Mathematical Foundations of Quantum Mechanics*, Princeton University Press, 1955.

S. Weinberg, *The Quantum Theory of Fields*, Cambridge University Press, 1995.

E. P. Wigner, *Group Theory and its Application to the Quantum Mechanics of Atomic Spectra*, Academic Press, Inc., 1959.

SELECTED REFERENCE JOURNAL ARTICLES

B. Bakamjian and L. H. Thomas, *Relativistic Particle Dynamics*, Phys. Rev. **92**, 1300 (1953).

J. Bardeen, L. N. Cooper, and J. R. Schrieffer, *Theory of Superconductivity*, Phys. Rev. **108**, 1175 (1957).

V. Bargmann, *Note on Wigner's Theorem on Symmetry Operators*, J. Math. Phys. **5**, 862 (1964).

——, *On Unitary Ray Representations of Continuous Groups*, Ann. Math. **59**, 1 (1954).

N. N. Bogoliubov, *A New Method in the Theory of Superconductivity*, Soviet Physics, JETP **7**, 41 (1958).

A. Bottino, A. Longoni, and T. Regge, *Potential Scattering for Complex Energy and Complex Angular Momentum*, Nuovo Cimento **23**, 954 (1962).

G. Breit and W. G. Bouricius, *Boundary Conditions and Range of Force for S State of Two Nucleons*, Phys. Rev. **75**, 1029 (1949).

G. F. Chew and S. Mandelstam, *Theory of Low-Energy Pion-Pion Interaction*, Phys. Rev. **119**, 467 (1960).

L. N. Cooper, *Bound Electron Pairs in a Degenerate Fermi Gas*, Phys. Rev. **104**, 1189 (1956).

D. G. Currie, T. F. Jordan, and E. C. G. Sudarshan, *Relativistic Invariance and Hamiltonian Theories of Interacting Particles*, Rev. Mod. Phys. **35**, 350 (1963).

P. A. M. Dirac, *A Theory of Electrons and Protons*, Proc. Roy. Soc. **A126**, 360 (1930).

——, *Forms of Relativistic Dynamics*, Rev. Mod. Phys. **21**, 392 (1949).

——, *The Conditions for a Quantum Field Theory to be Relativistic*, Rev. Mod. Phys. **34**, 592 (1962).

——, *The Quantum Theory of the Electron*, Proc. Roy. Soc. **A117**, 610 (1928).

U. Fano, *Description of States in Quantum Mechanics by Density Matrix and Operator Techniques*, Rev. Mod. Phys. **29**, 74 (1957).

H. Feshbach and E. Lomon, *Nucleon-Nucleon Scattering*, Phys. Rev. **102**, 891 (1956).

L. L. Foldy, *Relativistic Particle Systems with Interaction*, Phys. Rev. **122**, 275 (1961).

L. L. Foldy and S. A. Wouthuysen, *On the Dirac Theory of Spin 1/2 Particles and Its Nonrelativistic Limit*, Phys. Rev. **78**, 29 (1950).

R. Fong and J. Sucher, *Relativistic Particle Dynamics and the S Matrix*, J. Math. Phys. **5**, 456 (1964).

J. Goldstone, *Derivation of the Brueckner Many-Body Theory*, Proc. Roy. Soc. **A239**, 267 (1957).

O. W. Greenberg, *Haag's Theorem and Clothed Operators*, Phys. Rev. **115**, 706 (1959).

O. W. Greenberg and S. S. Schweber, *Clothed Particle Operators in Simple Models of Quantum Field Theory*, Nuovo Cimento **8**, 378 (1958).

D. H. Hearn, M. McMillan, and A. Raskin, *Dressing the Cloudy Bag Model: Second-order Nucleon-Nucleon Potential*, Phys. Rev. **C28**, 2489 (1983).

T. F. Jordan, A. J. MacFarlane, and E. C. G. Sudarshan, *Hamiltonian Model of Lorentz Invariant Particle Interactions*, Phys. Rev. **133**, B487 (1964).

W. E. Lamb and R. C. Retherford, *Fine Structure of the Hydrogen Atom by a Microwave Method*, Phys. Rev. **72**, 241 (1947).

M. McMillan, *Regge Cuts in a Nonrelativistic Three-Body Problem Involving Separable Potentials*, Nuovo Cimento **32**, 919 (1964).

——, *Separable Nonlocal Potentials and Regge Poles*, Nuovo Cimento **29**, 1043 (1963).

M. McMillan and E. L. Lomon, *The Nuclear Many-Body Problem Using the Boundary Condition Model*, Ann. Phys. (N.Y.) **23**, 439 (1963).

T. D. Newton and E. P. Wigner, *Localized States for Elementary Systems*, Rev. Mod. Phys. **21**, 400 (1949).

W. Pauli, *The Connection Between Spin and Statistics*, Phys. Rev. **58**, 716 (1940).

M. H. L. Pryce, *Commuting Coodinates in the New Field Theory*, Proc. Roy. Soc. **A150**, 166 (1935).

——, *The Mass-Centre in the Restricted Theory of Relativity and its Connexion with the Quantum Theory of Elementary Particles*, Proc. Roy. Soc. **A195**, 621 (1948).

T. Regge, *Introduction to Complex Orbital Momentum*, Nuovo Cimento **14**, 951, (1959).

F. Rohrlich, *Many-Body Forces and the Cluster Decompositon*, Phys. Rev. **D23**, 1305 (1981).

J. Schwinger, *Non-Abelian Gauge Fields. Relativistic Invariance*, Phys. Rev. **127**, 324 (1962).

E. C. G. Sudarshan, *The Nature of the Axioms of Relativistic Quantum Field Theory*, J. Math. Phys. **4**, 1029 (1963).

E. C. G. Sudarshan and K. Bardacki, *Nature of the Axioms of Relativistic Quantum Field Theory*, J. Math. Phys. **2**, 767 (1961).

L. H. Thomas, *The Relativistic Dynamics of a System of Particles Interacting at a Distance*, Phys. Rev. **85**, 868 (1952).

J. G. Valatin, *Comments on the Theory of Superconductivity*, Nuovo Cimento **7**, 843 (1958).

S. Weinberg, *Feynman Rules for Any Spin*, Phys. Rev. **B133**, 1318 (1964).

——, *Feynman Rules for Any Spin. Massless Particles*, Phys. Rev. **B134**, 882 (1964).

E. P. Wigner, *Relativistic Invariance and Quantum Phenomena*, Rev. Mod. Phys. **29**, 255 (1957).

——, *Relativistic Invariance in Quantum Mechanics*, Nuovo Cimento **3**, 517 (1956).

SELECTED REFERENCE THESES

R. Cropp, *Dressing a Model of Interacting Fermions, Antifermions and Bosons*, B.Sc. thesis, U.B.C., 1996.

D. H. Hearn, *The Dressing Transformation and its Application to a Fermion-Boson Trilinear Interaction*, M.Sc. thesis, U.B.C., 1981.

W. W. Hsieh, *A New Hamiltonian for Systems of Pions and Nucleons*, M.Sc. thesis, U.B.C., 1979.

H. N. James, *The Dressing Transformation and Scattering of Dressed Particles in a Model of Fermions and Bosons*, M.Sc. thesis, U.B.C., 1982.

P. A. Kalyniak, *Consequences of Space-Time Invariances in Quantum Mechanics and Direct Interactions Theories*, M.Sc. thesis, U.B.C., 1978.

M. McMillan, *The Nuclear Many-Body Problem Using the Boundary Condition Model*, Ph.D. thesis, McGill, 1961.

A. H. Monahan, *Relativistic Few-Body Quantum Mechanics*, M.Sc. thesis, U.B.C., 1995.