

# **QUANTUM LEAPS AND BOUNDS**

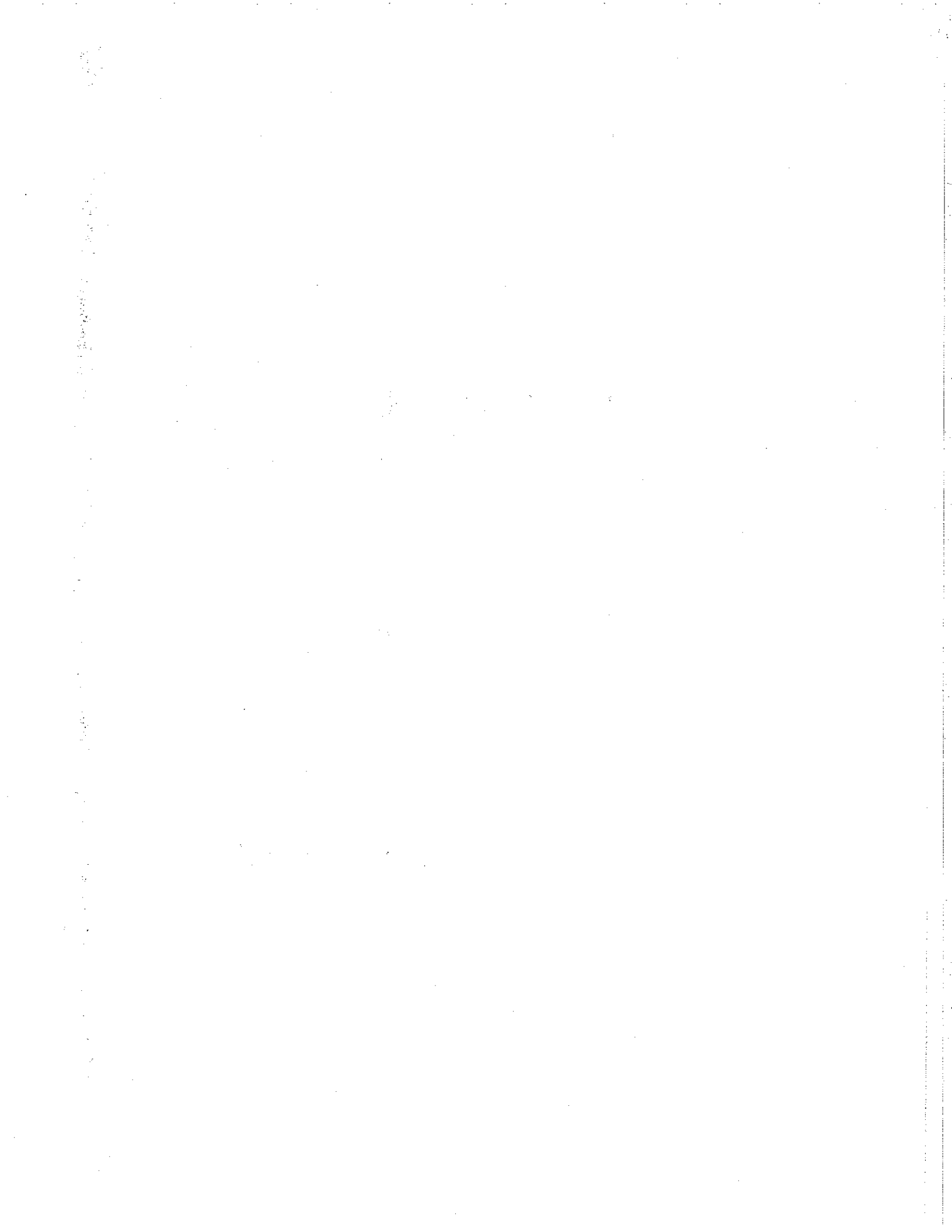
## **PART II**

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**PART II**  
**QUANTUM MECHANICS**  
**IN FOCK SPACE**



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In Part I of *Quantum Leaps and Bounds*, we explored consequences of incorporating the Principles of Special Relativity (SR) in Quantum Mechanics (QM).

Incorporating SR in QM, that is, setting up the QM of a Lorentz invariant physical system, led to the Poincare Algebra, which is a set of commutation relations for the Hamiltonian and the three components of the total momentum, total angular momentum and boost operators for the system. These commutation relations are the basic equations of relativistic QM.

The last five Chapters of Part I contain examples of Lorentz invariant systems.

Relativistic quantum field theory (RQFT) is not discussed in Part I. That is, not discussed is a physical system whose fundamental dynamical variables are quantum fields. This example does not appear in Part I because we had not developed the tools to do this. These tools are developed here in Part II.

In developing quantum field theory, we develop a language (second quantization) which provides techniques for studying nonrelativistic many-body systems.

The Lagrangian method in RQFT receives a very brief introduction in Part II. The many excellent books on RQFT should be consulted for further discussion of the Lagrangian method.<sup>1</sup>

The QM of a system of identical particles is considered in Chapter 2. Occupation number representation for systems of fermions and bosons is discussed in Chapter 3.

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<sup>1</sup> A list of selected reference books, journal articles and theses follows Chapter 12 of Part I.

The Fock space description of the QM of a system of fermions is given in Chapters 4 and 5. A derivation of the Hartree-Fock potential is given in Chapter 4. Quantum fields for fermions are introduced in Chapter 5. Chapter 5 contains expressions for the Poincare generators for a Lorentz invariant system of noninteracting fermions.

A system of noninteracting spin  $\frac{1}{2}$  fermions and antifermions is considered in Chapter 6. Fermions and antifermions are treated on the same footing; both appear in the theory with positive energy. It is shown how Dirac's hole theory arises from the finite theory developed.

The Dirac field  $\psi(x)$  is constructed in Chapter 6 and the Poincare generators are expressed in terms of a Lorentz invariant Lagrangian density. The Dirac field satisfies the Dirac equation.

The Fock space description of the QM of a system of bosons is given in Chapter 7. Quantum fields for bosons are introduced and expressions for the Poincare generators for a Lorentz invariant system of noninteracting bosons are given.

The scalar field  $\phi(x)$  is constructed in Chapter 7 and the Poincare generators are expressed in terms of a Lorentz invariant Lagrangian density. The scalar field satisfies the Klein—Gordon equation.

A Lorentz invariant system of noninteracting photons is described in Chapter 8. Quantum electric and magnetic fields are defined in terms of transverse photon creators and annihilators. The quantum electromagnetic field satisfies Maxwell's equations in free space. Each momentum component of the electromagnetic field corresponds to a transverse wave moving with the speed of light in the direction of the photon momentum.

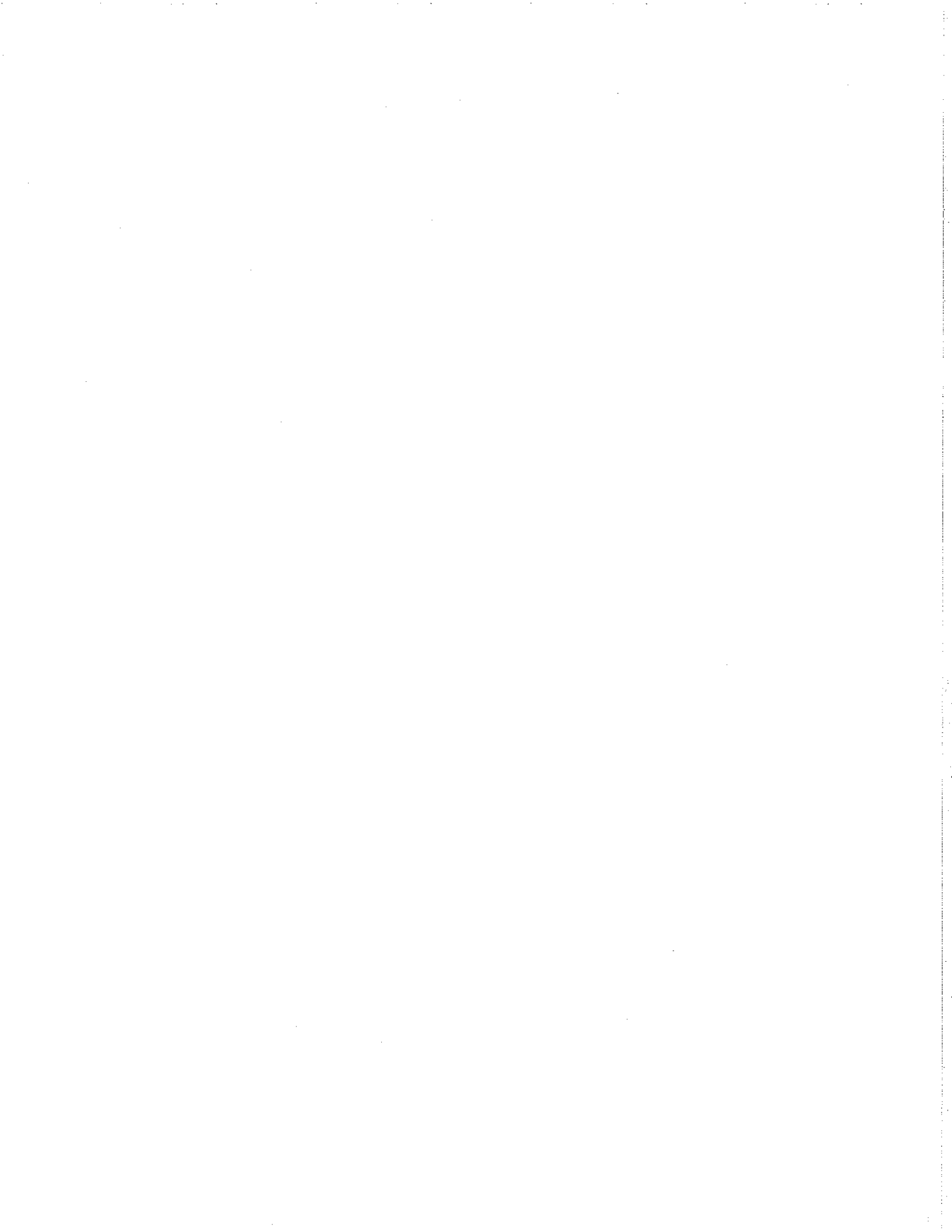
A manifestly covariant and gauge invariant theory of electromagnetism is developed in Section 8.7. The development is accomplished through introduction of creators and annihilators for fictitious longitudinal and time-like photons. The

Lagrangian for the free electromagnetic field is also given in Section 8.7.

A very brief introduction to quantum electrodynamics is given in Section 8.8.

The instant and point forms of dynamics for a system of interacting fermions and bosons are considered in Chapter 9. The dressing transformation method for expressing the Hamiltonian in terms of creators and annihilators for physical particles is given in Section 9.7. The Yukawa potential for interacting physical fermions is derived using the dressing transformation method in Topic 9.8.4.

An Appendix containing some useful commutation relations for fermion and boson variables follows Chapter 9.





## Chapter 2

## IDENTICAL PARTICLES

### Section 2.1 Introductory remarks

The quantum mechanics of a Lorentz invariant system of  $n$  distinguishable particles in interaction was considered in Chapter 11 of Part I. We now consider the case when the  $n$  particles are indistinguishable (identical).

Indistinguishability of the particles places restrictions on the form of the observables of the system and the states of the system.

### Section 2.2 $n$ particles

We consider the physical system to be a system of  $n$  particles with rest masses

$$m_1, m_2, \dots, m_n \quad (2.1)$$

and spins

$$s_1, s_2, \dots, s_n \quad (2.2)$$

The Hilbert space  ${}_n\mathfrak{H}^{s_1 s_2 \dots s_n}$  for the system is the direct product of  $n$  one-particle Hilbert spaces.

$${}_n\mathfrak{H}^{s_1 s_2 \dots s_n} = \mathfrak{H}_1^{s_1} \otimes \mathfrak{H}_1^{s_2} \otimes \dots \otimes \mathfrak{H}_1^{s_n} \quad (2.3)$$

$\mathfrak{H}_1^{s_\alpha}$  denotes the Hilbert space for particle  $\alpha$  and  $\otimes$  denotes direct product.

We take the fundamental dynamical variables of the system to be the Cartesian coordinates, momenta and spin of the individual particles

$$X_1^j, P_1^j, S_1^j, X_2^j, P_2^j, S_2^j, \dots, X_n^j, P_n^j, S_n^j \quad (2.4)$$

where  $j = 1, 2, 3$ . These variables satisfy

$$[X_\alpha^j, X_\beta^k] = 0 \quad (2.5)$$

$$[P_\alpha^j, P_\beta^k] = 0 \quad (2.6)$$

$$[X_\alpha^j, P_\beta^k] = i\hbar\delta_{\alpha\beta}\delta_{jk} \quad (2.7)$$

$$[S_\alpha^j, S_\beta^k] = i\hbar\delta_{\alpha\beta}\epsilon_{jkl}S_\alpha^l \quad (2.8)$$

$$S_\alpha^2 = s_\alpha(s_\alpha + 1)\hbar^2 \quad (2.9)$$

$$[X_\alpha^j, S_\beta^k] = [P_\alpha^j, S_\beta^k] = 0 \quad (2.10)$$

where

$$\alpha, \beta = 1, 2, \dots, n \quad (2.11)$$

The Hamiltonian for the system is

$$H = H_0 + V \quad (2.12)$$

$$H_0 = \sum_{\alpha=1}^n \sqrt{P_{\alpha}^2 c^2 + m_{\alpha}^2 c^4} \quad (2.13)$$

$$V = V(\xi_1, \xi_2, \dots, \xi_n) \quad (2.14)$$

$$\xi_{\alpha} = \{X_{\alpha}^1, P_{\alpha}^1, S_{\alpha}^1, \dots, X_{\alpha}^3, P_{\alpha}^3, S_{\alpha}^3\} \quad (2.15)$$

### Comments

1. Expression for  $H_0$

(2.13) is the relativistic expression for  $H_0$ . The nonrelativistic expression

$$H_0 = \sum_{\alpha=1}^n \frac{P_{\alpha}^2}{2m_{\alpha}} \quad (2.16)$$

is used in Section 4.6.

## 2. Interaction potential $V$

General expressions for the interaction potential  $V$  for Lorentz invariant and Galilei invariant systems are given in Chapter 11 of Part I.

$V$  need not be specified further here. A more specific expression for  $V$  is considered in Section 4.6.

## Section 2.3 $n$ identical particles

We now assume that the  $n$  particles are indistinguishable (identical). That is, we assume that there is no observable change in the system when particles are interchanged. There are two consequences of indistinguishability; they will be treated separately below.

### 2.3.1 Invariance of observables

The first consequence of indistinguishability is:

#### Consequence 1: Invariance of observables

Every observable of the system is invariant under all permutations of the particles.

#### Proof of Consequence 1

Indistinguishability of all particles of the system implies that

$$\langle \psi_{perm} | A | \psi_{perm} \rangle = \langle \psi | A | \psi \rangle \quad (2.17)$$

for all observables  $A$  and states  $|\psi\rangle$  where

$$|\psi_{perm}\rangle = \Pi |\psi\rangle \quad (2.18)$$

where  $\Pi$  is the operator corresponding to the permutation of the particles.

It follows from (2.17) and (2.18) that

$$\Pi A \Pi^\dagger = A \quad (2.19)$$

for all  $A$ . Equation (2.19) is Consequence 1.

### Comments

#### 1. Permutation operators and interchange operators

Every permutation operator  $\Pi$  can be written as a product of one or more interchange operators

$$\Pi_{\alpha\beta} \quad (2.20)$$

where  $\alpha, \beta = 1, 2, \dots, n$ .

$\Pi_{\alpha\beta}$  corresponds to the replacement

$$\xi_\alpha \leftrightarrow \xi_\beta \quad (2.21)$$

for particles  $\alpha$  and  $\beta$ .

## 2. Interchange operator for a two-particle system

For a system of two spinless particles,

$$\Pi = \Pi_{12} = \Pi_{21} \quad (2.22)$$

$$\Pi_{21} = \int d^3x_1 d^3x_2 |x_2, x_1\rangle \langle x_1, x_2| \quad (2.23)$$

For example, since

$$X_\alpha^j = \int d^3x_1 d^3x_2 |x_1, x_2\rangle x_\alpha^j \langle x_1, x_2| \quad (2.24)$$

it follows that

$$\Pi X_1^j \Pi^\dagger = X_2^j \quad (2.25)$$

$$\Pi X_2^j \Pi^\dagger = X_1^j \quad (2.26)$$

## Consequences of Consequence 1

### 1. Values of rest mass and spin

It follows from Consequence 1 that all  $n$  particles have the same rest mass  $m$  and same spin  $s$ .

### 2. Hilbert space

It follows from Consequence 1 that the Hilbert space  ${}_n\mathfrak{H}^s$  for the system consists of  $n$  copies of a one-particle Hilbert space.

$${}_n\mathfrak{H}^s = \mathfrak{H}_1^s \otimes \mathfrak{H}_1^s \otimes \cdots \otimes \mathfrak{H}_1^s \quad (2.27)$$

We write (2.27) more compactly as

$${}_n\mathfrak{H}^s = \otimes^n \mathfrak{H}_1^s \quad (2.28)$$

### 2.3.2 Symmetry of states

The second consequence of indistinguishability is

#### Consequence 2: Symmetry of states

The states of the system are either all symmetrical or all antisymmetrical with respect to the permutations of the  $n$  particles, this property depending upon the species of particle.

## Proof of Consequence 2

It follows from (2.17) that no experiment can determine whether the system is in the state  $|\psi_{perm}\rangle$  or the state  $|\psi\rangle$ . The vectors  $|\psi_{perm}\rangle$  and  $|\psi\rangle$  therefore correspond to the same ray in  $n\mathbb{K}^s$ . That is,

$$|\psi_{perm}\rangle = e^{i\delta} |\psi\rangle \quad (2.29)$$

where  $\delta$  is a real number.

It follows from properties of the symmetric group of order  $n$  that every  $|\psi\rangle$  can be written as

$$|\psi\rangle = |\psi_s\rangle + |\psi_a\rangle + |\psi_m\rangle \quad (2.30)$$

$$\Pi |\psi_s\rangle = + |\psi_s\rangle \quad (2.31)$$

$$\Pi |\psi_a\rangle = - |\psi_a\rangle \quad (2.32)$$

$|\psi_s\rangle$ ,  $|\psi_a\rangle$  and  $|\psi_m\rangle$  are mutually orthogonal and are, respectively, the symmetric, antisymmetric and mixed components of  $|\psi\rangle$ .

It follows from (2.18) and (2.30) to (2.32) that

$$|\psi_{perm}\rangle = |\psi_s\rangle - |\psi_a\rangle + \Pi |\psi_m\rangle \quad (2.33)$$



It follows from (2.30) and (2.33) that in order to satisfy (2.29),

either

$$|\psi\rangle = |\psi_s\rangle \quad (2.34)$$

in which case  $e^{i\delta} = +1$  and  $|\psi_{perm}\rangle = +|\psi\rangle$ ,

or

$$|\psi\rangle = |\psi_a\rangle \quad (2.35)$$

in which case  $e^{i\delta} = -1$  and  $|\psi_{perm}\rangle = -|\psi\rangle$ .

That is, the states of the system are either all symmetrical or all antisymmetrical, which is Consequence 2.

### Comments about Consequence 2

#### 1. Classical Mechanics

Consequence 2 has no analog in Classical Mechanics.

#### 2. Bosons and fermions

Particles whose many-particle states are symmetric are called bosons.

Particles whose many-particle states are antisymmetric are called fermions.

#### 3. Statistics and spin

Experiment shows that

Bosons have integral spin.

Fermions have half-odd integral spin.

The correspondence between the symmetry of many-particle states and the intrinsic spin of the constituent particles is a remarkable experimental fact.

#### 4. Spin-Statistics Theorem

The above correspondence between the symmetry of many-particle states and the intrinsic spin of the constituent particles has been proven in RQFT (The Spin-Statistics Theorem).

#### 5. Restrictions on the Hilbert space

Consequence 2 further restricts the Hilbert space of the system.

Let  ${}^b_n\mathfrak{H}^s$  be the  $n$ -boson Hilbert space.

Let  ${}^f_n\mathfrak{H}^s$  be the  $n$ -fermion Hilbert space.

${}^b_n\mathfrak{H}^s$  is the subspace of  ${}_n\mathfrak{H}^s$  spanned by symmetric basis vectors.

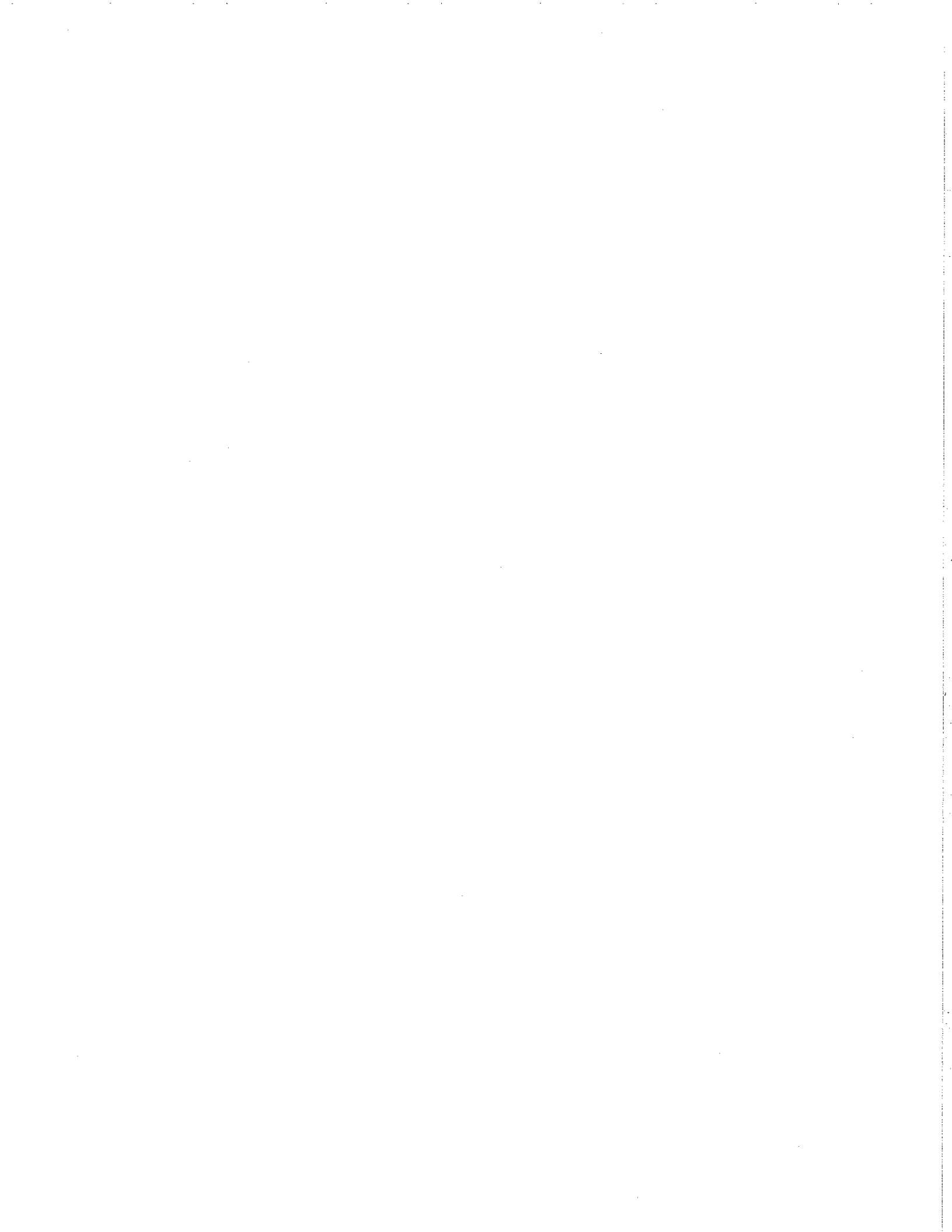
${}^f_n\mathfrak{H}^s$  is the subspace of  ${}_n\mathfrak{H}^s$  spanned by antisymmetric basis vectors.

We write

$${}^b_n\mathfrak{H}^s = \mathbb{S} {}_n\mathfrak{H}^s = \mathbb{S} \otimes^n \mathfrak{H}_1^s \quad (2.36)$$

$${}^f_n\mathfrak{H}^s = \mathbb{A} {}_n\mathfrak{H}^s = \mathbb{A} \otimes^n \mathfrak{H}_1^s \quad (2.37)$$

$\mathbb{S}$  and  $\mathbb{A}$  are the symmetrizing and antisymmetrizing operators, respectively, for the  $n$ -particle system.



## Chapter 3

# OCCUPATION NUMBER REPRESENTATION

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### Section 3.1 Introductory remarks

In this Chapter we construct basis vectors for a system of identical fermions and for a system of identical bosons. We need not specify details of the fermion or boson system under consideration.

The fermion system may, for example, be:

- electrons bound to a single atom
- conduction electrons in a metal
- nucleons in an atomic nucleus
- quarks in a nucleon

Whatever the system, each fermion has half-odd integral spin and all states of the system are antisymmetric under permutation of the particles.

The boson system may, for example, be:

- photons characterizing an electromagnetic field
- phonons characterizing the lattice vibrations of a crystal
- pions or kaons created in collisions of nuclear projectiles
- gluons in nuclear matter

Whatever the system, each boson has integral spin and all states of the system are symmetric under permutation of the particles.

In view of the symmetry requirement on the states of the system, the basis vectors we construct will be labelled by specifying which single-particle states are occupied. We thus construct the occupation number representation for fermion and boson systems.

## Section 3.2 System of identical fermions

In this Section we construct a set of basis vectors for a system of  $n$  identical fermions.

### 3.2.1 Basis vectors for the one-fermion system

Let

$$|\phi_r\rangle_\alpha \quad (3.1)$$

where  $r = 1, 2, \dots, \infty$  be a complete orthonormal set of vectors spanning the Hilbert space  $\mathfrak{H}_1^\alpha$  for fermion  $\alpha$ .

$$\sum_{r=1}^{\infty} |\phi_r\rangle_\alpha \langle \phi_r|_\alpha = 1_\alpha \quad (3.2)$$

$$\langle \phi_r |_\alpha \phi_s \rangle_\alpha = \delta_{rs} \quad (3.3)$$

$1_\alpha$  is the unit operator in the one-particle space for fermion number  $\alpha$ .

## Comments

### 1. Subscript $\alpha$

The subscript  $\alpha$  on  $|\phi_r\rangle_\alpha$  and  $1_\alpha$  has been added to serve as a reminder that the vectors and operators are in the Hilbert space  $\mathfrak{H}_1^\alpha$  for fermion number  $\alpha$ .

### 2. Denumerable set of basis vectors

It is convenient to use a denumerable set of vectors to span the one-fermion Hilbert space.

The coordinate-space/spin kets  $|x, m_s\rangle$ , the momentum-space/spin kets  $|p, m_s\rangle$  and the momentum/helicity kets  $|h^\lambda(p)\rangle$  given in Chapter 9 of Part I are each labelled by a continuous variable and will be used in Chapter 5 to span the one-fermion Hilbert space.

### 3. Example of the basis vectors $|\phi_r\rangle$

The  $|\phi_r\rangle$  may, for example, be chosen to be the simultaneous eigenvectors of the one-particle operators<sup>1</sup>

$$\frac{1}{2}(\mathcal{P}^2 + \mathcal{X}^2 - 3) \tag{3.4}$$

$$J \cdot J \tag{3.5}$$

$$J^3 \tag{3.6}$$

where

---

<sup>1</sup> The  $|\phi_r\rangle$  may also be simultaneous eigenvectors of the internal variables charge, baryon number, lepton number, isospin, strangeness and charm. We need not specify these variables here.

$$J^j = (X \times P)^j + S^j \quad (3.7)$$

$X^j$ ,  $P^j$  and  $S^j$  are the  $j$ -component of the Cartesian position, momentum and spin of the particle.

As in Chapter 8 of Part I, the dimensionless operators  $\mathcal{P}^j$  and  $\mathcal{X}^j$  are defined by

$$P^j = mc\mathcal{P}^j \quad (3.8)$$

$$X^j = \frac{\hbar}{mc}\mathcal{X}^j \quad (3.9)$$

The eigenvalue of  $S \cdot S$  is  $s(s+1)\hbar^2$  where  $s$  is a half-odd integer.  $s$  is the intrinsic spin of the elementary fermion.

The eigenvalues of (3.4) are the positive integers. The coordinate representatives of the eigenvectors of (3.4) involve Hermite polynomials.

Apart from multiplicative and additive constants, (3.4) is the Hamiltonian of a nonrelativistic harmonic oscillator. The eigenvalues of (3.4) can be determined by introducing the ladder operators

$$A^j = \frac{1}{\sqrt{2}}(\mathcal{X}^j + i\mathcal{P}^j) \quad (3.10)$$

$$A^{j\dagger} = \frac{1}{\sqrt{2}}(\mathcal{X}^j - i\mathcal{P}^j) \quad (3.11)$$



These one-particle operators satisfy

$$[\mathcal{A}^j, \mathcal{A}^k] = 0 \quad (3.12)$$

$$[\mathcal{A}^{j\dagger}, \mathcal{A}^{k\dagger}] = 0 \quad (3.13)$$

$$[\mathcal{A}^j, \mathcal{A}^{k\dagger}] = \delta_{jk} \quad (3.14)$$

#### 4. General one-fermion state

The general one-fermion state at time  $t$  is

$$|\psi(t)\rangle_\alpha = \sum_{r=1}^{\infty} \psi_r(t) |\phi_r\rangle_\alpha \quad (3.15)$$

$$\psi_r(t) = \langle \phi_r | \psi(t) \rangle_\alpha \quad (3.16)$$

is the probability amplitude that the fermion is in the state  $|\phi_r\rangle_\alpha$  at time  $t$ .

### 3.2.2 Basis vectors for the $n$ -fermion system

We recall from (2.28) that the  $n$ -particle Hilbert space  ${}_n\mathfrak{H}^s$  is a tensor product of  $n$  identical spaces. When  $s$  is a half-odd integer, it is spanned by vectors of the form

$$|\phi_r \rangle_1 |\phi_s \rangle_2 \cdots |\phi_t \rangle_n \quad (3.17)$$

where particle 1 is in single-particle state  $|\phi_r \rangle$ , particle 2 is in single-particle state  $|\phi_s \rangle$  and particle  $n$  is in single-particle state  $|\phi_t \rangle$ .

The  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{H}^s$  (2.37) is spanned by antisymmetric combinations of vectors of the form (3.17). That is,  ${}^f_n\mathfrak{H}^s$  is spanned by vectors of the form

$$|n\{n_1 n_2 \cdots\}\rangle_n = \left(\frac{1}{n!}\right)^{\frac{1}{2}} \det \begin{vmatrix} |\phi_r \rangle_1 & |\phi_r \rangle_2 & \cdots & |\phi_r \rangle_n \\ |\phi_s \rangle_1 & |\phi_s \rangle_2 & \cdots & |\phi_s \rangle_n \\ \vdots & \vdots & \vdots & \vdots \\ |\phi_t \rangle_1 & |\phi_t \rangle_2 & \cdots & |\phi_t \rangle_n \end{vmatrix} \quad (3.18)$$

$$r < s < \cdots < t \quad (3.19)$$

where det denotes determinant.

### Comments

#### 1. Slater determinant

(3.18) is a Slater determinant.

#### 2. Manifest antisymmetry

(3.18) is manifestly antisymmetric under particle interchange because the value of a determinant changes sign when any two columns are interchanged.

### 3. Occupied states

The set of single-particle labels  $r, s, \dots, t$  tells which single-particle states are occupied.

Because of the antisymmetrizing, one cannot specify which particle occupies which state.

### 4. Pauli Exclusion Principle

Since a determinant vanishes if all elements of one row are equal to all elements of another row, that is, if two or more of  $r, s, \dots, t$  are equal, it follows that no two fermions can occupy the same single-particle state.

This is the Pauli Exclusion Principle.

### 5. Occupation numbers

Let  $n_r$  be the number of particles occupying the single-particle state  $|\phi_r\rangle$ . Then

$$n_r = 0 \text{ or } 1 \quad (3.20)$$

$$\sum_{r=1}^{\infty} n_r = n \quad (3.21)$$

$n_r$  is the occupation number for the single-particle state  $|\phi_r\rangle$ .

(3.18) is labelled by the occupation numbers  $n_1, n_2, \dots$ .

## 6. Basis for the $n$ -fermion Hilbert space

The set of vectors (3.18) is an orthonormal basis for the  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{F}^s$  (2.37).

$$\sum_{n_1 n_2 \dots}^f |n\{n_1 n_2 \dots\}\rangle \langle n\{n_1 n_2 \dots\}| = I_n \quad (3.22)$$

$$\sum_{n_1 n_2 \dots}^f = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \dots \delta_{n_1+n_2+\dots n} \quad (3.23)$$

$$\langle n\{n_1 n_2 \dots\} | n\{n'_1 n'_2 \dots\} \rangle = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \dots \quad (3.24)$$

The representation provided by the set of vectors (3.18) is called the occupation number representation for the  $n$ -fermion system.

## 7. General $n$ -fermion state

The general  $n$ -fermion state has the form<sup>1</sup>

$$|\psi(t)\rangle_n = \sum_{n_1 n_2 \dots}^f |n\{n_1 n_2 \dots\}\rangle \langle n\{n_1 n_2 \dots\} | \psi(t)\rangle_n \quad (3.25)$$

<sup>1</sup> The subscript  $n$  is appended to state vectors in  ${}^f_n\mathfrak{F}^s$  as a reminder that we are considering an  $n$ -particle system.

$$\langle n\{n_1 n_2 \dots\} | \psi(t) \rangle_n \quad (3.26)$$

is the probability amplitude at time  $t$  that  $n_1$  fermions occupy the single-particle state  $|\phi_1\rangle$ , and  $n_2$  fermions occupy the single-particle state  $|\phi_2\rangle$ , and so on.

### Section 3.3 System of identical bosons

In this Section we construct a set of basis vectors for a system of  $n$  identical bosons. The boson system is treated analogously to the treatment of the fermion system in Section 3.2.

#### 3.3.1 Basis vectors for the one-boson system

Let

$$|\beta_r\rangle_\alpha \quad (3.27)$$

where  $r = 1, 2, \dots$  be a complete orthonormal set of vectors spanning the Hilbert space  $\mathfrak{H}_1^\alpha$  for boson  $\alpha$ .

$$\sum_{r=1}^{\infty} |\beta_r\rangle_\alpha \langle \beta_r|_\alpha = 1_\alpha \quad (3.28)$$

$$\langle \beta_r | \beta_s \rangle_\alpha = \delta_{rs} \quad (3.29)$$

$1_\alpha$  is the unit operator in the one-particle space for boson number  $\alpha$ .

### Comments

1. Example of the basis vectors  $|\beta_r\rangle$

The  $|\beta_r\rangle$  may be taken to be the eigenvectors of (3.4) to (3.6)<sup>1</sup>. The eigenvalue of  $S^2$  is  $s(s+1)\hbar^2$  where  $s$  is an integer.  $s$  is the intrinsic spin of the elementary boson.

2. General one-boson state

The general one-boson state at time  $t$  is

$$|\psi(t)\rangle_\alpha = \sum_{r=1}^{\infty} \psi_r(t) |\beta_r\rangle_\alpha \quad (3.30)$$

$$\psi_r(t) = \langle \beta_r | \psi(t) \rangle_\alpha \quad (3.31)$$

is the probability amplitude that the boson is in the state  $|\beta_r\rangle_\alpha$  at time  $t$ .

### 3.3.2 Basis vectors for the $n$ -boson system

The  $n$ -particle Hilbert space  ${}_n\mathfrak{H}^s$  is a tensor product of  $n$  identical spaces. When  $s$  is an integer, it is spanned by vectors of the form

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<sup>1</sup> The  $|\beta_r\rangle$  may also be simultaneous eigenvectors of the internal variables charge, baryon number, lepton number, isospin, strangeness and charm. As in the fermion case in Section 3.2, we need not specify these variables here.

$$|\beta_r \rangle_1 |\beta_s \rangle_2 \cdots |\beta_t \rangle_n \quad (3.32)$$

where particle 1 is in single-particle state  $|\beta_r \rangle$ , particle 2 is in single-particle state  $|\beta_s \rangle$  and particle  $n$  is in single-particle state  $|\beta_t \rangle$ .

The  $n$ -boson Hilbert space  ${}^b_n\mathfrak{H}^s$  (2.36) is spanned by symmetric combinations of vectors of the form (3.32). That is,  ${}^b_n\mathfrak{H}^s$  is spanned by vectors of the form

$$|n[n_1 n_2 \cdots]\rangle_n = \left( \frac{n_1! n_2! \cdots}{n!} \right)^{\frac{1}{2}} \text{sym det} \begin{vmatrix} |\beta_r \rangle_1 & |\beta_r \rangle_2 & \cdots & |\beta_r \rangle_n \\ |\beta_s \rangle_1 & |\beta_s \rangle_2 & \cdots & |\beta_s \rangle_n \\ \vdots & \vdots & \ddots & \vdots \\ |\beta_t \rangle_1 & |\beta_t \rangle_2 & \cdots & |\beta_t \rangle_n \end{vmatrix} \quad (3.33)$$

$$r \leq s \leq \cdots \leq t \quad (3.34)$$

where *sym det* denotes a determinant which has plus signs in its definition rather than minus signs.

### Comments

#### 1. Notation

The left side of (3.18) has  $\{\cdots\}$ . The left side of (3.33) has  $[\cdots]$ .

2. **Manifest symmetry**

(3.33) is manifestly symmetric under particle interchange.

3. **Occupied states**

The set of single-particle labels  $r, s, \dots, t$  tells which single-particle states are occupied.

Because of the symmetrizing, one cannot specify which particle occupies which state.

4. **No Pauli Exclusion Principle**

There is no Pauli Exclusion Principle for a system of identical bosons because *sym det* does not vanish if all elements of one row are equal to all elements of another row, that is, if two or more of  $r, s, \dots, t$  are equal.

5. **Occupation numbers**

Let  $n_r$  be the number of particles occupying the single-particle state  $|\beta_r\rangle$ .  
Then

$$0 \leq n_r \leq n \quad (3.35)$$

$$\sum_{r=1}^{\infty} n_r = n \quad (3.36)$$

$n_r$  is the occupation number for the single-particle state  $|\beta_r\rangle$ .

(3.33) is labelled by the occupation numbers  $n_1, n_2, \dots$ .



In contradistinction to the fermion case, all bosons may occupy one single-particle state.

## 6. Basis for the $n$ -boson Hilbert space

The set of vectors (3.33) is orthonormal and spans the  $n$ -boson Hilbert space  ${}^b_n\mathfrak{H}^s$ .

$$\sum_{n_1 n_2 \dots}^b |n[n_1 n_2 \dots]\rangle \langle_n n[n_1 n_2 \dots]| = 1_n \quad (3.37)$$

$$\sum_{n_1 n_2 \dots}^b = \sum_{n_1=0}^n \sum_{n_2=0}^n \dots \delta_{n_1+n_2+\dots, n} \quad (3.38)$$

$$\langle_n n[n_1 n_2 \dots] | n[n'_1 n'_2 \dots] \rangle = \delta_{n_1 n'_1} \delta_{n_2 n'_2} \dots \quad (3.39)$$

The representation provided by the set of vectors (3.33) is called the occupation number representation for the  $n$ -boson system.

## 7. General $n$ -boson state

The general  $n$ -boson state has the form<sup>1</sup>

$$|\psi(t)\rangle_n = \sum_{n_1 n_2 \dots}^b |n[n_1 n_2 \dots]\rangle \langle_n n[n_1 n_2 \dots] | \psi(t)\rangle_n \quad (3.40)$$

<sup>1</sup> The subscript  $n$  is appended to state vectors in  ${}^b_n\mathfrak{H}^s$  as a reminder that we are considering an  $n$ -particle system.

$$\langle n[n_1 n_2 \dots] | \psi(t) \rangle_n \quad (3.41)$$

is the probability amplitude at time  $t$  that  $n_1$  bosons occupy the single-particle state  $|\beta_1\rangle$ , and  $n_2$  bosons occupy the single-particle state  $|\beta_2\rangle$ , and so on.

### Section 3.4 System of identical fermions and bosons

The Hilbert space

$${}_{nn'}^{fb} \mathfrak{H}^{ss'} \quad (3.42)$$

for a system of  $n$  identical fermions each with spin  $s$  and  $n'$  identical bosons each with spin  $s'$  is the direct product of the  $n$ -fermion and  $n'$ -boson Hilbert spaces.

$${}_{nn'}^{fb} \mathfrak{H}^{ss'} = {}_n^f \mathfrak{H}^s \otimes {}_{n'}^b \mathfrak{H}^{s'} \quad (3.43)$$

${}_{nn'}^{fb} \mathfrak{H}^{ss'}$  is spanned by vectors of the form

$$|n\{n_1 n_2 \dots\}\rangle_n |n'[n'_1 n'_2 \dots]\rangle_{n'} \quad (3.44)$$

$|n\{n_1 n_2 \dots\}\rangle_n$  is the Slater determinant (3.18) and  $|n'[n'_1 n'_2 \dots]\rangle_{n'}$  is the symmetric determinant (3.33).

### Section 4.1 Introductory remarks

So far in Part II we have considered the total number of particles in each system to be fixed for all time. We now drop this restriction and consider a larger system with an unspecified number of particles. This is handled mathematically by considering the Hilbert space for the system to be Fock space.

Fock space is the natural mathematical arena for accommodating particle creation and annihilation, that is, for allowing the conversion of energy to mass and *vice versa* which is allowed by RQM.

It is not necessary to use Fock space to describe a fermion system because it is an experimental fact that every fermion system has a definite number of fermions. We will see, however, that using Fock space allows an elegant and intuitive reformulation of the QM of an  $n$ -fermion system. This reformulation (second quantization) is the standard language of nonrelativistic condensed matter physics and low-energy nuclear physics.

Fock space for a system of fermions is the subject of this and the next Chapter. Fock space for a system of bosons is the subject of Chapter 7.

Creation and annihilation operators for fermions are defined in Section 4.3. These operators are defined in terms of a denumerable set of vectors which form an orthonormal basis for the one-fermion system. Fermion creation and annihilation operators which are labelled by a continuous variable are defined in Chapter 5.

General expressions for observables in terms of fermion creation and annihilation operators are given in Sections 4.4 and 4.5. The fermion Fock space

expression for the Hamiltonian for a system with two-body interactions is given in Section 4.6. The Hartree-Fock potential is derived in Section 4.6.

Hole creation and annihilation operators and particle-hole states of an  $n$ -fermion system are discussed in Section 4.7. Correlated particle-hole states of an  $n$ -fermion system are discussed in Section 4.8.

## Section 4.2 Fermion Fock space defined

1. Let

$$\psi = (\psi_0, \psi_1, \dots, \psi_n, \dots) \quad (4.1)$$

where  $\psi_n$  is a vector in  $n$ -fermion Hilbert space  ${}^f_n\mathbb{H}^s$  (2.37).<sup>1</sup>

Each  $\psi_n$  is of the form (3.25).  $\psi_n$  is the component of  $\psi$  in  ${}^f_n\mathbb{H}^s$ .

2. Addition of  $\psi$  and  $\chi = (\chi_0, \chi_1, \dots, \chi_n, \dots)$  is defined as

$$\psi + \chi = (\psi_0 + \chi_0, \psi_1 + \chi_1, \dots, \psi_n + \chi_n, \dots) \quad (4.2)$$

3. Multiplication of  $\psi$  by a scalar  $a$  is defined as

$$a\psi = (a\psi_0, a\psi_1, \dots, a\psi_n, \dots) \quad (4.3)$$

4. The scalar product of  $\psi$  and  $\chi$  is defined as

$$(\psi, \chi) = \sum_{n=0}^{\infty} (\psi_n, \chi_n) \quad (4.4)$$

It is required that  $(\psi, \psi) < \infty$  for all  $\psi$ .

5. The set of elements  $\psi$  is a separable Hilbert space.

---

<sup>1</sup>  ${}^f_0\mathbb{H}^s$  is defined in item 5 of the Comments list.

## Comments

### 1. Fermion Fock space $f_{\mathfrak{H}^s}$

The above Hilbert space is called fermion Fock space. It will be denoted by  $f_{\mathfrak{H}^s}$ .

$f_{\mathfrak{H}^s}$  is the direct sum of the Hilbert spaces  $f_n^{\mathfrak{H}^s}$  (2.37) for all  $n$ .

$$f_{\mathfrak{H}^s} = f_0^{\mathfrak{H}^s} \oplus f_1^{\mathfrak{H}^s} \oplus \cdots \oplus f_n^{\mathfrak{H}^s} \oplus \cdots \quad (4.5)$$

$\oplus$  denotes direct sum.

### 2. States of the system

The unit norm vectors (4.1) in  $f_{\mathfrak{H}^s}$  correspond to states of the system.

The probability  $P_n$  that the system has  $n$  particles in it is

$$P_n = \langle \psi_n | \psi_n \rangle \quad (4.6)$$

$$\sum_{n=0}^{\infty} P_n = 1 \quad (4.7)$$

### 3. Components of $\psi$

$\psi$  can have only one nonzero component. For example,

$$\psi = (0, 0, \dots, 0, \psi_n, 0, \dots) \quad (4.8)$$

One can construct boson Fock space  ${}^b\mathfrak{H}^s$  analogous to the construction of  ${}^f\mathfrak{H}^s$ . (This is done in Chapter 7). States in  ${}^b\mathfrak{H}^s$  can have more than one nonzero component.

4. Hilbert space  ${}_{2 \leftrightarrow 3}\mathfrak{H}$

The Hilbert space  ${}_{2 \leftrightarrow 3}\mathfrak{H}$  defined in Chapter 12 of Part I is the analog of Fock space for a  $2 \leftrightarrow 3$  particle system.

5. Hilbert space  ${}^f_0\mathfrak{H}^s$

${}^f_0\mathfrak{H}^s$  is defined to be a one-dimensional space.

The unit norm vector spanning  ${}^f_0\mathfrak{H}^s$  is labelled  $|0\{00\dots\}\rangle_0$ .

6. Basis vectors for  ${}^f\mathfrak{H}^s$

A basis for  ${}^f\mathfrak{H}^s$  is the set of vectors

$$|n\{n_1 n_2 \dots\}\rangle \quad (4.9)$$

defined by

$$|0\{00\dots\}\rangle = \left( |0\{00\dots\}\rangle_0, 0, \dots \right) \quad (4.10)$$

$$|1\{n_1 n_2 \dots\}\rangle = \left( 0, |1\{n_1 n_2 \dots\}\rangle_1, 0, \dots \right) \quad (4.11)$$

$$|2\{n_1 n_2 \dots\}\rangle = \left( 0, 0, |2\{n_1 n_2 \dots\}\rangle_2, 0, \dots \right) \quad (4.12)$$

⋮

where

$$|n\{n_1 n_2 \dots\}\rangle_n \quad (4.14)$$

for all  $n = 1, 2, \dots$  is the Slater determinant (3.18).

Then

$$\sum_{nn_1n_2\cdots}^f |n\{n_1n_2\cdots\}\rangle \langle n\{n_1n_2\cdots}\rangle = 1 \quad (4.15)$$

$$\sum_{nn_1n_2\cdots}^f = \sum_{n=0}^{\infty} \sum_{n_1n_2\cdots}^f \quad (4.16)$$

$$\sum_{n_1n_2\cdots}^f = \sum_{n_1=0}^1 \sum_{n_2=0}^1 \cdots \delta_{n_1+n_2+\cdots n} \quad (4.17)$$

$$\langle n\{n_1n_2\cdots\} | n'\{n'_1n'_2\cdots\} \rangle = \delta_{nn'} \delta_{n_1n'_1} \delta_{n_2n'_2} \cdots \quad (4.18)$$

## 7. Vacuum state

(4.10) is the vacuum state of the system. It will be denoted by  $|0\rangle$ .

$$|0\rangle = |0\{00\cdots\}\rangle \quad (4.19)$$

## 8. General state of the system

The general state of the system has the form

$$|\psi(t)\rangle = \sum_{nn_1n_2\cdots}^f |n\{n_1n_2\cdots\}\rangle \langle n\{n_1n_2\cdots\} | \psi(t)\rangle \quad (4.20)$$



$$\langle n\{n_1 n_2 \dots\} | \psi(t) \rangle \quad (4.21)$$

is the probability amplitude that at time  $t$  there are  $n$  fermions in the system with  $n_1$  fermions occupying the single-particle state  $|\phi_1\rangle$ , and  $n_2$  fermions occupying the single-particle state  $|\phi_2\rangle$ , and so on.

## Section 4.3 Creators and annihilators

We define fermion creation and annihilation operators in this Section. These operators are fundamental dynamical variables for a system of identical fermions. They obey anticommutation relations.

Introduction of creation and annihilation operators yields intuitive and elegant expressions for observables and basis states.

Matrix elements of observables are expressed as vacuum expectation values of products of creation and annihilation operators. These vacuum expectation values are calculated through a simple strategy.

### 4.3.1 Definitions

For each  $r = 1, 2, \dots$ , we define

$$F_r^\dagger = \sum_{nn_1n_2\cdots}^f |n+1\{n_1n_2\cdots n_r+1\cdots\} \rangle (-)^{m_r} (1-n_r) \langle n\{n_1n_2\cdots n_r\cdots\} | \quad (4.22)$$

$$F_r = \sum_{nn_1n_2\cdots}^f |n\{n_1n_2\cdots n_r-1\cdots\} \rangle (-)^{m_r} n_r \langle n+1\{n_1n_2\cdots n_r\cdots\} | \quad (4.23)$$

$$m_r = \sum_{s=1}^{r-1} n_s \quad (4.24)$$

where  $|n\{n_1n_2\cdots\} \rangle$  is the basis vector (4.9) in fermion Fock space  ${}^f\mathfrak{F}^s$ .

It follows from (4.22) and (4.23) that<sup>1</sup>

$$F_r^\dagger |n\{n_1n_2\cdots 0\cdots\} \rangle = (-)^{m_r} |n+1\{n_1n_2\cdots 1\cdots\} \rangle \quad (4.25)$$

$$F_r |n+1\{n_1n_2\cdots 1\cdots\} \rangle = (-)^{m_r} |n\{n_1n_2\cdots 0\cdots\} \rangle \quad (4.26)$$

$$F_r^\dagger |n\{n_1n_2\cdots 1\cdots\} \rangle = 0 \quad (4.27)$$

$$F_r |n\{n_1n_2\cdots 0\cdots\} \rangle = 0 \quad (4.28)$$

<sup>1</sup> The 0 and 1 in the basis states in (4.25) to (4.28) occur in the  $r$ th place.

In particular,

$$F_r |0\rangle = 0 \quad (4.29)$$

$$\langle 0 | F_r^\dagger = 0 \quad (4.30)$$

$$F_r^\dagger |0\rangle = \left(0, \left|\phi_r\right\rangle_1, 0, \dots\right) \quad (4.31)$$

### Comments

#### 1. Fermion creator

$F_r^\dagger$  is a fermion creation operator or fermion creator.

When acting on an  $n$ -fermion basis vector (4.9) with  $n_r = 0$ ,  $F_r^\dagger$  yields an  $n + 1$ -fermion basis vector with  $n_r = 1$ .

#### 2. Fermion annihilator

$F_r$  is a fermion annihilation operator or fermion annihilator.

When acting on an  $n + 1$ -fermion basis vector (4.9) with  $n_r = 1$ ,  $F_r$  yields an  $n$ -fermion basis vector with  $n_r = 0$ .

#### 3. Value of $m_r$

$m_r$  is the number of occupied single-particle states in  $|n\{n_1 n_2 \dots n_r \dots\}\rangle$  up to single-particle state number  $r$ .

#### 4. Pauli Exclusion Principle

(4.27) is a manifestation of the Pauli Exclusion Principle: a fermion cannot be put in an occupied single-particle state.

#### 5. Creating an elementary fermion

When acting on the vacuum state  $|0\rangle$ ,  $F_r^\dagger$  creates an elementary fermion with rest mass  $m$  and spin  $s$  in single-particle state  $|\phi_r\rangle$ .

#### 6. One-fermion state

The general one-fermion state at time  $t$  is

$$|\psi(t)\rangle = \sum_{r=1}^{\infty} \psi_r(t) F_r^\dagger |0\rangle \quad (4.32)$$

$$\psi_r(t) = \langle 0 | F_r \psi(t) \rangle \quad (4.33)$$

is the probability amplitude that the fermion is in the state  $|\phi_r\rangle$  at time  $t$ .

### 4.3.2 Anticommutation relations

It follows from (4.22) and (4.23) that<sup>1</sup>

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<sup>1</sup> (4.34) to (4.36) are proven in Topic 4.3.5

$$\{F_r, F_s\} = 0 \quad (4.34)$$

$$\{F_r^\dagger, F_s^\dagger\} = 0 \quad (4.35)$$

$$\{F_r, F_s^\dagger\} = \delta_{rs} \quad (4.36)$$

where  $\{A, B\} = AB + BA$  is the anticommutator of  $A$  and  $B$ .

### Comment

#### 1. Pauli Exclusion Principle

(4.34) and (4.35) with  $r = s$ ,

$$(F_r)^2 = (F_r^\dagger)^2 = 0 \quad (4.37)$$

are operator forms of the Pauli Exclusion Principle.

(4.37) imply that no state can have occupation number greater than one.

### 4.3.3 Basis vectors

It follows from (4.25) that the basis vector (4.9) may be expressed as  $n$  creators acting on the vacuum state.

In particular, if (4.9) corresponds to the  $n$  single-particle levels  $r, s, \dots, t$  occupied with  $r < s < \dots < t$ , then

$$|n\{n_1 n_2 \dots\}\rangle = F_r^\dagger F_s^\dagger \dots F_t^\dagger |0\rangle \quad (4.38)$$

### Proof of (4.38)

The proof is straightforward. The state may be expressed as

$$|n\{\dots 1 \dots 1 \dots \dots 1 \dots\}\rangle \quad (4.39)$$

where the 1's appear in the  $r$ th,  $s$ th and  $t$ th, places. Using (4.25) repeatedly yields

$$\begin{aligned} |n\{\dots 1 \dots 1 \dots \dots 1 \dots\}\rangle &= F_r^\dagger |n-1\{\dots 0 \dots 1 \dots \dots 1 \dots\}\rangle \\ &= F_r^\dagger F_s^\dagger |n-2\{\dots 0 \dots 0 \dots \dots 1 \dots\}\rangle \\ &= \dots = F_r^\dagger F_s^\dagger \dots F_t^\dagger |0\rangle \end{aligned} \quad (4.40)$$

which is (4.38).

### Comments

#### 1. Form of the basis vector

(4.38) is a compact, intuitive and elegant expression for the basis vector (4.9).

#### 2. Manifest antisymmetry

In view of (4.35), (4.38) changes sign when any two of  $r, s, \dots, t$  are interchanged.

(4.38) is manifestly antisymmetric under particle interchange.

### 3. Fundamental dynamical variables

Each basis vector (4.9) can be expressed in terms of fermion creators acting on the vacuum state. The set of creators and annihilators defined by (4.22) and (4.23) is a set of fundamental dynamical variables for a system of identical fermions.

Anticommutation relations (4.34) to (4.36) are a fundamental algebra for the system.

#### 4.3.4 Evaluating vacuum expectation values

Evaluation of matrix elements of observables involves evaluating matrix elements of combinations of products of fermion creators and annihilators between states of the form (4.38).

That is, it involves calculating the average value in the vacuum state of products of fermion creators and annihilators. These vacuum expectation values may be evaluated using the following strategy:

Express products of creators and annihilators in normal order using (4.34) to (4.36) and (A.1) to (A.9), then use (4.29) and (4.30).

### Example

The norm of the vector  $F_r^\dagger F_s^\dagger |0\rangle$  is calculated as follows:

$$\begin{aligned}
 & \langle F_r^\dagger F_s^\dagger 0 | F_r^\dagger F_s^\dagger |0\rangle = \langle 0 | F_s F_r F_r^\dagger F_s^\dagger |0\rangle \\
 & = \langle 0 | \left( -[F_r^\dagger F_s^\dagger, F_s F_r] + F_r^\dagger F_s^\dagger F_s F_r \right) |0\rangle = (\delta_{rr}\delta_{ss} - \delta_{rs}\delta_{rs}) \langle 0 |0\rangle \\
 & = 1 - \delta_{rs}
 \end{aligned} \tag{4.41}$$

### 4.3.5 Proof of the anticommutation relations

#### Proof of (4.36)

It follows from (4.22) and (4.23) that

$$F_r F_r^\dagger = \sum_{nn_1 n_2 \dots}^f |n\{\dots 0 \dots\}\rangle \langle n\{\dots 0 \dots\}| \tag{4.42}$$

and

$$F_r^\dagger F_r = \sum_{nn_1 n_2 \dots}^f |n+1\{\dots 1 \dots\}\rangle \langle n+1\{\dots 1 \dots\}| \tag{4.43}$$

and thus

$$F_r F_r^\dagger + F_r^\dagger F_r = 1 \tag{4.44}$$

which is (4.36) when  $r = s$ . When  $r < s$ ,

$$F_r F_s^\dagger = \sum_{nn_1 n_2 \dots}^f |n+1\{\dots 0 \dots 1 \dots\}\rangle (-)^{2m_r+1+\lambda_{rs}} \langle n+1\{\dots 1 \dots 0 \dots\}| \tag{4.45}$$



where we have written

$$m_s = m_r + n_r + \lambda_{rs} \quad (4.46)$$

where

$$\lambda_{rs} = \sum_{t=r+1}^{s-1} n_t \quad (4.47)$$

is the number of occupied levels between levels  $r$  and  $s$ . Then

$$\begin{aligned} F_s^\dagger F_r &= \sum_{n_1 n_2 \dots}^f |n+1\{\dots 0 \dots 1 \dots\} \rangle (-)^{2m_r+0+\lambda_{rs}} \langle n+1\{\dots 1 \dots 0 \dots\} | \\ &= -F_r F_s^\dagger \end{aligned} \quad (4.48)$$

(4.48) also holds when  $s < r$ . This completes the proof of (4.36).

(4.34) follows similarly, and (4.35) follows from (4.34).

## Section 4.4 Number operators

We define

$$N_r = F_r^\dagger F_r \quad (4.49)$$

for each  $r = 1, 2, \dots$ , and

$$N = \sum_{r=1}^{\infty} N_r \quad (4.50)$$

It follows that

$$N_r^\dagger = N_r \quad (4.51)$$

$$(N_r)^2 = N_r \quad (4.52)$$

$$[N_r, N_s] = 0 \quad (4.53)$$

for all  $r, s = 1, 2, \dots$ . Moreover,

$$N_r = \sum_{n_1 n_2 \dots}^f |n\{n_1 n_2 \dots\} \rangle n_r \langle n\{n_1 n_2 \dots\} | \quad (4.54)$$

$$N = \sum_{n_1 n_2 \dots}^f |n\{n_1 n_2 \dots\} \rangle n \langle n\{n_1 n_2 \dots\} | \quad (4.55)$$

### Comments

#### 1. Compatible observables

It follows from (4.51) and (4.52) that  $N_r$  is an observable with eigenvalues 0 and 1.

It follows from (4.53) that the  $N_r$  are an infinite set of compatible observables.

#### 2. Eigenvalue decomposition

(4.54) and (4.55) are the eigenvalue decompositions of  $N_r$  and  $N$ .

The basis vector (4.9) is a simultaneous eigenvector of  $N, N_1, N_2, \dots$  belonging to eigenvalues  $n, n_1, n_2, \dots$ .

$N, N_1, N_2, \dots$  are a complete set of compatible observables.

### 3. Nomenclature

When operating on the basis vector (4.38),  $N_r$  gives zero if the single-particle state  $|\phi_r\rangle$  is unoccupied and one if it is occupied.

$N_r$  is the number operator for the single-particle state  $|\phi_r\rangle$ .

$N$  is the number operator for the system.

## Section 4.5 Observables

An observable on the Hilbert space  $\mathcal{H}_n^f$  for  $n$  identical fermions each of spin  $s$  is a function<sup>1</sup>

$$A_n(\xi_1, \dots, \xi_n) \quad (4.56)$$

$$\xi_\alpha = \{X_\alpha^1, P_\alpha^1, S_\alpha^1, \dots, X_\alpha^3, P_\alpha^3, S_\alpha^3\} \quad (4.57)$$

$X_\alpha^j, P_\alpha^j, S_\alpha^j$  are the Cartesian position, momentum and spin, respectively, for particle  $\alpha$ . (4.56) is invariant under  $\xi_\alpha \leftrightarrow \xi_\beta$  for all  $\alpha$  and  $\beta$ .

<sup>1</sup> The subscript  $n$  has been added to the observable to serve as a reminder that the observable is on  $\mathcal{H}_n^f$ .

In this Section, we construct operators on fermion Fock space  ${}^f\mathfrak{F}^s$  which are equal to (4.56) for all  $n$ . Of particular interest are one-particle operators and two-particle operators on  ${}^f_n\mathfrak{F}^s$ .

### 4.5.1 One-particle operators

The operator

$$A = \sum_{r,s=1}^{\infty} \langle r | A | s \rangle F_r^\dagger F_s \quad (4.58)$$

on fermion Fock space  ${}^f\mathfrak{F}^s$  where

$$\langle r | A | s \rangle = \langle \phi_r | A(\xi) | \phi_s \rangle \quad (4.59)$$

is equal to

$$A_n(\xi_1, \dots, \xi_n) = \sum_{\alpha=1}^n A(\xi_\alpha) \quad (4.60)$$

on  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{F}^s$  for every  $n = 1, 2, \dots$ .

### Comments

1. One-particle operator

(4.60) is a one-particle operator on  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{F}^s$ .

Matrix element (4.59) is a one-particle matrix element.

2. Free-particle Hamiltonian on  $n$ -fermion Hilbert space

The free-particle Hamiltonian (2.13)

$$H_{0n}(\xi_1, \dots, \xi_n) = \sum_{\alpha=1}^n \sqrt{P_{\alpha}^2 c^2 + m^2 c^4} \quad (4.61)$$

is a one-particle operator on  $n$ -fermion Hilbert space  ${}^f_n \mathbb{F}^s$ .

### 3. Free—particle Hamiltonian on Fock space

It follows from (4.58) that the operator on fermion Fock space  ${}^f_{\mathbb{F}}^s$  which is equal to (4.61) for all  $n$  is

$$H_0 = \sum_{r,t=1}^{\infty} \langle r | H_0 | t \rangle F_r^{\dagger} F_t \quad (4.62)$$

$$\begin{aligned} \langle r | H_0 | t \rangle &= \langle \phi_r | \sqrt{P^2 c^2 + m^2 c^4} | \phi_t \rangle \\ &= \sum_{m_s=-s}^{+s} \int d^3 p \phi_{r m_s}^*(p) \epsilon(p) \phi_{t m_s}(p) \end{aligned} \quad (4.63)$$

$$\epsilon(p) = \sqrt{p^2 c^2 + m^2 c^4} \quad (4.64)$$

$\phi_{r m_s}(p)$  in (4.63) is the momentum—space/spin representative of the single-particle vector  $|\phi_r\rangle$ .

$$\phi_{r m_s}(p) = \langle p, m_s | \phi_r \rangle \quad (4.65)$$

The momentum-space/spin ket  $|p, m_s\rangle$  is discussed in Chapter 9 of Part I.

**Proof of (4.58)**

We show that

$$\langle n\{n_1 n_2 \dots\} | A_n | n\{n'_1 n'_2 \dots\} \rangle = \langle n\{n_1 n_2 \dots\} | A | n\{n'_1 n'_2 \dots\} \rangle \quad (4.66)$$

where the left side involves quantities in  $n$ -fermion Hilbert space  ${}^f_n \mathfrak{H}^s$  and the right side involves quantities in fermion Fock space  ${}^f \mathfrak{F}^s$ .

That is,  $|n\{n_1 n_2 \dots\}\rangle_n$  is a Slater determinant (3.18) and  $A_n$  is given by (4.60), and  $|n\{n_1 n_2 \dots\}\rangle$  is given by (4.38) and  $A$  is given by (4.58).

Now

$$\sum_{r=1}^{\infty} |\phi_r\rangle_{\alpha} \langle \phi_r|_{\alpha} = 1_{\alpha} \quad (4.67)$$

so

$$A(\xi_{\alpha}) = \sum_{r,s=1}^{\infty} |\phi_r\rangle_{\alpha} \langle \phi_r|_{\alpha} | A(\xi_{\alpha}) | \phi_s\rangle_{\alpha} \langle \phi_s| \quad (4.68)$$

But

$$\langle \phi_r | A(\xi_1) | \phi_s \rangle = \langle \phi_r | A(\xi_2) | \phi_s \rangle = \dots = \langle \phi_r | A(\xi_n) | \phi_s \rangle \quad (4.69)$$

since the  $n$  single-particle spaces are identical, so

$$\langle \phi_r | A(\xi_{\alpha}) | \phi_s \rangle = \langle r | A | s \rangle \quad (4.70)$$

and thus

$$A(\xi_{\alpha}) = \sum_{r,s=1}^{\infty} \langle r | A | s \rangle |\phi_r\rangle_{\alpha} \langle \phi_s| \quad (4.71)$$

and

$$A_n(\xi_1, \xi_2, \dots, \xi_n) = \sum_{r,s=1}^{\infty} \langle r | A | s \rangle \sum_{\alpha=1}^n | \phi_r \rangle_{\alpha} \langle \phi_s |_{\alpha} \quad (4.72)$$

Now consider  $r < s$ . Then

$$\sum_{\alpha=1}^n | \phi_r \rangle_{\alpha} \langle \phi_s |_{\alpha} | n\{n_1 n_2 \dots\} \rangle_n = (-)^{\lambda_{rs}} | n\{\dots 1 \dots 0 \dots\} \rangle_n \delta_{0n_r} \delta_{1n_s} \quad (4.73)$$

where

$$\lambda_{rs} = \sum_{t=r+1}^{s-1} n_t \quad (4.74)$$

The factor  $(-)^{\lambda_{rs}}$  in (4.73) accounts for interchanging rows of the Slater determinant (3.18) to get them into the order  $r < s < \dots < t$ .

Thus

$$\sum_{\alpha=1}^n | \phi_r \rangle_{\alpha} \langle \phi_s |_{\alpha} = \sum_{n_1 n_2 \dots}^f | n\{\dots 1 \dots 0 \dots\} \rangle_n (-)^{\lambda_{rs}} \langle n\{\dots 0 \dots 1 \dots\} | \quad (4.75)$$

(4.58) follows since

$$F_r^{\dagger} F_s = \sum_{nn_1 n_2 \dots}^f | n\{\dots 1 \dots 0 \dots\} \rangle (-)^{\lambda_{rs}} \langle n\{\dots 0 \dots 1 \dots\} | \quad (4.76)$$

## 4.5.2 Two-particle operators

The operator

$$A = \sum_{r,s,t,u=1}^{\infty} \langle rs | A | ut \rangle F_r^\dagger F_s^\dagger F_t F_u \quad (4.77)$$

on fermion Fock space  ${}^f\mathfrak{H}^s$  where

$$\langle rs | A | ut \rangle = \langle \phi_r | \langle \phi_s | A(\xi_\alpha, \xi_\beta) | \phi_u \rangle | \phi_t \rangle_\beta \quad (4.78)$$

is equal to<sup>1</sup>

$$A_n(\xi_1, \dots, \xi_n) = \sum_{\alpha, \beta=1}^n A(\xi_\alpha, \xi_\beta) \quad (4.79)$$

on  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{H}^s$  where

$$A(\xi_\alpha, \xi_\beta) = A(\xi_\beta, \xi_\alpha) \quad (4.80)$$

for every  $n = 2, 3, \dots$

### Comments

1. No misprint in (4.77)

(4.77) is not misprinted. The subscripts on the creators and annihilators are in alphabetic order whereas the labels in (4.78) are not.

2. Two-particle operator

---

<sup>1</sup> The double summation in (4.79) is restricted to  $\alpha \neq \beta$ .



(4.79) is a two-particle operator on  $n$ -fermion Hilbert space  $f_n^s$ .

(4.78) is a two-particle matrix element. It follows from (4.80) that

$$\langle rs | A | ut \rangle = \langle sr | A | tu \rangle \quad (4.81)$$

### 3. Total potential energy on $n$ -fermion Hilbert space

The total potential energy of a system of fermions interacting via two-body potentials<sup>2</sup>

$$V_n(\xi_1, \dots, \xi_n) = \frac{1}{2} \sum_{\alpha, \beta=1}^n V(\xi_\alpha, \xi_\beta) \quad (4.82)$$

$$V(\xi_\alpha, \xi_\beta) = V(\xi_\beta, \xi_\alpha) \quad (4.83)$$

is a two-particle operator on  $n$ -fermion Hilbert space  $f_n^s$ .

When the two-body potential is a function only of position,

$$V(\xi_\alpha, \xi_\beta) = V(X_\alpha, X_\beta) \quad (4.84)$$

### 4. Total potential energy on Fock space

<sup>2</sup> The double summation in (4.82) is restricted to  $\alpha \neq \beta$ .

It follows from (4.77) that the operator on fermion Fock space  $f_{\mathbb{H}^s}$  which is equal to (4.82) with (4.84) for all  $n$  is

$$V = \frac{1}{2} \sum_{r,t,u,v=1}^{\infty} \langle rt | V | vu \rangle F_r^\dagger F_t^\dagger F_u F_v \quad (4.85)$$

$$\begin{aligned} & \langle rt | V | vu \rangle \\ = & \sum_{m_1, m_2 = -s}^{+s} \int d^3x d^3y \phi_{rm_1}^*(x) \phi_{tm_2}^*(y) V(x, y) \phi_{vm_1}(x) \phi_{um_2}(y) \end{aligned} \quad (4.86)$$

$\phi_{rm_s}(x)$  in (4.86) is the coordinate—space/spin representative of the single-particle vector  $|\phi_r\rangle$ .

$$\phi_{rm_s}(x) = \langle x, m_s | \phi_r \rangle \quad (4.87)$$

The coordinate-space/spin ket  $|x, m_s\rangle$  is discussed in Chapter 9 of Part I.

#### 5. Proof of (4.77)

The proof of (4.77) and is similar to the proof of (4.58) and is not given here.

#### 6. Three checks of (4.77)

We check (4.77) by considering three examples which use the one-particle equations (4.58) and (4.59).

Example 1:

When

$$A(\xi_\alpha, \xi_\beta) = 1 \quad (4.88)$$

(4.79) becomes

$$A_n(\xi_1, \dots, \xi_n) = n(n-1) \quad (4.89)$$

The Fock space operator corresponding to (4.89) is

$$A = N(N-1) \quad (4.90)$$

Substitution of (4.88) into (4.78) yields (4.90) as required.

Example 2:

When

$$A(\xi_\alpha, \xi_\beta) = [A(\xi_\alpha) + A(\xi_\beta)] / 2 \quad (4.91)$$

(4.79) becomes

$$A_n(\xi_1, \dots, \xi_n) = (n-1) \sum_{\alpha=1}^n A(\xi_\alpha) \quad (4.92)$$

The Fock space operator corresponding to (4.92) is

$$A = (N-1) \sum_{r,s=1}^{\infty} \langle r | A | s \rangle F_r^\dagger F_s \quad (4.93)$$

Substitution of (4.91) into (4.78) yields (4.93) as required.

Example 3:

When

$$A(\xi_\alpha, \xi_\beta) = A(\xi_\alpha)A(\xi_\beta) \quad (4.94)$$

(4.79) becomes

$$A_n(\xi_1, \xi_2, \dots, \xi_n) = \left[ \sum_{\alpha=1}^n A(\xi_\alpha) \right]^2 - \sum_{\alpha=1}^n [A(\xi_\alpha)]^2 \quad (4.95)$$

The Fock space operator corresponding to (4.95) is

$$A = \left[ \sum_{r,s=1}^{\infty} \langle r | A | s \rangle F_r^\dagger F_s \right]^2 - \sum_{r,s=1}^{\infty} \langle r | A^2 | s \rangle F_r^\dagger F_s \quad (4.96)$$

(4.96) may be written in the form (4.77) where

$$\langle rs | A | ut \rangle = \langle r | A | u \rangle \langle s | A | t \rangle \quad (4.97)$$

as required.

## Section 4.6 Hamiltonian

We consider a system of identical fermions each with rest mass  $m$  and spin  $s$  interacting with each other via two-body potentials. Each fermion is also be subjected to an external potential.

### 4.6.1 Hamiltonian on $n$ -fermion Hilbert space

The Hamiltonian for the system on  $n$ -fermion Hilbert space  $f_n^s$  is<sup>12</sup>

$$H_n = \sum_{\alpha=1}^n H_0(\xi_\alpha) + \frac{1}{2} \sum_{\alpha,\beta=1}^n V(\xi_\alpha, \xi_\beta) \quad (4.98)$$

$$H_0(\xi_\alpha) = \frac{P_\alpha^2}{2m} + V(\xi_\alpha) \quad (4.99)$$

$$V(\xi_\alpha, \xi_\beta) = V(\xi_\beta, \xi_\alpha) \quad (4.100)$$

<sup>1</sup> (4.99) contains the nonrelativistic expression  $P_\alpha^2/2m$ . The relativistic expression  $\sqrt{P_\alpha^2 c^2 + m^2 c^4}$  can be used instead as needed.

<sup>2</sup> The double summation in (4.98) is restricted to  $\alpha \neq \beta$ .

We need not specify the details of the external potential  $V(\xi_\alpha)$  or the two-body potential  $V(\xi_\alpha, \xi_\beta)$ .

We assume that the one-particle vectors  $|\phi_r\rangle$  (3.1) are the eigenvectors of

$$H_0(\xi_\alpha) + \mathcal{V}(\xi_\alpha) \quad (4.101)$$

for some  $\mathcal{V}(\xi_\alpha)$ . That is, we assume that

$$[H_0(\xi_\alpha) + \mathcal{V}(\xi_\alpha)] |\phi_r\rangle = \epsilon_r |\phi_r\rangle \quad (4.102)$$

can be solved for  $|\phi_r\rangle$  and  $\epsilon_r$  ( $r = 1, 2, \dots$ ).

We write (4.98) in the form

$$H_n = \mathcal{H}_{0n} + \mathcal{V}_n \quad (4.103)$$

$$\mathcal{H}_{0n} = \sum_{\alpha=1}^n [H_0(\xi_\alpha) + \mathcal{V}(\xi_\alpha)] \quad (4.104)$$

$$\mathcal{V}_n = \frac{1}{2} \sum_{\alpha, \beta=1}^n V(\xi_\alpha, \xi_\beta) - \sum_{\alpha=1}^n \mathcal{V}(\xi_\alpha) \quad (4.105)$$

## Comments

### 1. Central potential and residual interactions

(4.103) describes  $n$  identical fermions moving independently in a central potential with residual interactions.

The central potential experienced by particle  $\alpha$  is

$$V(\xi_\alpha) + \mathcal{V}(\xi_\alpha) \tag{4.106}$$

$\mathcal{V}_n$  is the total residual interaction.

In practice,  $\mathcal{V}(\xi_\alpha)$  is chosen such that  $\mathcal{H}_{0n}$  is a good approximation to  $H_n$ .

### 2. Nucleons in an atomic nucleus

(4.103) describes  $n$  nucleons in an atomic nucleus.

$V(\xi_\alpha, \xi_\beta)$  is the sum of the strong interaction potential and the repulsive Coulomb potential between nucleon  $\alpha$  and nucleon  $\beta$ .

There is no external potential experienced by nucleon  $\alpha$ . That is,  $V(\xi_\alpha) = 0$

$\mathcal{V}(\xi_\alpha)$  is the nuclear shell model potential.

### 3. Conduction electrons in a metal

(4.103) describes  $n$  conduction electrons in a metal.

$V(\xi_\alpha, \xi_\beta)$  is the repulsive Coulomb potential between electron  $\alpha$  and electron  $\beta$ .

$V(\xi_\alpha)$  is the attractive Coulomb potential experienced by electron  $\alpha$  due to

the positive ions in the lattice.  $V(\xi_\alpha)$  is invariant under a space displacement equal to the distance between the ions.

$\mathcal{V}(\xi_\alpha)$  may be taken to be zero. The single-particle states are Bloch states.

#### 4.6.2 Hamiltonian on fermion Fock space

It follows from (4.58) and (4.77) that

$$H = \mathcal{H}_0 + \mathcal{V} \quad (4.107)$$

$$\mathcal{H}_0 = \sum_{r=1}^{\infty} \epsilon_r F_r^\dagger F_r \quad (4.108)$$

$$\mathcal{V} = \frac{1}{2} \sum_{r,s,t,u=1}^{\infty} \langle rs | V | ut \rangle F_r^\dagger F_s^\dagger F_t F_u - \sum_{r,s=1}^{\infty} \langle r | \mathcal{V} | s \rangle F_r^\dagger F_s \quad (4.109)$$

on fermion Fock space  ${}^f \mathfrak{F}^s$  is equal to (4.103) on  $n$ -fermion Hilbert space  ${}^f_n \mathfrak{H}^s$  for all  $n = 2, 3, \dots$ .

## Comments

### 1. Fock space description

In going from (4.103) to (4.107) the description of  $n$  fermions in interaction has been carried over from a description in  $n$ -fermion Hilbert space  ${}^f_n\mathbb{H}^s$  to a description in fermion Fock space  ${}^f\mathbb{H}^s$ .

(4.107) contains no reference to the number of particles. It can be used for every system of fermions interacting via two-body potentials and with an external potential.

### 2. Two-fermion system

To describe the deuteron, for example, or two electrons, one prepares

$$|\psi\rangle = \sum_{r,s=1}^{\infty} \psi_{rs} F_r^\dagger F_s^\dagger |0\rangle \quad (4.110)$$

at time zero where

$$\psi_{rs} = \langle 0 | F_r F_s | \psi \rangle \quad (4.111)$$

is the probability amplitude that the system is in the state

$$F_r^\dagger F_s^\dagger |0\rangle \quad (4.112)$$

at time zero.  $|\psi\rangle$  is an eigenvector of the number operator (4.50) belonging to eigenvalue two.

The state at time  $t$  is

$$\begin{aligned} |\psi(t)\rangle &= U(t) |\psi\rangle = e^{-iHt/\hbar} |\psi\rangle \\ &= \sum_{r,s=1}^{\infty} \psi_{rs} e^{-iHt/\hbar} F_r^\dagger F_s^\dagger |0\rangle \end{aligned} \quad (4.113)$$



$H$  is given by (4.107). Since

$$[N, H] = 0 \quad (4.114)$$

it follows that  $|\psi(t)\rangle$  is an eigenvector of (4.50) belonging to eigenvalue two for all time.

### 3. An advantage of the Fock space description

One advantage of the Fock space description of the  $n$ -fermion system over the  $n$ -particle Hilbert space description [that is, in using (4.107) instead of (4.103)] is that (4.107) is expressed explicitly in terms of the eigenvectors and eigenvalues of (4.104).

That is, (4.107) incorporates an approximate solution of its eigenvalue problem.

### 4. Goldstone diagrams

The potential terms in (4.107) can be represented by diagrams.

J. Goldstone [J 18] was the first to develop and apply diagrammatic methods with the Fock space description of the  $n$ -fermion system.

Goldstone gave the first proof of the unlinked cluster expansion for the ground state wave function and energy of an  $n$ -fermion system.

#### 4.6.3 Hartree-Fock potential

The creation operator  $F_r^\dagger$  (4.22) and the annihilation operator  $F_r$  (4.23) are defined in terms of basis vectors in Fock space  $^f\mathfrak{H}^s$ . These basis vectors are constructed from Slater determinants (3.18) in  $n$ -fermion Hilbert space  $^f_n\mathfrak{H}^s$ . The Slater determinants are constructed from the eigenvectors  $|\phi_r\rangle$  of the single-particle Hamiltonian (4.101) for some single-particle potential  $\mathcal{V}(\xi_\alpha)$ .

Different choices of  $\mathcal{V}(\xi_\alpha)$  lead to different  $F_r$  and  $F_r^\dagger$ .

The choice of  $\mathcal{V}(\xi_\alpha)$  which gives the best independent-particle approximation to the  $n$ -particle ground state of  $H$  is

$$\langle r | \mathcal{V} | s \rangle = \sum_{t=1}^n \{ \langle rt | V | st \rangle - \langle rt | V | ts \rangle \} \quad (4.115)$$

(4.115) defines the the Hartree-Fock potential.

The Hartree-Fock approximation to the ground state of the  $n$ -particle system is

$$| F \rangle = F_1^\dagger F_2^\dagger \cdots F_n^\dagger | 0 \rangle \quad (4.116)$$

The Hartree-Fock approximation to the ground state energy of the  $n$ -particle system is

$$\langle F | H | F \rangle = \sum_{r=1}^n \langle r | H_0 | r \rangle + \frac{1}{2} \sum_{r=1}^n \langle r | \mathcal{V} | r \rangle \quad (4.117)$$

### Comments

#### 1. Ground state energy

The energy  $E_{ground}$  of the  $n$ -particle ground state of  $H$  is less than or equal

to every  $n$ -particle expectation value of  $H$ .

(4.117) is the best approximation to  $E_{ground}$  for independent-particle states.

## 2. Fermi level and Fermi energy

The single-particle energies are labelled

$$\epsilon_1 \leq \epsilon_2 \leq \dots \leq \epsilon_n \quad (4.118)$$

(4.116) is the state having the lowest  $n$  single-particle states occupied.

The highest occupied level is the Fermi level for the system.

$\epsilon_n$  is the Fermi energy of the system.

## 3. Determining the Hartree-Fock potential

$\mathcal{V}$  is determined by a self-consistent process. The steps in the process are

- i. choose  $\mathcal{V}$
- ii. determine the  $|\phi_r\rangle$  by solving (4.102)
- iii. determine a new  $\mathcal{V}$  using (4.115)
- iv. repeat the process until output  $\mathcal{V}$  equals input  $\mathcal{V}$

### Proof of (4.115)

Let  $F_r^\dagger$  and  $\hat{F}_r^\dagger$  be creation operators determined, respectively, from single-

particle potentials  $\mathcal{V}(\xi_\alpha)$  and  $\mathcal{U}(\xi_\alpha)$ . Then

$$\hat{F}_r^\dagger = F_r^\dagger + \sum_{r'=1}^{\infty} a_{rr'} F_{r'}^\dagger + \dots \quad (4.119)$$

for some numbers  $a_{rr'}$ , where  $a_{rr} = 0$  for all  $r$ .

Let

$$|F\rangle = F_1^\dagger F_2^\dagger \dots F_n^\dagger |0\rangle \quad (4.120)$$

and

$$|\hat{F}\rangle = \hat{F}_1^\dagger \hat{F}_2^\dagger \dots \hat{F}_n^\dagger |0\rangle \quad (4.121)$$

(4.120) and (4.121) are the independent-particle vectors corresponding to single-particle potentials  $\mathcal{V}(\xi_\alpha)$  and  $\mathcal{U}(\xi_\alpha)$ , respectively.

We assume that (4.120) yields the minimum expectation value of (4.107) for all independent-particle vectors. In view of (4.119), this assumption may be expressed as

$$\frac{\partial}{\partial a_{rr'}} \langle \hat{F} | H | \hat{F} \rangle |_{a_{rr'}=0} = 0 \quad (4.122)$$

(4.122) will be used to determine the single-particle potential  $\mathcal{V}(\xi_\alpha)$ .

It follows from (4.122) that the term in

$$\langle \hat{F} | H | \hat{F} \rangle \quad (4.123)$$

which is linear in the  $a_{rr'}$  must vanish. The term in (4.121) which is linear in the  $a_{rr'}$  is

$$|\hat{F}_{lin}\rangle = \sum_{r=1}^n \sum_{r'=1}^{\infty} a_{rr'} F_1^\dagger F_2^\dagger \dots F_{r'}^\dagger \dots F_n^\dagger |0\rangle \quad (4.124)$$

where the  $F_{r'}^\dagger$  occurs in the  $r$ th place. Now since

$$\left(F_r^\dagger\right)^2 = 0 \quad (4.125)$$

it follows that

$$|\tilde{F}_{lin}\rangle = \sum_{r=1}^n \sum_{r'=n+1}^{\infty} a_{rr'} F_1^\dagger F_2^\dagger \cdots F_{r'}^\dagger \cdots F_n^\dagger |0\rangle \quad (4.126)$$

It follows from (4.34) and (4.36) that (4.126) can be rewritten as

$$|\tilde{F}_{lin}\rangle = \sum_{r=1}^n \sum_{r'=n+1}^{\infty} a_{rr'} F_{r'}^\dagger F_r |F\rangle \quad (4.127)$$

Requiring the term in (4.123) which is linear in the  $a_{rr'}$  to vanish thus yields

$$\mathcal{R}_2(\langle F | H F_{r'}^\dagger F_r | F \rangle) = 0 \quad (4.128)$$

for all

$$r \leq n < r' \quad (4.129)$$

Now<sup>1</sup>

$$F_r^\dagger |F\rangle = F_{r'} |F\rangle = 0 \quad (4.130)$$

$$\langle F | F_r = \langle F | F_{r'}^\dagger = 0 \quad (4.131)$$

It follows that (4.128) can be written as

$$\langle F | [F_{r'}^\dagger, H] F_r | F \rangle = 0 \quad (4.132)$$

We now evaluate the left side of (4.132) when  $H$  is given by (4.107). It follows using (A.2) and (A.9) that

$$[F_{r'}^\dagger, H] = -\epsilon_{r'} F_{r'}^\dagger + \sum_{s=1}^{\infty} \langle s | \mathcal{V} | r' \rangle F_s^\dagger \quad (4.133)$$

$$+\frac{1}{2} \sum_{s,t,u=1}^{\infty} \{ \langle st | \mathcal{V} | ur' \rangle - \langle st | \mathcal{V} | r'u \rangle \} F_s^\dagger F_t^\dagger F_u$$

<sup>1</sup> Unprimed and primed single-particle labels obey (4.129) in the following.

Using

$$\langle F | F_r^\dagger F_r | F \rangle = 0 \quad (4.134)$$

$$\langle F | F_s^\dagger F_r | F \rangle = \delta_{rs} \quad (4.135)$$

$$\langle F | F_s^\dagger F_t^\dagger F_u F_r | F \rangle = \delta_{rs} \delta_{tu} - \delta_{rt} \delta_{us} \quad (4.136)$$

it follows that

$$\begin{aligned} & \langle F | [F_r^\dagger, H] F_r | F \rangle = \\ & = \langle r | \mathcal{V} | r' \rangle - \sum_{t=1}^n \{ \langle rt | V | r't \rangle - \langle rt | V | tr' \rangle \} \end{aligned} \quad (4.137)$$

It follows that (4.132) is satisfied if

$$\langle r | \mathcal{V} | r' \rangle = \sum_{t=1}^n \{ \langle rt | V | r't \rangle - \langle rt | V | tr' \rangle \} \quad (4.138)$$

Extending (4.138) to define all matrix elements of  $\mathcal{V}$  completes the proof of (4.115).

## Section 4.7 Particle-hole states of an $n$ -fermion system

The Hartree-Fock state  $|F\rangle$  (4.116) is the the best independent-particle approximation to the ground state of an  $n$ -particle system.  $|F\rangle$  is constructed from  $n$  creators acting on the vacuum state  $|0\rangle$ ;  $|F\rangle$  has the lowest  $n$  single-particle states  $|\phi_r\rangle$  occupied. The  $|\phi_r\rangle$  are determined by solving an eigenvalue problem involving the Hartree-Fock potential (4.115).

The single-particle states occupied in  $|F\rangle$  are called unexcited states. The single-particle states not occupied in  $|F\rangle$  are called excited states.

An improved approximation to the Hartree-Fock approximation to the ground state includes

$$\sum_{r=1}^n \sum_{r'=n+1}^{\infty} a_{rr'} F_{r'}^{\dagger} F_r |F\rangle \quad (4.139)$$

(4.139) is an  $n$ -particle state which is a linear combination of states with one particle created in the excited state  $|\phi_{r'}\rangle$  and one particle annihilated in the unexcited state  $|\phi_r\rangle$ .

It is convenient to express states like (4.139) in terms of particles and holes. This is accomplished by defining

$$H_r^{\dagger} = F_r \quad (4.140)$$

$$r = 1, 2, \dots, n \quad (4.141)$$

### Comments

#### 1. Hole creator

$H_r^{\dagger}$  is a hole creation operator or hole creator.

When operating on a basis vector  $|n\{\dots n_r \dots\}\rangle$  with  $n_r = 1$ ,  $H_r^{\dagger}$  yields a basis vector with  $n_r = 0$ .

$H_r^{\dagger}$  creates a hole in the state  $|n\{\dots 1 \dots\}\rangle$ .

## 2. Hole annihilator

$H_r$  is a hole annihilation operator or hole annihilator.

When operating on a basis vector  $|n\{\dots n_r \dots\}\rangle$  with  $n_r = 0$ ,  $H_r$  yields a basis vector with  $n_r = 1$ .

$H_r$  annihilates a hole in the state  $|n\{\dots 0 \dots\}\rangle$ .

## 3. Fundamental dynamical variables

Fundamental dynamical variables appropriate for describing an  $n$ -particle system are creators and annihilators for particles in excited states

$$F_{n+r}^\dagger \quad \text{and} \quad F_{n+r} \quad r = 1, 2, \dots, \infty \quad (4.142)$$

and creators and annihilators for holes in unexcited states

$$H_{n+1-r}^\dagger \quad \text{and} \quad H_{n+1-r} \quad r = 1, 2, \dots, n \quad (4.143)$$

The fundamental dynamical variables obey the anticommutation relations

$$\{F_{n+r}, F_{n+s}\} = \{F_{n+r}^\dagger, F_{n+s}^\dagger\} = 0 \quad (4.144)$$

$$\{F_{n+r}, F_{n+s}^\dagger\} = \delta_{rs} \quad (4.145)$$

$$r, s = 1, 2, \dots, \infty \quad (4.146)$$



$$\{H_{n+1-r}, H_{n+1-s}\} = \{H_{n+1-r}^\dagger, H_{n+1-s}^\dagger\} = 0 \quad (4.147)$$

$$\{H_{n+1-r}, H_{n+1-s}^\dagger\} = \delta_{rs} \quad (4.148)$$

$$r, s = 1, 2, \dots, n \quad (4.149)$$

$$\{F, H\} = 0 \quad (4.150)$$

$$F = F_{n+s} \quad \text{or} \quad F_{n+s}^\dagger \quad (4.151)$$

$$H = H_{n+1-r} \quad \text{or} \quad H_{n+1-r}^\dagger \quad (4.152)$$

#### 4. Symmetry in the concepts of creation and annihilation

(4.147) and (4.148) have the same form as (4.144) and (4.145).

The theory of fermions is symmetric in the concepts of creation and annihilation.

The theory of fermions deals with particles and holes on equal footing.

#### 5. Number operators

We define particle and hole number operators

$$N_{pr} = F_{n+r}^\dagger F_{n+r} \quad (4.153)$$

$$N_{hr} = H_{n+1-r}^\dagger H_{n+1-r} \quad (4.154)$$

$$N_p = \sum_{r=1}^{\infty} N_{pr} \quad (4.155)$$

$$N_h = \sum_{r=1}^n N_{hr} \quad (4.156)$$

$N_p$  is the excited-particle number operator.  $N_h$  is the hole number operator.

The number operator  $N$  (4.50) is expressed in terms of particle and hole number operators as follows:

$$N = n + N_p - N_h \quad (4.157)$$

## 6. Hamiltonian $\mathcal{H}_0$

The independent-particle Hamiltonian  $\mathcal{H}_0$  (4.108) is expressed in terms of particle and hole number operators as follows:

$$\mathcal{H}_0 = \epsilon_F + \sum_{r=1}^{\infty} \epsilon_{pr} N_{pr} - \sum_{r=1}^n \epsilon_{hr} N_{hr} \quad (4.158)$$

$$\epsilon_F = \sum_{r=1}^n \epsilon_r = \sum_{r=1}^n \epsilon_{hr} \quad (4.159)$$

$$\epsilon_{pr} = \epsilon_{n+r} \quad (4.160)$$

$$\epsilon_{hr} = \epsilon_{n+1-r} \quad (4.161)$$

$\epsilon_{pr}$  is the energy of a particle in an excited state.

$\epsilon_{hr}$  is the energy of a hole in an unexcited state.

## 7. Properties of the Hartree-Fock state

The Hartree-Fock state  $|F\rangle$  satisfies

$$F_{n+r} |F\rangle = 0 \quad (4.162)$$

$$r = 1, 2, \dots, \infty \quad (4.163)$$

$$H_{n+1-r} |F\rangle = 0 \quad (4.164)$$

$$r = 1, 2, \dots, n \quad (4.165)$$

$$N |F\rangle = n |F\rangle \quad (4.166)$$

$$N_p |F\rangle = N_h |F\rangle = 0 \quad (4.167)$$

$$\mathcal{H}_0 | F \rangle = \epsilon_F | F \rangle \quad (4.168)$$

$| F \rangle$  is an  $n$ -particle state which contains no excited particles and no holes.

$| F \rangle$  is an eigenvector of the independent-particle Hamiltonian  $\mathcal{H}_0$  belonging to eigenvalue  $\epsilon_F$ .

We call  $| F \rangle$  the Hartree-Fock sea.

### 8. Creating particles and holes

When acting on  $| F \rangle$ ,  $F_{n+r}^\dagger$  yields an  $(n+1)$ -particle basis vector with one particle in an excited state.

When acting on  $| F \rangle$ ,  $H_{n+1-r}^\dagger$  yields an  $(n-1)$ -particle basis vector with one particle removed from an unexcited state.

One says that  $F_{n+r}^\dagger$  creates a particle above the Hartree-Fock sea and  $H_{n+1-r}^\dagger$  creates a hole in the Hartree-Fock sea.

### 9. Particle-hole creators

$F_{n+r}^\dagger H_{n+1-s}^\dagger$  is a particle-hole creator.

When acting on  $| F \rangle$ ,  $F_{n+r}^\dagger H_{n+1-s}^\dagger$  yields an  $n$ -particle basis vector with one particle in an excited state and one particle removed from an unexcited state.

One says that  $F_{n+r}^\dagger H_{n+1-s}^\dagger$  creates a particle above the Hartree-Fock sea and a hole in the Hartree-Fock sea.

### 10. Particle-hole states

The  $n$ -particle state  $F_{n+r}^\dagger H_{n+1-s}^\dagger | F \rangle$  is a one-particle-one-hole state.

One can similarly construct two-particle-two-hole states, etc.

The general  $n$ -particle state is a linear combination of  $| F \rangle$  and of many-particle-many-hole states.

## Section 4.8 Correlated particle-hole states

The state  $F_{n+r}^\dagger H_{n+1-s}^\dagger | F \rangle$  of an  $n$ -particle system has a hole in single-particle state  $|\phi_{n+1-s}\rangle$  in the Hartree-Fock sea  $| F \rangle$  and a particle in excited single-particle state  $|\phi_{n+r}\rangle$  above the Hartree-Fock sea. The state labels  $r$  and  $s$  are independent; the particle-hole pair which is created is uncorrelated.

In this Section we consider states of an  $n$ -particle system which involve correlated particle-hole pairs.

Correlated particle-hole pairs are called Cooper pairs following the work of L.N. Cooper, *Bound Electron Pairs in a Degenerate Fermi Gas*, Phys. Rev. **104**, 1189 (1956).

$n$ -particle states involving Cooper pairs are called BCS states following the work of J. Bardeen, L.N. Cooper, J.R. Schrieffer, *Theory of Superconductivity*, Phys. Rev. **108**, 1175 (1957).

Each BCS state is defined as a unitary transformation of  $| F \rangle$ . The transformation is called a Bogoliubov transformation following the work of N.N. Bogoliubov, *A New Method in the Theory of Superconductivity*, Soviet Physics JETP **7**, 41 (1958) and J.G. Valatin, *Comments on the Theory of Superconductivity*, Nuovo Cimento **7**, 843 (1958).<sup>1</sup>

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<sup>1</sup> Reprints of the above papers are in D. Pines, *The Many-Body Problem*, W.A. Benjamin, Inc., 1961.

The best BCS state is defined as the BCS state which minimizes the average value of an approximate Hamiltonian involving interaction between Cooper pairs. The average obtained is less than the energy  $\epsilon_F$  for the Hartree-Fock state for any attractive interaction between Cooper pairs. That is, there is an energy gap between the best BCS state and the Hartree-Fock state. The expression for the energy gap cannot be obtained from any finite-order perturbation theory involving the pair interaction.

### 4.8.1 Cooper pairs

We define a particle-hole annihilator  $G_r$  by

$$G_r = H_{n+1-r} F_{n+r} \quad (4.169)$$

$$r = 1, 2, \dots, n \quad (4.170)$$

The corresponding particle-hole creator  $G_r^\dagger$  is

$$G_r^\dagger = F_{n+r}^\dagger H_{n+1-r}^\dagger \quad (4.171)$$

$G_r$  and  $G_r^\dagger$  satisfy:

$$(G_r)^2 = (G_r^\dagger)^2 = 0 \quad (4.172)$$

$$G_r G_r^\dagger G_r = G_r \quad (4.173)$$

$$G_r^\dagger G_r = N_{pr} N_{hr} \quad (4.174)$$

$$G_r G_r^\dagger = 1 - N_{pr} - N_{hr} + N_{pr} N_{hr} \quad (4.175)$$

### Comments

#### 1. Nomenclature

$G_r^\dagger$  creates a particle in an excited state and a hole in an unexcited state.

$G_r^\dagger$  is constructed such that the energy difference between the excited and unexcited states increases with increasing  $r$ .

$G_r^\dagger$  creates a correlated particle-hole pair.

We say that  $G_r^\dagger$  creates a Cooper pair.

#### 4.8.2 Bogoliubov transformation

We define

$$U = U_1 U_2 \cdots U_n \quad (4.176)$$

$$U_r = A_r + B_r \quad (4.177)$$

$$A_r = 1 - S_{0r} \quad (4.178)$$

$$B_r = u_r S_{0r} - i v_r S_{2r} \quad (4.179)$$

where

$$S_{0r} = G_r G_r^\dagger + G_r^\dagger G_r \quad (4.180)$$

$$S_{2r} = i(G_r^\dagger - G_r) \quad (4.181)$$

$$u_r^* = u_r \quad (4.182)$$

$$v_r^* = v_r \quad (4.183)$$

$$u_r^2 + v_r^2 = 1 \quad (4.184)$$

### Comments

#### 1. Unitary operators

The  $U_r$  are a family of number-conserving commuting unitary operators.



$$[N, U_r] = 0 \quad (4.185)$$

$$[U_r, U_s] = 0 \quad (4.186)$$

$$U_r U_r^\dagger = U_r^\dagger U_r = 1 \quad (4.187)$$

$U$  is a number-conserving unitary operator.

$$[N, U] = 0 \quad (4.188)$$

$$U U^\dagger = U^\dagger U = 1 \quad (4.189)$$

## 2. Nomenclature

Parameters analogous to  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  were introduced by Bogoliubov in his theory of superconductivity.

Accordingly, we call  $U$  a Bogoliubov transformation.

We call  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  Bogoliubov parameters.

### 4.8.3 BCS states

We define

$$|G\rangle = U |F\rangle \quad (4.190)$$

It follows from (4.176) that

$$|G\rangle = (u_n + v_n G_n^\dagger) \cdots (u_2 + v_2 G_2^\dagger) (u_1 + v_1 G_1^\dagger) |F\rangle \quad (4.191)$$

### Comments

#### 1. Nomenclature

$|G\rangle$  is an  $n$ -particle state which is linear combination of the Hartree-Fock state  $|F\rangle$  and of states formed by creating up to  $n$  Cooper pairs from  $|F\rangle$ .

A correlated pair state analogous to  $|G\rangle$  for a system of electrons forms the basis of the Bardeen, Cooper and Schrieffer (BCS) theory of superconductivity.

Accordingly, we call  $|G\rangle$  a BCS state.

Each BCS state  $|G\rangle$  is labelled by a set of Bogoliubov parameters.

#### 4.8.4 Quasiparticles and quasiholes

We define

$$Q_{pr} = U F_{n+r} U^\dagger \quad (4.192)$$

$$r = 1, 2, \dots, \infty \quad (4.193)$$

$$Q_{hr} = UH_{n+1-r}U^\dagger \quad (4.194)$$

$$r = 1, 2, \dots, n \quad (4.195)$$

It follows that

$$Q_{pr} = u_r F_{n+r} - v_r H_{n+1-r}^\dagger \quad (4.196)$$

$$Q_{hr} = u_r H_{n+1-r} + v_r F_{n+r}^\dagger \quad (4.197)$$

$$r = 1, 2, \dots, n \quad (4.198)$$

$$Q_{pr} = F_{n+r} \quad (4.199)$$

$$r = n + 1, \dots, \infty \quad (4.200)$$

It follows from (4.196) and (4.197) that

$$F_{n+r} = u_r Q_{pr} + v_r Q_{hr}^\dagger \quad (4.201)$$

$$H_{n+1-r} = u_r Q_{hr} - v_r Q_{pr}^\dagger \quad (4.202)$$

$$r = 1, 2, \dots, n \quad (4.203)$$

It follows from (4.144) to (4.152) and from (4.192) to 4.195) that

$$\{Q_{pr}, Q_{ps}\} = \{Q_{pr}^\dagger, Q_{ps}^\dagger\} = 0 \quad (4.204)$$

$$\{Q_{pr}, Q_{ps}^\dagger\} = \delta_{rs} \quad (4.205)$$

$$\{Q_{hr}, Q_{hs}\} = \{Q_{hr}^\dagger, Q_{hs}^\dagger\} = 0 \quad (4.206)$$

$$\{Q_{hr}, Q_{hs}^\dagger\} = \delta_{rs} \quad (4.207)$$

$$\{Q_{pr}, Q_{hs}\} = \{Q_{pr}, Q_{hs}^\dagger\} = 0 \quad (4.208)$$

### Comments

#### 1. Fundamental dynamical variables

(4.192) and (4.194) define fundamental dynamical variables  $Q_{pr}$  and  $Q_{hr}$ .

(4.204) to (4.208) is a fundamental algebra for the system.

#### 2. Transformation equations

(4.201) and (4.202) allow transformations from  $Q_{pr}$  and  $Q_{hr}$  to  $F_{n+r}$  and  $H_{n+1-r}$ .

### 3. Nomenclature

$Q_{pr}$  and  $Q_{hr}$  are linear combinations of particle and hole operators.

$Q_{pr}$  is a quasiparticle annihilator.  $Q_{hr}$  is a quasihole annihilator.

### 4. Number operators

We define quasiparticle and quasihole number operators

$$N_{qpr} = UN_{pr}U^\dagger = Q_{pr}^\dagger Q_{pr} \quad (4.209)$$

$$N_{qhr} = UN_{hr}U^\dagger = Q_{hr}^\dagger Q_{hr} \quad (4.210)$$

$$N_{qp} = UN_pU^\dagger = \sum_{r=1}^{\infty} N_{qpr} \quad (4.211)$$

$$N_{qh} = UN_hU^\dagger = \sum_{r=1}^n N_{qhr} \quad (4.212)$$

$N_{qp}$  is the quasiparticle number operator.  $N_{qh}$  is the quasihole number operator.

It follows from (4.157) and (4.188) that the number operator  $N$  (4.50) is expressed in terms of quasiparticle and quasihole number operators as follows:

$$N = n + N_{qp} - N_{qh} \quad (4.213)$$

## 5. Hamiltonian $U\mathcal{H}_0U^\dagger$

The independent-quasiparticle-quasihole Hamiltonian  $U\mathcal{H}_0U^\dagger$  is expressed in terms of quasiparticle and quasihole number operators as follows:

$$U\mathcal{H}_0U^\dagger = \epsilon_F + \sum_{r=1}^{\infty} \epsilon_{pr} N_{qpr} - \sum_{r=1}^n \epsilon_{hr} N_{qhr} \quad (4.214)$$

The eigenvectors of  $U\mathcal{H}_0U^\dagger$  form a basis for the Hilbert space of the system.

## 6. Properties of the BCS state

$|G\rangle$  satisfies

$$Q_{pr} |G\rangle = 0 \quad (4.215)$$

$$r = 1, 2, \dots, \infty \quad (4.216)$$

$$Q_{hr} |G\rangle = 0 \quad (4.217)$$

$$r = 1, 2, \dots, n \quad (4.218)$$

$$N |G\rangle = n |G\rangle \quad (4.219)$$

$$N_{qp} |G\rangle = N_{qh} |G\rangle = 0 \quad (4.220)$$

$$U\mathcal{H}_0U^\dagger |G\rangle = \epsilon_F |G\rangle \quad (4.221)$$

$|G\rangle$  is an  $n$ -particle state which contains no quasiparticles and no quasiholes.

$|G\rangle$  is an eigenvector of the independent-quasiparticle-quasihole Hamiltonian  $U\mathcal{H}_0U^\dagger$  belonging to eigenvalue  $\epsilon_F$ .

### 7. Quasiparticle-quasihole states

The  $n$ -particle state  $Q_{pr}^\dagger Q_{hs}^\dagger |G\rangle$  is a one-quasiparticle-one-quasihole state.

One can similarly construct two-quasiparticle-two-quasihole states, etc.

The general  $n$ -particle state is a linear combination of  $|G\rangle$  and of many-particle-many-hole states.

## 4.8.5 The best BCS state and the energy gap

Further discussion of the equations derived in this Topic is found in J. Bardeen, L.N. Cooper, J.R. Schrieffer, *Theory of Superconductivity*, Phys. Rev. **108**, 1175 (1957).

We assume that the Hamiltonian for the system can be approximated by

$$H_{pair} = \mathcal{H}_0 + V_{pair} \quad (4.222)$$

$$V_{pair} = - \sum_{r,s=1}^n g_{rs} G_r^\dagger G_s \quad (4.223)$$

$$g_{rs}^* = g_{rs} \quad g_{sr} = g_{rs} \quad g_{rr} = 0 \quad (4.224)$$

$\mathcal{H}_0$  is given by (4.158).

$V_{pair}$  is a Hermitian interaction between different Cooper pairs.

The best BCS state is defined as the state  $|G\rangle$  which minimizes

$$\bar{E}_G = \langle G | H_{pair} | G \rangle \quad (4.225)$$

It follows using (4.201) and (4.202) that

$$N_{pr} = v_r^2 + \dots \quad (4.226)$$

$$N_{hr} = v_r^2 + \dots \quad (4.227)$$

$$G_r^\dagger G_s = u_r v_r u_s v_s + \dots \quad (4.228)$$

where  $\dots$  are terms for which  $\langle G | \dots | G \rangle = 0$  and therefore



$$\bar{E}_G = \sum_{r=1}^n (u_r^2 \epsilon_{hr} + v_r^2 \epsilon_{pr} - a_r u_r v_r) \quad (4.229)$$

$$a_r = \sum_{s=1}^n g_{rs} u_s v_s \quad (4.230)$$

Minimizing

$$\bar{E}_G + \sum_{r=1}^n \lambda_r (u_r^2 + v_r^2) \quad (4.231)$$

with respect to  $u_1, u_2, \dots, u_n$  and  $v_1, v_2, \dots, v_n$  where the  $\lambda_r$  are Lagrange multipliers yields

$$2(\epsilon_{hr} + \lambda_r)u_r - a_r v_r = 0 \quad (4.232)$$

$$2(\epsilon_{pr} + \lambda_r)v_r - a_r u_r = 0 \quad (4.233)$$

$$r = 1, 2, \dots, n \quad (4.234)$$

It follows from (4.232) and (4.233) that

$$(\epsilon_{hr} + \lambda_r)u_r^2 = (\epsilon_{pr} + \lambda_r)v_r^2 \quad (4.235)$$

and therefore

$$u_r^2 = \frac{\epsilon_{pr} + \lambda_r}{\epsilon_{pr} + \epsilon_{hr} + 2\lambda_r} \quad (4.236)$$

$$v_r^2 = \frac{\epsilon_{hr} + \lambda_r}{\epsilon_{pr} + \epsilon_{hr} + 2\lambda_r} \quad (4.237)$$

We define  $\varepsilon_{0r}$  by

$$\varepsilon_{0r} = 2(\epsilon_{pr} + \lambda_r)^{\frac{1}{2}}(\epsilon_{hr} + \lambda_r)^{\frac{1}{2}} \quad (4.238)$$

It follows that

$$\epsilon_{pr} + \epsilon_{hr} + 2\lambda_r = (\varepsilon_r^2 + \varepsilon_{0r}^2)^{\frac{1}{2}} \quad (4.239)$$

$$\varepsilon_r = \epsilon_{pr} - \epsilon_{hr} \quad (4.240)$$

$$u_r^2 = \frac{1}{2} \left[ 1 + \frac{\epsilon_r}{(\epsilon_r^2 + \epsilon_{0r}^2)^{\frac{1}{2}}} \right] \quad (4.241)$$

$$v_r^2 = \frac{1}{2} \left[ 1 - \frac{\epsilon_r}{(\epsilon_r^2 + \epsilon_{0r}^2)^{\frac{1}{2}}} \right] \quad (4.242)$$

$$2u_r v_r = \frac{\epsilon_{0r}}{(\epsilon_r^2 + \epsilon_{0r}^2)^{\frac{1}{2}}} \quad (4.243)$$

$$a_r = \epsilon_{0r} \quad (4.244)$$

$$\overline{E}_G - \epsilon_F = \frac{1}{2} \sum_{r=1}^n \left[ \epsilon_r - (\epsilon_r^2 + \epsilon_{0r}^2)^{\frac{1}{2}} \right] \quad (4.245)$$

It follows from (4.230), (4.243) and (4.244) that the  $\epsilon_{0r}$  satisfy

$$\epsilon_{0r} = \frac{1}{2} \sum_{s=1}^n \frac{g_{rs} \epsilon_{0s}}{(\epsilon_s^2 + \epsilon_{0s}^2)^{\frac{1}{2}}} \quad (4.246)$$

Determining the  $\epsilon_{0r}$  from (4.246) determines  $\overline{E}_G$ . The values of the  $\epsilon_{0r}$  depend upon the values of the parameters  $g_{rs}$  which specify the interaction (4.223) between Cooper pairs.

We assume that

$$g_{rs} = 0 \quad \text{if} \quad r \text{ or } s > n_c \quad (4.247)$$

$$g_{rs} = 2g \quad \text{if} \quad r \quad s \leq n_c \quad (4.248)$$

The integer  $n_c$  specifies the number of Cooper pairs which participate in the pairing interaction  $V_{pair}$  (4.223).

It follows from (4.247) and (4.248) that

$$\varepsilon_{0r} = 0 \quad \text{if} \quad r > n_c \quad (4.249)$$

and

$$\varepsilon_{0r} = \varepsilon_0 \quad \text{if} \quad r \leq n_c \quad (4.250)$$

where

$$\frac{1}{g} = \sum_{r=1}^{n_c-1} \frac{1}{(\varepsilon_r^2 + \varepsilon_0^2)^{\frac{1}{2}}} \quad (4.251)$$

We assume that the number of energy levels is sufficiently high near the Fermi level  $\varepsilon_n$  that the summations in (4.245) and (4.251) can be replaced by

integrals.

$$\overline{E}_G - \epsilon_F = \frac{1}{2} \int_0^{\hbar\omega} \left[ \epsilon - (\epsilon^2 + \epsilon_0^2)^{\frac{1}{2}} \right] n(\epsilon) d\epsilon \quad (4.252)$$

$$\frac{1}{g} = \int_0^{\hbar\omega} \frac{n(\epsilon) d\epsilon}{(\epsilon^2 + \epsilon_0^2)^{\frac{1}{2}}} \quad (4.253)$$

$\hbar\omega = \epsilon_{n_c}$  and  $n(\epsilon)d\epsilon$  is the number of single-particle states with energy  $\epsilon$  between  $\epsilon$  and  $\epsilon + d\epsilon$ .

We further assume that

$$n(\epsilon) = n(0) \quad \text{for} \quad 0 \leq \epsilon \leq \hbar\omega \quad (4.254)$$

$n(0)$  is the energy-level density at the Fermi level.

It follows from (4.252) to (4.254) that

$$\epsilon_0 = \hbar\omega / \sinh \left[ \frac{1}{n(0)g} \right] \quad (4.255)$$

$$\bar{E}_G - \epsilon_F = \frac{1}{4}n(0)(\hbar\omega)^2 \left\{ 1 - \left[ 1 + \left( \frac{\epsilon_0}{\hbar\omega} \right)^2 \right]^{\frac{1}{2}} \right\} - \frac{\epsilon_0^2}{4g} \quad (4.256)$$

That is,

$$\bar{E}_G - \epsilon_F = -\frac{1}{2}n(0)(\hbar\omega)^2 \left[ \frac{1}{e^{2x} - 1} + \frac{2x}{e^{2x} - 2 + e^{-2x}} \right] \quad (4.257)$$

$$x = \frac{1}{n(0)g} \quad (4.258)$$

### Comments

#### 1. Energy gap

It follows from (4.257) that

$$\bar{E}_G < \epsilon_F \quad \text{if} \quad g > 0 \quad (4.259)$$

That is, if there is an attractive force between Cooper pairs, no matter how weak, the energy of the best BCS state  $|G\rangle$  is less than the energy of Hartree-Fock state  $|F\rangle$ .

There is an energy gap between  $|G\rangle$  and  $|F\rangle$ .

#### 2. Nonperturbative result

The right side of (4.257) has an essential singularity when  $g = 0$ .

(4.257) cannot be obtained from any finite-order perturbation theory involving expansion in powers of  $g$ .





## Chapter 5

## FOCK SPACE FOR FERMIONS: PART 2

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### Section 5.1 Introductory remarks

In Chapter 4 we considered a system of identical fermions with Fock space as the Hilbert space for the system. The fundamental dynamical variables are creators  $F_r^\dagger$  (4.22) and annihilators  $F_r$  (4.23) satisfying the fundamental algebra (4.34) to (4.36). The  $F_r^\dagger$  and  $F_r$  are constructed using a denumerable set of vectors which span the one-fermion Hilbert space.

In this Chapter we continue the consideration of a system of identical fermions with Fock space as the Hilbert space for the system. In this Chapter we consider creators and annihilators constructed using nondenumerable sets of eigenkets which span the one-fermion Hilbert space.

In Section 5.2 we construct creators and annihilators using coordinate-space/spin eigenkets. This leads to quantum field theory for fermions.

In Sections 5.3 and 5.4 we construct creators and annihilators using momentum-space/spin and momentum-space/helicity eigenkets, respectively.

The Poincare generators for a Lorentz invariant system of free fermions are given in Section 5.5.

The Galilei generators for a Galilei invariant system of interacting fermions are given in Section 5.6.

## Section 5.2 Creators and annihilators labelled by position and spin

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### 5.2.1 Definitions

The creator  $F_r^\dagger$  (4.22) and the annihilator  $F_r$  (4.23) are defined in terms of the denumerable set of vectors  $|\phi_r\rangle_\alpha$  (3.1) which form an orthonormal basis for the Hilbert space for a one-fermion system with rest mass  $m$  and spin  $s$ .<sup>1</sup> The  $|\phi_r\rangle_\alpha$  are simultaneous eigenvectors of specified one-particle operators.

We recall from Chapter 9 of Part I that the simultaneous eigenkets  $|x, m_s\rangle_\alpha$  of the Cartesian position  $X_\alpha^1, X_\alpha^2, X_\alpha^3$  and of the  $z$ -component of spin  $S_\alpha^3$  may be used as an orthonormal basis

$$1_\alpha = \sum_{m_s=-s}^{+s} \int d^3x |x, m_s\rangle_\alpha \langle x, m_s| \quad (5.1)$$

$$\langle x, m_s|_\alpha |x', m'_s\rangle_\alpha = \delta(x - x') \delta_{m_s m'_s} \quad (5.2)$$

for the Hilbert space for a one-particle system with rest mass  $m$  and spin  $s$ .

The function

$$\phi_{rm_s}(x) = \langle x, m_s|_\alpha |\phi_r\rangle_\alpha \quad (5.3)$$

is the coordinate-space/spin representative of  $|\phi_r\rangle_\alpha$ .

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<sup>1</sup> The subscript  $\alpha$  serves as a reminder that the states are defined in the Hilbert space for fermion  $\alpha$ .

These functions satisfy

$$\sum_{m_s=-s}^{+s} \int d^3x \phi_{rm_s}^*(x) \phi_{um_s}(x) = \delta_{ru} \quad (5.4)$$

$$\sum_{r=1}^{\infty} \phi_{rm_s}^*(x) \phi_{rm'_s}(x') = \delta(x-x') \delta_{m_s m'_s} \quad (5.5)$$

We define

$$F_{m_s}^\dagger(x) = \sum_{r=1}^{\infty} \phi_{rm_s}^*(x) F_r^\dagger \quad (5.6)$$

$$F_{m_s}(x) = \sum_{r=1}^{\infty} \phi_{rm_s}(x) F_r \quad (5.7)$$

It follows using (4.29) to (4.31) that

$$F_{m_s}(x) |0\rangle = 0 \quad (5.8)$$

$$\langle 0 | F_{m_s}^\dagger(x) = 0 \quad (5.9)$$

$$F_{m_s}^\dagger(x) |0\rangle = \left( 0, |x, m_s, \underset{1}{>}, 0, \dots \right) \quad (5.10)$$

It follows using (5.4) and (5.5) that

$$F_r^\dagger = \sum_{m_s=-s}^{+s} \int d^3x \phi_{r m_s}(x) F_{m_s}^\dagger(x) \quad (5.11)$$

$$F_r = \sum_{m_s=-s}^{+s} \int d^3x \phi_{r m_s}^*(x) F_{m_s}(x) \quad (5.12)$$

### Comments

#### 1. Fundamental dynamical variables

(5.6) and (5.7) define fundamental dynamical variables  $F_{m_s}^\dagger(x)$  and  $F_{m_s}(x)$  for a system of identical fermions each with rest mass  $m$  and spin  $s$ .

#### 2. Transformation equations

(5.11) and (5.12) allow transformations from  $F_{m_s}^\dagger(x)$  and  $F_{m_s}(x)$  to  $F_r^\dagger$  and  $F_r$  given by (4.22) and (4.23).

#### 3. Quantum field theory

$F_{m_s}^\dagger(x)$  and  $F_{m_s}(x)$  are labelled by the continuous variable  $x$ .

$F_{m_s}^\dagger(x)$  and  $F_{m_s}(x)$  are quantum field operators in the Schrodinger picture.<sup>2</sup>

The description of a system of identical fermions using quantum fields as fundamental dynamical variables is called quantum field theory (QFT) of fermions.

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<sup>2</sup> A field is a function of the coordinates of a point in space.

#### 4. One-fermion state

When acting on the vacuum state  $|0\rangle$ ,  $F_{m_s}^\dagger(x)$  creates an elementary fermion at position  $x$  with rest mass  $m$ , spin  $s$  and  $z$ -component of spin  $m_s$ .

The general one-fermion state at time  $t$  is

$$|\psi(t)\rangle = \sum_{m_s=-s}^{+s} \int d^3x \psi_{m_s}(x, t) F_{m_s}^\dagger(x) |0\rangle \quad (5.13)$$

$$\psi_{m_s}^*(x, t) \psi_{m_s}(x, t) d^3x \quad (5.14)$$

is the probability that the fermion is in the volume  $dx dy dz$  about the point  $(x, y, z)$  with  $z$ -component of spin  $m_s$  at time  $t$ .

$\psi_{m_s}(x, t)$  is the coordinate-space/spin wave function of the fermion.

#### 5. Second quantization

Definitions (5.6) and (5.7) have led to the notion that the field operators (5.6) and (5.7) are given by a process of “second quantization” applied to the functions  $\phi_{r m_s}(x)$ .

The phrase “second quantization” arises because one describes the quantum mechanics of a single fermion in terms of  $\phi_{r m_s}(x)$  representing the state of the particle then one uses these same functions to construct operators appropriate for describing the quantum mechanics of an assembly of identical fermions.

“Second quantization” is misleading. It gives the impression that the description of a physical system using field theory is beyond quantum mechanics or that it requires a modification of quantum mechanics. It doesn't.

People say “second quantization” when they mean “the Hilbert space is Fock space”.

### 5.2.2 Anticommutation relations

It follows from (5.6) and (5.7) and (4.34) to (4.35) that

$$\{F_{m_s}(x), F_{m'_s}(x')\} = 0 \quad (5.15)$$

$$\{F_{m_s}^\dagger(x), F_{m'_s}^\dagger(x')\} = 0 \quad (5.16)$$

$$\{F_{m_s}(x), F_{m'_s}^\dagger(x')\} = \delta(x - x')\delta_{m_s m'_s} \quad (5.17)$$

(5.15) to (5.17) are a fundamental algebra for the system of fermions.

Some commutators of products of fermion creators and annihilators are given in the Appendix.

### 5.2.3 One- and two-particle operators

The operators (4.58) and (4.77) on fermion Fock space  ${}^f\mathfrak{H}^s$  are equal, respectively, to the one- and two-particle operators (4.60) and (4.79) on  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{H}^s$  for all  $n$ .

It follows using (5.11) and (5.12) that (4.58) may be expressed as

$$A = \int d^3x d^3x' F^\dagger(x) A(x, x') F(x') \quad (5.18)$$

where

$$F(x) = \begin{pmatrix} F_s(x) \\ F_{s-1}(x) \\ \vdots \\ F_{-s}(x) \end{pmatrix} \quad (5.19)$$

$F^\dagger(x)$  is the corresponding  $2s + 1$  row matrix and  $A(x, x')$  is the  $2s + 1$  by  $2s + 1$  square matrix

$$(A(x, x'))_{m_s m'_s} = \langle x, m_s | A(\xi_\alpha) | x', m'_s \rangle_\alpha \quad (5.20)$$

of one-particle matrix elements.

It follows using (5.11) and (5.12) that (4.77) may be expressed as

$$A = \int d^3 x d^3 y d^3 x' d^3 y' F^\dagger(x) F^\dagger(y) A(x, y, x', y') F(y') F(x') \quad (5.21)$$

where  $A(x, y, x', y')$  is the tensor

$$(A(x, y, x', y'))_{m_\alpha m_\beta m'_\alpha m'_\beta} = \langle x, m_\alpha | \langle y, m_\beta | A(\xi_\alpha, \xi_\beta) | x', m'_\alpha \rangle_\alpha | y', m'_\beta \rangle_\beta \quad (5.22)$$

of two-particle matrix elements.

## Section 5.3 Creators and annihilators labelled by momentum and spin

### 5.3.1 Definitions

In Section 5.2 we used the simultaneous eigenkets  $|x, m_s\rangle_\alpha$  of the Cartesian position  $X_\alpha^1, X_\alpha^2, X_\alpha^3$  and of the  $z$ -component of spin  $S_\alpha^3$  for fermion  $\alpha$  to define the creation operator  $F_{m_s}^\dagger(x)$  (5.6) and the annihilation operator  $F_{m_s}(x)$  (5.7). The  $|x, m_s\rangle_\alpha$  may be used as an orthonormal basis for the Hilbert space for a one-fermion system with rest mass  $m$  and spin  $s$ .

We recall from Chapter 9 of Part I that the simultaneous eigenkets  $|p, m_s\rangle_\alpha$  of the Cartesian momentum  $P_\alpha^1, P_\alpha^2, P_\alpha^3$  and of the  $z$ -component of spin  $S_\alpha^3$  may be used as an orthonormal basis

$$1_\alpha = \sum_{m_s=-s}^{+s} \int d^3p |p, m_s\rangle_\alpha \langle p, m_s| \quad (5.23)$$

$$\langle p, m_s|_\alpha |p', m'_s\rangle_\alpha = \delta(p - p') \delta_{m_s m'_s} \quad (5.24)$$

for the Hilbert space for a one-particle system with rest mass  $m$  and spin  $s$ .

The  $|p, m_s\rangle_\alpha$  and  $|x, m_s\rangle_\alpha$  eigenkets are related according to

$$\langle x, m_s|_\alpha |p, m'_s\rangle_\alpha = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{ip \cdot x/\hbar} \delta_{m_s m'_s} \quad (5.25)$$



We define

$$F_{m_s}^\dagger(p) = \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \int d^3x e^{ip \cdot x / \hbar} F_{m_s}^\dagger(x) \quad (5.26)$$

$$F_{m_s}(p) = \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \int d^3x e^{-ip \cdot x / \hbar} F_{m_s}(x) \quad (5.27)$$

It follows using (5.8) to (5.10) that

$$F_{m_s}(p) | 0 \rangle = 0 \quad (5.28)$$

$$\langle 0 | F_{m_s}^\dagger(p) = 0 \quad (5.29)$$

$$F_{m_s}^\dagger(p) | 0 \rangle = \left( 0, | p, m_s \rangle_1, 0, \dots \right) \quad (5.30)$$

It follows from (5.26) and (5.27) that

$$F_{m_s}^\dagger(x) = \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \int d^3p e^{-ip \cdot x / \hbar} F_{m_s}^\dagger(p) \quad (5.31)$$

$$F_{m_s}(x) = \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \int d^3p e^{ip \cdot x / \hbar} F_{m_s}(p) \quad (5.32)$$

## Comments

### 1. Fundamental dynamical variables

(5.26) and (5.27) define fundamental dynamical variables  $F_{m_s}^\dagger(p)$  and  $F_{m_s}(p)$  for a system of identical fermions each with rest mass  $m$  and spin  $s$ .

### 2. Transformation equations

(5.31) and (5.32) allow transformations from  $F_{m_s}^\dagger(p)$  and  $F_{m_s}(p)$  to  $F_{m_s}^\dagger(x)$  and  $F_{m_s}(x)$  given by (5.6) and (5.7).

### 3. One-fermion state

When acting on the vacuum state  $|0\rangle$ ,  $F_{m_s}^\dagger(p)$  creates an elementary fermion with rest mass  $m$ , spin  $s$ ,  $z$ -component of spin  $m_s$  and momentum  $p$ .

The general one-fermion state at time  $t$  is

$$|\psi(t)\rangle = \sum_{m_s=-s}^{+s} \int d^3p \psi_{m_s}(p, t) F_{m_s}^\dagger(p) |0\rangle \quad (5.33)$$

$$\psi_{m_s}^*(p, t) \psi_{m_s}(p, t) d^3p \quad (5.34)$$

is the probability that the fermion has momentum in the volume  $dp^1 dp^2 dp^3$  about  $(p^1, p^2, p^3)$  with  $z$ -component of spin  $m_s$  at time  $t$ .

$\psi_{m_s}(p, t)$  is the momentum-space/spin wave function for the fermion.

### 5.3.2 Anticommutation relations

It follows from (5.26) and (5.27) and from (5.15) to (5.17) that

$$\{F_{m_s}(p), F_{m'_s}(p')\} = 0 \quad (5.35)$$

$$\{F_{m_s}^\dagger(p), F_{m'_s}^\dagger(p')\} = 0 \quad (5.36)$$

$$\{F_{m_s}(p), F_{m'_s}^\dagger(p')\} = \delta(p - p') \delta_{m_s m'_s} \quad (5.37)$$

It follows also that

$$\{F_{m_s}(x), F_{m'_s}^\dagger(p)\} = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{ip \cdot x/\hbar} \delta_{m_s m'_s} \quad (5.38)$$

(5.35) to (5.37) are a fundamental algebra for a system of identical fermions each with rest mass  $m$  and spin  $s$ .

### 5.3.3 One- and two-particle operators

The operators (5.18) and (5.21) on fermion Fock space  ${}^f\mathfrak{F}^s$  are equal, respectively, to the one- and two-particle operators (4.60) and (4.79) on  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{F}^s$  for all  $n$ .

It follows using (5.31) and (5.32) that (5.18) can be written as

$$A = \int d^3 p d^3 p' F^\dagger(p) A(p, p') F(p') \quad (5.39)$$

where

$$F(p) = \begin{pmatrix} F_s(p) \\ F_{s-1}(p) \\ \vdots \\ F_{-s}(p) \end{pmatrix} \quad (5.40)$$

$F^\dagger(p)$  is the corresponding  $2s + 1$  row matrix and  $A(p, p')$  is the  $2s + 1$  by  $2s + 1$  square matrix

$$(A(p, p'))_{m_s m'_s} = \langle p, m_s | A(\xi_\alpha) | p', m'_s \rangle_\alpha \quad (5.41)$$

of one-particle matrix elements.

It follows using (5.31) and (5.32) that (5.21) can be written as

$$A = \int d^3 p d^3 q d^3 p' d^3 q' F^\dagger(p) F^\dagger(q) A(p, q, p', q') F(q') F(p') \quad (5.42)$$

where  $A(p, q, p', q')$  is the tensor

$$\begin{aligned}
& (A(p, q, p', q'))_{m_\alpha m_\beta m'_\alpha m'_\beta} \\
& = \langle p, m_\alpha | \langle q, m_\beta | A(\xi_\alpha, \xi_\beta) | p', m'_\alpha \rangle | q', m'_\beta \rangle
\end{aligned} \tag{5.43}$$

of two-particle matrix elements.

## Section 5.4 Creators and annihilators labelled by momentum and helicity

### 5.4.1 Definitions

In Section 5.3 we used the simultaneous eigenkets  $|p, m_s\rangle_\alpha$  of the Cartesian momentum  $P_\alpha^1, P_\alpha^2, P_\alpha^3$  and of the  $z$ -component of spin  $S_\alpha^3$  for particle  $\alpha$  to define the creation operator  $F_{m_s}^\dagger(p)$  (5.26) and the annihilation operator  $F_{m_s}(p)$  (5.27). The  $|p, m_s\rangle_\alpha$  may be used as an orthonormal basis for the Hilbert space for a one-fermion system with rest mass  $m$  and spin  $s$ .

We recall from Chapter 9 of Part I that the simultaneous eigenkets  $|h^\lambda(p)\rangle_\alpha$  of the Cartesian momentum  $P_\alpha^1, P_\alpha^2, P_\alpha^3$  and the helicity  $\Lambda$  may be used as an orthonormal basis

$$1_\alpha = \sum_{\lambda=-s}^{+s} \int d^3p |h^\lambda(p)\rangle_\alpha \langle h^\lambda(p)| \tag{5.44}$$

$$\langle h^\lambda(p) | h^{\lambda'}(p') \rangle_\alpha = \delta(p - p') \delta_{\lambda\lambda'} \tag{5.45}$$

for the Hilbert space for a one-particle system with rest mass  $m$  and spin  $s$ .

The  $|h^\lambda(p)\rangle_\alpha$  and  $|p, m_s\rangle_\alpha$  eigenkets are related according to

$$\langle p', m_s | h^\lambda(p) \rangle_\alpha = \delta(p - p') D_{m_s \lambda}^s(\varphi, \theta, 0) \quad (5.46)$$

$(\theta, \varphi)$  are the spherical polar coordinates of  $\vec{p}$  in the fixed inertial frame and

$$D_{m' m}^j(\alpha, \beta, \gamma) \quad (5.47)$$

are rotation matrices.

We define

$$F^{\lambda\dagger}(p) = \sum_{m_s=-s}^{+s} D_{m_s \lambda}^s(\varphi, \theta, 0) F_{m_s}^\dagger(p) \quad (5.48)$$

$$F^\lambda(p) = \sum_{m_s=-s}^{+s} D_{m_s \lambda}^{s*}(\varphi, \theta, 0) F_{m_s}(p) \quad (5.49)$$

It follows using (5.28) to (5.30) that

$$F^\lambda(p) |0\rangle = 0 \quad (5.50)$$

$$\langle 0 | F^{\lambda\dagger}(p) = 0 \quad (5.51)$$

$$F^{\lambda\dagger}(p) | 0 \rangle = \left( 0, | h^\lambda(p) \rangle_1, 0, \dots \right) \quad (5.52)$$

It follows from (5.48) and (5.49) that

$$F_{m_s}^\dagger(p) = \sum_{\lambda=-s}^{+s} D_{m_s\lambda}^{s*}(\varphi, \theta, 0) F^{\lambda\dagger}(p) \quad (5.53)$$

$$F_{m_s}(p) = \sum_{\lambda=-s}^{+s} D_{m_s\lambda}^s(\varphi, \theta, 0) F^\lambda(p) \quad (5.54)$$

### Comments

#### 1. Fundamental dynamical variables

(5.48) and (5.49) define fundamental dynamical variables  $F^{\lambda\dagger}(p)$  and  $F^\lambda(p)$  for a system of identical fermions each with rest mass  $m$  and spin  $s$ .

#### 2. Transformation equations

(5.53) and (5.54) allow transformations from  $F^{\lambda\dagger}(p)$  and  $F^\lambda(p)$  to  $F_{m_s}^\dagger(p)$  and  $F_{m_s}(p)$  given by (5.26) and (5.27).

#### 3. One-fermion state

When acting on the vacuum state  $| 0 \rangle$ ,  $F^{\lambda\dagger}(p)$  creates an elementary fermion with rest mass  $m$ , spin  $s$ , momentum  $p$  and helicity  $\lambda$ .

The general one-fermion state at time  $t$  is

$$|\psi(t)\rangle = \sum_{\lambda=-s}^{+s} \int d^3p \psi^\lambda(p, t) F^{\lambda\dagger}(p) |0\rangle \quad (5.55)$$

$$\psi^{\lambda*}(p, t) \psi^\lambda(p, t) d^3p \quad (5.56)$$

is the probability that the fermion has momentum in the volume  $dp^1 dp^2 dp^3$  about  $(p^1, p^2, p^3)$  with helicity  $\lambda$  at time  $t$ .

$\psi^\lambda(p, t)$  is the momentum-space/helicity wave function for the fermion.

We call from Chapter 9 of Part I that for a fermion with zero rest mass which is described by a Hamiltonian which is not invariant under space inversion

$$\psi^\lambda(p, t) = 0 \quad \text{unless either } \lambda = +s \quad \text{or} \quad \lambda = -s \quad (5.57)$$

### 5.4.2 Anticommutation relations

It follows from (5.48) and (5.49) and from (5.35) to (5.37) that

$$\{F^\lambda(p), F^{\lambda'}(p')\} = 0 \quad (5.58)$$

$$\{F^{\lambda\dagger}(p), F^{\lambda'\dagger}(p')\} = 0 \quad (5.59)$$

$$\{F^\lambda(p), F^{\lambda'\dagger}(p')\} = \delta(p - p') \delta_{\lambda\lambda'} \quad (5.60)$$



It follows also that

$$\left\{ F_{m_s}(x), F^{\lambda\dagger}(p) \right\} = \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} e^{ip \cdot x / \hbar} D_{m_s \lambda}^s(\varphi, \theta, 0) \quad (5.61)$$

$$\left\{ F_{m_s}(p), F^{\lambda\dagger}(p') \right\} = \delta(p - p') D_{m_s \lambda}^s(\varphi, \theta, 0) \quad (5.62)$$

(5.58) to (5.60) are a fundamental algebra for a system of identical fermions each with rest mass  $m$  and spin  $s$ .

### 5.4.3 One- and two-particle operators

The operators (5.39) and (5.42) on fermion Fock space  ${}^f \mathfrak{F}^s$  are equal, respectively, to the one- and two-particle operators (4.60) and (4.79) on  $n$ -fermion Hilbert space  ${}^f_n \mathfrak{F}^s$  for all  $n$ .

It follows using (5.53) and (5.54) that (5.39) can be written as

$$A = \int d^3 p d^3 p' \mathcal{F}^\dagger(p) \mathcal{A}(p, p') \mathcal{F}(p') \quad (5.63)$$

where

$$\mathcal{F}(p) = \begin{pmatrix} F^s(p) \\ F^{s-1}(p) \\ \vdots \\ F^{-s}(p) \end{pmatrix} \quad (5.64)$$

$\mathcal{F}^\dagger(p)$  is the corresponding  $2s + 1$  row matrix and  $\mathcal{A}(p, p')$  is the  $2s + 1$  by  $2s + 1$  square matrix

$$(\mathcal{A}(p, p'))_{\lambda\lambda'} = \langle h^\lambda(p) | A(\xi_\alpha) | h^{\lambda'}(p') \rangle_\alpha \quad (5.65)$$

of one-particle matrix elements.

It follows using (5.53) and (5.54) that (5.42) can be written as

$$A = \int d^3 p d^3 q d^3 p' d^3 q' \mathcal{F}^\dagger(p) \mathcal{F}^\dagger(q) \mathcal{A}(p, q, p', q') \mathcal{F}(q') \mathcal{F}(p') \quad (5.66)$$

$\mathcal{A}(p, q, p', q')$  is the tensor

$$(\mathcal{A}(p, q, p', q'))_{\lambda_\alpha \lambda_\beta \lambda'_\alpha \lambda'_\beta} = \langle h^{\lambda_\alpha}(p) | \langle h^{\lambda_\beta}(q) | A(\xi_\alpha, \xi_\beta) | h^{\lambda'_\alpha}(p') \rangle_\alpha | h^{\lambda'_\beta}(q') \rangle_\beta \quad (5.67)$$

of two-particle matrix elements.

## Section 5.5 Lorentz invariant system of free fermions

Equations (5.18), (5.39), (5.63) and (5.21), (5.42), (5.66) are general expressions for operators on fermion Fock space  $f\mathfrak{F}^s$  which are equal, respectively, to one- and two-particle operators (4.60) and (4.79) on  $n$ -fermion Hilbert space  $f_n\mathfrak{F}^s$  for all  $n$ .

These general expressions involve different sets of fundamental dynamical variables. The variables are expressed as column matrices  $F(x)$  (5.19),  $F(p)$  (5.40),  $\mathcal{F}(p)$  (5.64) and corresponding row matrices  $F^\dagger(x)$ ,  $F^\dagger(p)$ ,  $\mathcal{F}^\dagger(p)$ .

We consider some examples of these general expressions in this Section. In particular, we give the Poincare generators of a Lorentz invariant system of free fermions.

### 5.5.1 Poincare Algebra

To describe a physical system which is Lorentz invariant, one must construct the Poincare generators from the fundamental dynamical variables. That is, one must construct ten Hermitian operators  $H, P^j, J^j, K^j$  (where  $j = 1, 2, 3$ ) which satisfy the Poincare Algebra

$$[P^j, P^k] = 0 \quad (5.68)$$

$$[P^j, H] = 0 \quad (5.69)$$

$$[J^j, P^k] = i\hbar\epsilon_{jkl}P^l \quad (5.70)$$

$$[J^j, H] = 0 \quad (5.71)$$

$$[J^j, J^k] = i\hbar\epsilon_{jkl}J^l \quad (5.72)$$

$$[K^j, P^k] = -i\hbar\delta_{jk}H/c^2 \quad (5.73)$$

$$[K^j, H] = -i\hbar P^j \quad (5.74)$$

$$[K^j, J^k] = i\hbar\epsilon_{jkl}K^l \quad (5.75)$$

$$[K^j, K^k] = -i\hbar\epsilon_{jkl}J^l/c^2 \quad (5.76)$$

The Poincare Algebra (5.68) to (5.76) may be written in the form

$$[P^\mu, P^\nu] = 0 \quad (5.77)$$

$$[M^{\mu\nu}, P^\sigma] = i\hbar(g^{\nu\sigma}P^\mu - g^{\mu\sigma}P^\nu) \quad (5.78)$$

$$[M^{\mu\nu}, M^{\sigma\tau}] = i\hbar(g^{\nu\sigma}M^{\mu\tau} - g^{\mu\sigma}M^{\nu\tau} + g^{\nu\tau}M^{\sigma\mu} - g^{\mu\tau}M^{\sigma\nu}) \quad (5.79)$$

where

$$P^\mu = \left( \frac{1}{c}H, P^1, P^2, P^3 \right) \quad (5.80)$$

$$(M^{23}, M^{31}, M^{12}) = (J^1, J^2, J^3) \quad (5.81)$$

$$(M^{01}, M^{02}, M^{03}) = (cK^1, cK^2, cK^3) \quad (5.82)$$

$$M^{\mu\nu} = -M^{\nu\mu} \quad (5.83)$$

### 5.5.2 Poincare generators

For a system of free identical fermions each with rest mass  $m$  and spin  $s$ , (5.68) to (5.76) are satisfied when

$$H = \int d^3p \epsilon(p) F^\dagger(p) F(p) = \int d^3p \epsilon(p) \mathcal{F}^\dagger(p) \mathcal{F}(p) \quad (5.84)$$

$$\epsilon(p) = \sqrt{p^2 c^2 + m^2 c^4} \quad (5.85)$$

$$\begin{aligned}
P^j &= \int d^3 p p^j F^\dagger(p) F(p) = \int d^3 p p^j \mathcal{F}^\dagger(p) \mathcal{F}(p) \\
&= -i\hbar \int d^3 x F^\dagger(x) \frac{\partial}{\partial x^j} F(x)
\end{aligned} \tag{5.86}$$

$$J^j = -i\hbar \int d^3 x F^\dagger(x) (x \times \nabla)^j F(x) + \int d^3 x F^\dagger(x) s^j F(x) \tag{5.87}$$

$$K^j = -\frac{i\hbar}{2c^2} \int d^3 p \epsilon(p) F^\dagger(p) \frac{\partial}{\partial p^j} F(p) + \int d^3 p F^\dagger(p) \frac{(s \times p)^j}{\epsilon(p) + mc^2} F(p) \tag{5.88}$$

The  $s^j$  in (5.87) and (5.88) are  $2s + 1$  by  $2s + 1$  matrices satisfying

$$[s^j, s^k] = i\hbar \epsilon_{jkl} s^l \tag{5.89}$$

The number operator  $N$  for the system is

$$N = \int d^3 x F^\dagger(x) F(x) = \int d^3 p F^\dagger(p) F(p) = \int d^3 p \mathcal{F}^\dagger(p) \mathcal{F}(p) \tag{5.90}$$

The helicity  $\Lambda$  for the system is

$$\Lambda = \int d^3 p \mathcal{F}^\dagger(p) s^3 \mathcal{F}(p) \quad (5.91)$$

when  $s^3$  is diagonal.

### Comment

#### 1. Conservation of number of fermions

The generators (5.84) to (5.88) and the helicity (5.91) commute with  $N$ . The number of fermions in the system is constant in time.

In Chapter 9 we consider a system of interacting fermions and bosons where the number of bosons in the system is not constant in time.

## Section 5.6 Galilei invariant system of interacting fermions

(5.84) to (5.88) are the Poincare generators for a Lorentz invariant system of free fermions. In this Section we give the Galilei generators for a Galilei invariant system of interacting fermions.

### 5.6.1 Galilei Algebra

To describe a physical system which is Galilean invariant, one must construct the Galilei generators from the fundamental dynamical variables. That is, one must construct ten Hermitian operators  $H, P^j, J^j, K^j$  (where  $j = 1, 2, 3$ ) which

satisfy the Galilei Algebra

$$[P^j, P^k] = 0 \quad (5.92)$$

$$[P^j, H] = 0 \quad (5.93)$$

$$[J^j, P^k] = i\hbar\epsilon_{jkl}P^l \quad (5.94)$$

$$[J^j, H] = 0 \quad (5.95)$$

$$[J^j, J^k] = i\hbar\epsilon_{jkl}J^l \quad (5.96)$$

$$[K^j, P^k] = -i\hbar m\delta_{jk} \quad (5.97)$$

$$[K^j, H] = -i\hbar P^j \quad (5.98)$$

$$[K^j, J^k] = i\hbar\epsilon_{jkl}K^l \quad (5.99)$$

$$[K^j, K^k] = 0 \quad (5.100)$$



## 5.6.2 Galilei generators

For a system of identical fermions each with rest mass  $m$  and spin  $s$  interacting via central spin-independent two-body potentials

$$V(\xi_\alpha, \xi_\beta) = V(|X_\alpha - X_\beta|) \quad (5.101)$$

(5.92) to (5.100) are satisfied when  $P^j$  and  $J^j$  are given by (5.86) and (5.87) and

$$K^j = -mX^j \quad (5.102)$$

$$X^j = \frac{1}{N} \int d^3x F^\dagger(x) x^j F(x) \quad (5.103)$$

$$H = H_0 + V \quad (5.104)$$

$$\begin{aligned} H_0 &= \int d^3p F^\dagger(p) \frac{p^2}{2m} F(p) = \int d^3p \mathcal{F}^\dagger(p) \frac{p^2}{2m} \mathcal{F}(p) \\ &= -\frac{\hbar^2}{2m} \int d^3x F^\dagger(x) \nabla^2 F(x) \end{aligned} \quad (5.105)$$

$$\begin{aligned}
V &= \frac{1}{2} \int d^3 r d^3 R \\
F^\dagger \left( R + \frac{1}{2} r \right) F^\dagger \left( R - \frac{1}{2} r \right) V(|r|) F \left( R - \frac{1}{2} r \right) F \left( R + \frac{1}{2} r \right) \\
&= \frac{1}{2} \int d^3 k d^3 k' d^3 K \\
F^\dagger \left( \frac{1}{2} K + k \right) F^\dagger \left( \frac{1}{2} K - k \right) V(|k' - k|) F \left( \frac{1}{2} K - k' \right) F \left( \frac{1}{2} K + k' \right) \\
&= \frac{1}{2} \int d^3 k d^3 k' d^3 K \\
\mathcal{F}^\dagger \left( \frac{1}{2} K + k \right) \mathcal{F}^\dagger \left( \frac{1}{2} K - k \right) V(|k' - k|) \mathcal{F} \left( \frac{1}{2} K - k' \right) \mathcal{F} \left( \frac{1}{2} K + k' \right)
\end{aligned} \tag{5.106}$$

$$V(|k' - k|) = \left( \frac{1}{2\pi\hbar} \right)^3 \int d^3 r e^{i(k' - k) \cdot r / \hbar} V(|r|) \tag{5.107}$$

### Comments

#### 1. Centre of mass position operator

(5.103) is the  $j$ —th component of the centre of mass position operator for the system.

It follows on calculation that

$$[X^j, X^k] = 0 \quad (5.108)$$

$$[P^j, P^k] = 0 \quad (5.109)$$

$$[X^j, P^k] = i\hbar\delta_{jk} \quad (5.110)$$

## 2. Generalization of the Galilei generators

(5.104) describes a Galilean invariant system of fermions interacting via central spin-independent two-body potentials given by (5.101).

(5.104) can be generalized to describe a Galilean invariant system of fermions interacting via noncentral spin-dependent two-body potentials.

Hsieh [T2] contains such a generalization for a system of nucleons.

## 3. Creating a physical fermion

It follows from (5.104) that

$$H F_{m_s}^\dagger(p) |0\rangle = \frac{p^2}{2m} F_{m_s}^\dagger(p) |0\rangle \quad (5.111)$$

$F_{m_s}^\dagger(p) |0\rangle$  is an eigenket of the Hamiltonian for the interacting system.

When acting on the vacuum state  $|0\rangle$ ,  $F_{m_s}^\dagger(p)$  creates a physical fermion

with mass  $m$ , spin  $s$ ,  $z$ -component of spin  $m_s$ , momentum  $p$  and energy  $p^2/2m$ .

#### 4. Physical particles and elementary particles

The elementary particle of the theory is a physical particle when the Hamiltonian is given by (5.104).

This is not surprising. For example, (5.104) and its generalization to include spin-dependent two-body interactions describe nuclear systems for energies below the threshold for producing pions. The elementary particles of the theory are physical nucleons: one says "Nuclei are composed of physical nucleons."

#### 5. Form of the interaction

There is no contribution from (5.106) to the right side of (5.111) because (5.106) contains two annihilators.

It follows that

$$VF_{m_s}^\dagger(p) | 0 \rangle = 0 \quad (5.112)$$

Elementary particles are physical particles for every  $V$  for which (5.112) holds.

In Chapter 9 we consider interaction in a system of fermions and bosons for which (5.112) does not hold.

For such a system, the elementary particles are not physical particles.

## Chapter 6

# SPIN $\frac{1}{2}$ FERMIONS AND ANTIFERMIONS

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### Section 6.1 Introductory remarks

In this Chapter we give the Fock space description of a system of free fermions and antifermions.

An antifermion is the antiparticle of a fermion. The particles of the fermion-antifermion pair have the same rest mass and spin, and opposite charge, parity and fermion number. The fermion is the particle with positive parity.

We do not consider a system of interacting fermions and antifermions in this Chapter because experiment indicates that a complete description of such systems must also involve boson variables. The Fock space description of a system of bosons is given in Chapter 7.

For definiteness, we consider a system of spin  $\frac{1}{2}$  fermions and antifermions. This could, for example, be a system of

- electrons and positrons
- muons and antimuons
- neutrinos and antineutrinos<sup>1</sup>
- quarks and antiquarks
- nucleons and antinucleons

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<sup>1</sup> Neutrinos and antineutrinos are considered explicitly in Section 6.6.

Fermions and antifermions are treated on the same footing. Both appear in the formalism with positive energy and both evolve forward in time. The theory is finite.

It will be recalled from Chapter 10 of Part I that the first theory of antiparticles was invented by Dirac in 1930. Dirac suggested that all negative energy states of his Hamiltonian for the electron were occupied with one particle in each state in accordance with the Pauli Principle. A hole in this negative energy sea is manifested as a positron state. This brilliant suggestion is a measure of Dirac's genius, but it is not a satisfactory picture of antiparticles. Associating an infinite negative energy sea with a single fermion (and not making this same association with a single boson) is an unnecessary complication to the description of systems of fermions and antifermions.

In Section 6.5 we show how Dirac's hole theory arises from an incorrect interpretation of the formalism for fermions and antifermions given in Sections 6.2 and 6.3.

In Section 6.6 we consider a system of free neutrinos and antineutrinos.

In Section 6.7 we give a covariant description of the fermion-antifermion system which involves construction of the Dirac field  $\psi(x)$  and determination of the Poincare generators from a Lorentz invariant Lagrangian density. The Dirac field satisfies the Dirac equation.

## Section 6.2 Fundamental dynamical variables

We consider the Hilbert space for a system of spin  $\frac{1}{2}$  fermions and antifermions to be

$$f\bar{f}_{\mathfrak{H}^{\frac{1}{2}}} = \mathcal{A} f_{\mathfrak{H}^{\frac{1}{2}}} \otimes \bar{f}_{\mathfrak{H}^{\frac{1}{2}}} \quad (6.1)$$

$f\bar{f}_{\mathbb{H}^{\frac{1}{2}}}$  is the antisymmetric direct product of Fock space  $f_{\mathbb{H}^{\frac{1}{2}}}$  for spin  $\frac{1}{2}$  fermions and Fock space  $\bar{f}_{\mathbb{H}^{\frac{1}{2}}}$  for spin  $\frac{1}{2}$  antifermions.

The fundamental dynamical variables are momentum/helicity creators and annihilators for spin  $\frac{1}{2}$  fermions

$$F^{\lambda\dagger}(p) \quad (6.2)$$

$$F^{\lambda}(p) \quad (6.3)$$

( $\lambda = \pm\frac{1}{2}$ ) and momentum/helicity creators and annihilators for spin  $\frac{1}{2}$  antifermions

$$\bar{F}^{\lambda\dagger}(p) \quad (6.4)$$

$$\bar{F}^{\lambda}(p) \quad (6.5)$$

These operators satisfy

$$\{F^{\lambda}(p), F^{\lambda'}(p')\} = 0 \quad (6.6)$$

$$\{F^{\lambda\dagger}(p), F^{\lambda'\dagger}(p')\} = 0 \quad (6.7)$$

$$\{F^{\lambda}(p), F^{\lambda'\dagger}(p')\} = \delta(p - p')\delta_{\lambda\lambda'} \quad (6.8)$$

$$\{\bar{F}^\lambda(p), \bar{F}^{\lambda'}(p')\} = 0 \quad (6.9)$$

$$\{\bar{F}^{\lambda\dagger}(p), \bar{F}^{\lambda'\dagger}(p')\} = 0 \quad (6.10)$$

$$\{\bar{F}^\lambda(p), \bar{F}^{\lambda'\dagger}(p')\} = \delta(p - p')\delta_{\lambda\lambda'} \quad (6.11)$$

$$\{F, \bar{F}\} = 0 \quad (6.12)$$

$$F = F^\lambda(p) \quad \text{or} \quad F^{\lambda\dagger}(p) \quad (6.13)$$

$$\bar{F} = \bar{F}^\lambda(p) \quad \text{or} \quad \bar{F}^{\lambda\dagger}(p) \quad (6.14)$$

### Comments

1. Equation (6.12): Pauli Exclusion Principle for fermions and antifermions

(6.12) expresses the unique relationship between a fermion and the corresponding antifermion. If the particles were unrelated, (6.12) would be replaced by a commutation relation.

(6.12) ensures the anticommutation relations (6.114) to (6.117) satisfied by the Dirac field  $\psi(x)$ .

(6.12) is consistent with the experimental observation that the  $^1S_0$  state of positronium decays to 2 photons and the  $^3S_1$  state decays to 3 photons.



## Section 6.3 Observables

We list a few observables for the system in this Section. In doing so, it is convenient to group the four annihilation operators (6.3) and (6.5) as the column matrix

$$\vec{F}(p) = \begin{pmatrix} F^{+\frac{1}{2}}(p) \\ F^{-\frac{1}{2}}(p) \\ \bar{F}^{+\frac{1}{2}}(p) \\ \bar{F}^{-\frac{1}{2}}(p) \end{pmatrix} \quad (6.15)$$

The four creation operators (6.2) and (6.4) are grouped as the corresponding row matrix.

### Four—momentum $P^\mu$

The four-momentum for a Lorentz invariant system of free fermions with nonzero rest mass  $m$  and spin  $s$  is given by (5.84) and (5.86).

The four-momentum for a Lorentz invariant system of free antifermions with nonzero rest mass  $m$  and spin  $\frac{1}{2}$  is given by (5.84) and (5.86) with  $F^\lambda(p)$  replaced by  $\bar{F}^\lambda(p)$ .

The four-momentum  $P^\mu$  for a Lorentz invariant system of free fermions and antifermions with nonzero rest mass  $m$  and spin  $\frac{1}{2}$  is the sum of the above expressions.

$$P^\mu = \int d^3p \bar{F}^\dagger(p) p^\mu \vec{F}(p) \quad (6.16)$$

$$p^\mu = \left( \frac{\epsilon(p)}{c}, p^1, p^2, p^3 \right) \quad (6.17)$$

$$\epsilon(p) = \sqrt{p^2 c^2 + m^2 c^4} \quad (6.18)$$

### Fermion number operator $N$

$$N = \int d^3 p \bar{F}^\dagger(p) \beta \bar{F}(p) \quad (6.19)$$

$$\beta = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \quad (6.20)$$

### Charge $Q$

$$Q = qN \quad (6.21)$$

For electrons and positrons,  $q = e = -1.60 \times 10^{-19} \text{C}$ .

### Helicity $\Lambda$

$$\Lambda = \frac{1}{2} \hbar \int d^3 p \bar{F}^\dagger(p) \Sigma^3 \bar{F}(p) \quad (6.22)$$

$$\Sigma^3 = \begin{pmatrix} \sigma^3 & 0 \\ 0 & \sigma^3 \end{pmatrix} \quad (6.23)$$

### Comments

#### 1. Vacuum state

The vacuum state  $|0\rangle$  contains no fermions or antifermions.

$$F^\lambda(p) |0\rangle = \bar{F}^\lambda(p) |0\rangle = 0 \quad (6.24)$$

$$\langle 0| F^{\lambda\dagger}(p) = \langle 0| \bar{F}^{\lambda\dagger}(p) = 0 \quad (6.25)$$

#### 2. Creating fermions and antifermions

When acting on the vacuum state  $|0\rangle$ ,  $F^{\lambda\dagger}(p)$  creates a spin  $\frac{1}{2}$  fermion with rest mass  $m$ , four-momentum  $p^\mu$  and helicity  $\lambda$ .

When acting on the vacuum state  $|0\rangle$ ,  $\bar{F}^{\lambda\dagger}(p)$  creates a spin  $\frac{1}{2}$  antifermion with rest mass  $m$ , four-momentum  $p^\mu$  and helicity  $\lambda$ .

#### 3. One-fermion state

The general one-fermion state has the form

$$|\psi(t)\rangle = \sum_{\lambda=-\frac{1}{2}}^{+\frac{1}{2}} \int d^3p \psi^\lambda(p, t) F^{\lambda\dagger}(p) |0\rangle \quad (6.26)$$

$$\psi^{\lambda*}(p, t)\psi^\lambda(p, t)d^3p \quad (6.27)$$

is the probability that the fermion is in the volume  $d^3p$  about the point  $(p^1, p^2, p^3)$  at time  $t$  with helicity  $\lambda$ .

#### 4. One-antifermion state

The general one-antifermion state has the form

$$|\psi(t)\rangle = \sum_{\lambda=-\frac{1}{2}}^{+\frac{1}{2}} \int d^3p \psi^\lambda(p, t) \bar{F}^{\lambda\dagger}(p) |0\rangle \quad (6.28)$$

$$\psi^{\lambda*}(p, t)\psi^\lambda(p, t)d^3p \quad (6.29)$$

is the probability that the antifermion is in the volume  $d^3p$  about the point  $(p^1, p^2, p^3)$  at time  $t$  with helicity  $\lambda$ .

#### 5. One-fermion one-antifermion state

The general one-fermion one-antifermion state has the form

$$|\psi(t)\rangle = \sum_{\lambda=-\frac{1}{2}}^{+\frac{1}{2}} \sum_{\lambda'=-\frac{1}{2}}^{+\frac{1}{2}} \int d^3p d^3p' \psi^{\lambda\lambda'}(p, p', t) F^{\lambda\dagger}(p) \bar{F}^{\lambda'\dagger}(p') |0\rangle \quad (6.30)$$

$$\psi^{\lambda\lambda'*}(p, p', t)\psi^{\lambda\lambda'}(p, p', t)d^3p d^3p' \quad (6.31)$$

is the probability that the fermion-antifermion pair is in the volume  $d^3p d^3p'$  about the point  $(p, p')$  at time  $t$  with helicities  $\lambda$  and  $\lambda'$ .

## 6. Observables

It is an experimental fact that the fermion number is conserved.

All observables involve pairs of fermion and antifermion creation operators and pairs of fermion and antifermion annihilation operators.

It follows from (6.12) that all fermion observables commute with all antifermion observables.

### Section 6.4 Space inversion, time reversal, charge conjugation

The space inversion and time reversal transformations for a general physical system are discussed in Chapter 6 of Part I. In this Section we specify how the fundamental dynamical variables (6.2) to (6.5) transform under space inversion, time reversal and charge conjugation.

#### 6.4.1 Space inversion

The space inversion operator  $P$  on fermion-antifermion Fock space  $\bar{f}\bar{f}_F^s$  is a linear operator satisfying

$$PP^\dagger = P^\dagger P = 1 \quad (6.32)$$

$$P^2 = 1 \quad (6.33)$$

The fundamental dynamical variables (6.3) and (6.5) transform under space inversion as follows:<sup>1</sup>

$$PF^\lambda(p)P^\dagger = F^{-\lambda}(-p) \quad (6.34)$$

$$P\bar{F}^\lambda(p)P^\dagger = -\bar{F}^{-\lambda}(-p) \quad (6.35)$$

Using Dirac's representation of the  $\gamma$ -matrices, (6.34) and (6.35) can be combined into

$$P\ddot{F}(p)P^\dagger = \gamma^1\gamma^5\ddot{F}(-p) \quad (6.36)$$

It follows using (6.36) that

$$PHP^\dagger = H \quad (6.37)$$

$$PP^jP^\dagger = -P^j \quad (6.38)$$

$$PAP^\dagger = -A \quad (6.39)$$

$$PNP^\dagger = N \quad (6.40)$$

$$PQP^\dagger = Q \quad (6.41)$$

<sup>1</sup> We recall that the fermion has positive parity and the antifermion has negative parity.

## 6.4.2 Time reversal

The time reversal operator  $T$  on fermion-antifermion Fock space  $f\bar{f}_{\mathbb{F}^s}$  is an antilinear operator satisfying

$$TT^\dagger = T^\dagger T = 1 \quad (6.42)$$

$$T^2 = 1 \quad (6.43)$$

The fundamental dynamical variables (6.3) and (6.5) transform under time reversal as follows:

$$TF^\lambda(p)T^\dagger = F^\lambda(-p) \quad (6.44)$$

$$T\bar{F}^\lambda(p)T^\dagger = \bar{F}^\lambda(-p) \quad (6.45)$$

(6.44) and (6.45) can be combined into

$$T\ddot{F}(p)T^\dagger = \ddot{F}(-p) \quad (6.46)$$

It follows using (6.46) that

$$THT^\dagger = H \quad (6.47)$$

$$TP^jT^\dagger = -P^j \quad (6.48)$$

$$TAT^\dagger = \Lambda \quad (6.49)$$

$$TNT^\dagger = N \quad (6.50)$$

$$TQT^\dagger = Q \quad (6.51)$$

### 6.4.3 Charge conjugation

The charge conjugation operator  $C$  on fermion-antifermion Fock space  $\bar{f}f_{\mathbb{H}^s}$  is a linear operator satisfying

$$CC^\dagger = C^\dagger C = 1 \quad (6.52)$$

$$C^2 = 1 \quad (6.53)$$

$C$  changes particles to antiparticles. The fundamental dynamical variables (6.3) and (6.5) transform under charge conjugation as follows:



$$CF^\lambda(p)C^\dagger = \bar{F}^\lambda(p) \quad (6.54)$$

$$C\bar{F}^\lambda(p)C^\dagger = F^\lambda(p) \quad (6.55)$$

Using Dirac's representation of the  $\gamma$ -matrices, (6.54) and (6.55) can be combined into

$$C\bar{F}(p)C^\dagger = \gamma^5 \bar{F}(p) \quad (6.56)$$

It follows using (6.56) that

$$CHC^\dagger = H \quad (6.57)$$

$$CP^jC^\dagger = P^j \quad (6.58)$$

$$C\Lambda C^\dagger = \Lambda \quad (6.59)$$

$$CNC^\dagger = -N \quad (6.60)$$

$$CQC^\dagger = -Q \quad (6.61)$$

#### 6.4.4 CP transformation

The fundamental dynamical variables (6.3) and (6.5) transform under the combined operations of charge conjugation and space inversion as follows:

$$(CP)F^\lambda(p)(CP)^\dagger = \bar{F}^{-\lambda}(-p) \quad (6.62)$$

$$(CP)\bar{F}^\lambda(p)(CP)^\dagger = -F^{-\lambda}(-p) \quad (6.63)$$

Using Dirac's representation of the  $\gamma$ -matrices, (6.62) and (6.63) can be combined into

$$(CP)\ddot{F}(p)(CP)^\dagger = \gamma^1 \ddot{F}(-p) \quad (6.64)$$

#### 6.4.5 CPT transformation

The fundamental dynamical variables (6.3) and (6.5) transform under the combined operations of charge conjugation, space inversion and time reversal as follows:

$$(CPT)F^\lambda(p)(CPT)^\dagger = \bar{F}^{-\lambda}(p) \quad (6.65)$$

$$(CPT)\bar{F}^\lambda(p)(CPT)^\dagger = -F^{-\lambda}(p) \quad (6.66)$$

Using Dirac's representation of the  $\gamma$ -matrices, (6.65) and (6.66) can be combined to

$$(CPT)\bar{F}(p)(CPT)^\dagger = \gamma^0\gamma^1\bar{F}(p) \quad (6.67)$$

### Comments

#### 1. Invariance of the Hamiltonian

$H$  given by (6.16) satisfies (6.37), (6.47) and (6.57).

The Hamiltonian for a Lorentz invariant system of free spin  $\frac{1}{2}$  fermions and antifermions with nonzero rest mass  $m$  is invariant under space inversion, time reversal and charge conjugation.

## Section 6.5 Dirac's hole theory

We recall from Chapter 10 of Part I that Dirac's hole theory was the first theory of antiparticles. In the hole theory, a positron is identified with a hole in the filled sea of electron negative energy states. This sea (the Dirac sea) has infinite negative energy and infinite negative charge.

In this Section we show how Dirac's hole theory arises from an incorrect interpretation of the formalism given in Section 6.3.

It follows from (6.11) that

$$\bar{F}^{\dagger\lambda}(p)\bar{F}^\lambda(p') = -\bar{F}^\lambda(p)\bar{F}^{\dagger\lambda}(p') + \delta(p-p') \quad (6.68)$$

Accordingly, we define  $\widehat{H}, \widehat{P}^j, \widehat{N}, \widehat{Q}, \widehat{\Lambda}$  analogous to (6.16) to (6.22) but with

$$\overline{F}^{\dagger\lambda}(p)\overline{F}^\lambda(p') \quad (6.69)$$

replaced by

$$-\overline{F}^\lambda(p)\overline{F}^{\dagger\lambda}(p') \quad (6.70)$$

We define  $F^r(p)$  ( $r = 1, 2, 3, 4$ ) by

$$\begin{pmatrix} F^1(p) \\ F^2(p) \\ F^3(p) \\ F^4(p) \end{pmatrix} = \begin{pmatrix} F^{+\frac{1}{2}}(p) \\ F^{-\frac{1}{2}}(p) \\ \overline{F}^{-\frac{1}{2}\dagger}(-p) \\ \overline{F}^{\dagger+\frac{1}{2}}(-p) \end{pmatrix} \quad (6.71)$$

It follows from (6.6) to (6.12) that

$$\{F^r(p), F^{r'}(p')\} = 0 \quad (6.72)$$

$$\{F^{r\dagger}(p), F^{r'\dagger}(p')\} = 0 \quad (6.73)$$

$$\{F^r(p), F^{r'\dagger}(p')\} = \delta(p - p')\delta_{rr'} \quad (6.74)$$

where  $r, r' = 1, 2, 3, 4$ .

We write

$$\hat{F}(p) = \begin{pmatrix} F^1(p) \\ F^2(p) \\ F^3(p) \\ F^4(p) \end{pmatrix} \quad (6.75)$$

Then

$$\hat{H} = \int d^3 p \hat{F}^\dagger(p) \beta \epsilon(p) \hat{F}(p) \quad (6.76)$$

$$\hat{P}^j = \int d^3 p \hat{F}^\dagger(p) p^j \hat{F}(p) \quad (6.77)$$

$$\hat{\Lambda} = \frac{1}{2} \hbar \int d^3 p \hat{F}^\dagger(p) \Sigma^3 \hat{F}(p) \quad (6.78)$$

$$\hat{N} = \int d^3 p \hat{F}^\dagger(p) \hat{F}(p) \quad (6.79)$$

$$\hat{Q} = q \hat{N} \quad (6.80)$$

### Comments

#### 1. Dirac's hole theory: observables

Dirac's hole theory corresponds to assuming (incorrectly) that (6.76) to (6.80)

are the observable Hamiltonian,  $j$ —th component of momentum, helicity, fermion number and charge of the system.

## 2. Dirac's hole theory: positive and negative energy states

Dirac's hole theory deals with a single species of particle characterized by the annihilation operators (6.75) and the corresponding creation operators. These operators obey (6.72) to (6.74).

The particle has positive fermion number, charge  $q$ , and possesses both positive energy states ( $r = 1, 2$ ) and negative energy states ( $r = 3, 4$ ).

The states with  $r = 1, 3$  have positive helicity; the states with  $r = 2, 4$  have negative helicity.

## 3. The vacuum state and the Dirac sea

It follows from (6.71) that

$$F^r(p) | 0 \rangle = 0 \quad r = 1, 2 \quad (6.81)$$

$$F^r(p) | 0 \rangle \neq 0 \quad r = 3, 4 \quad (6.82)$$

In Dirac's hole theory, the vacuum state  $| 0 \rangle$  is the state with all positive energy states unoccupied and all negative energy states occupied.

The vacuum state  $| 0 \rangle$  is the Dirac sea.

Consistent with this,  $\langle 0 | \hat{N} | 0 \rangle$ ,  $\langle 0 | \hat{Q} | 0 \rangle$  and  $\langle 0 | \hat{H} | 0 \rangle$  are all infinite.

## 4. Holes in the Dirac sea and antiparticles

$F^3(p)$  and  $F^4(p)$  create holes in the Dirac sea.

It follows from (6.71) that

$$F^3(p) | 0 \rangle = \overline{F}^{-\frac{1}{2}\dagger}(-p) | 0 \rangle \quad (6.83)$$

$$F^4(p) | 0 \rangle = \overline{F}^{+\frac{1}{2}\dagger}(-p) | 0 \rangle \quad (6.84)$$

As assumed by Dirac, a hole in the negative energy sea corresponds to an antiparticle.

#### 5. Dirac's hole theory and relativistic quantum field theory

Dirac's hole theory was a brilliant invention in 1930. It paved the way for the development of relativistic quantum field theory in the 1930's, but it is not an appropriate description of fermions and antifermions since it is based on the incorrect assumption that the observable Hamiltonian,  $j$ -th component of momentum, fermion number, charge and helicity of the system are given by (6.76) to (6.78).

### Section 6.6 Neutrinos and antineutrinos

In this Section we consider a system of noninteracting neutrinos and antineutrinos. Neutrinos and antineutrinos are spin  $\frac{1}{2}$  fermions and antifermions. We assume each has zero rest mass.

There are three neutrino-antineutrino families:

$$(\nu_e, \bar{\nu}_e) \quad (\nu_\mu, \bar{\nu}_\mu) \quad (\nu_\tau, \bar{\nu}_\tau) \quad (6.85)$$

It is an experimental fact that in each family the neutrino is left-handed and the antineutrino is right-handed.

Fundamental dynamical variables for a system of a single family of neutrinos and antineutrinos are momentum/helicity creation and annihilation operators (6.2) to (6.5) satisfying anticommutation relations (6.6) to (6.12).

Observables for the system are constructed analogously to (6.16) to (6.22). Instead of involving the four-element column matrix (6.15) and its adjoint, however, they involve the two-element column matrix

$$\hat{F}(p) = \begin{pmatrix} F^{-\frac{1}{2}}(p) \\ \overline{F}^{+\frac{1}{2}}(p) \end{pmatrix} \quad (6.86)$$

and its adjoint.

States of the system involve

$$F^{-\frac{1}{2}\dagger}(p) \quad \text{and} \quad \overline{F}^{+\frac{1}{2}\dagger}(p) \quad (6.87)$$

acting on the vacuum state  $|0\rangle$ .

### Comments

#### 1. Helicity

We recall from Chapter 7 of Part I that a single-particle state of a particle with zero rest mass can be characterized by a unique value of the helicity.



A one-neutrino state has helicity  $-\frac{1}{2}\hbar$ . A one-antineutrino state has helicity  $+\frac{1}{2}\hbar$ .

## 2. Fictitious neutrino and antineutrino creators

The operators

$$F^{+\frac{1}{2}\dagger}(p) \quad \text{and} \quad \overline{F}^{-\frac{1}{2}\dagger}(p) \quad (6.88)$$

and their adjoints are mathematical objects devoid of physical meaning.

(6.88) create fictitious right-handed neutrinos and left-handed antineutrinos.

A smaller Hilbert space without (6.88) and their adjoints can also be used for the description of neutrino-antineutrino systems.

### 6.6.1 Hamiltonian

The Hamiltonian for a Lorentz invariant system of noninteracting neutrinos and antineutrinos is

$$H = \int d^3p \dot{F}^\dagger(p) \epsilon(p) \dot{F}(p) \quad (6.89)$$

$$\epsilon(p) = pc \quad (6.90)$$

Transformation of momentum/helicity creators and annihilators under space inversion, time reversal and charge conjugation has been considered in Section 6.4. It follows from (6.34), (6.35), (6.44), (6.45), (6.54) and (6.55) that for the

Hamiltonian (6.89)

$$PHP^\dagger \neq H \quad (6.91)$$

$$THT^\dagger = H \quad (6.92)$$

$$CHC^\dagger \neq H \quad (6.93)$$

$$(CP)H(CP)^\dagger = H \quad (6.94)$$

$$(CPT)H(CPT)^\dagger = H \quad (6.95)$$

### Comments

#### 1. Nonconservation of parity

(6.91) states that parity is not conserved for a system of neutrinos and antineutrinos.

Nonconservation of parity is implied by the experimental fact that the neutrino is left-handed and the antineutrino is right-handed. (6.89) accommodates this fact.

#### 2. Invariance under T, CP and CPT

It follows from (6.92), (6.94) and (6.95) that a system of neutrinos and antineutrinos is time-reversal invariant and invariant under the combined operations of CP and CPT.

## Section 6.7 The Dirac field and the Lagrangian method

We return to the consideration of a Lorentz invariant system of free spin  $\frac{1}{2}$  fermions and antifermions with nonzero rest mass  $m$ . The four-momentum of the system is given by (6.16).

The formalism developed so far in this Chapter is not manifestly covariant. That is, it has not been expressed in terms of quantities which transform under Lorentz transformations like scalars, vectors, tensors, etc.

The noncovariance of the formalism does not present any problem in the description of a system of free fermions and antifermions. It does, however, complicate the description of a system of fermions and antifermions interacting with themselves or with bosons.

With this point in mind, in this Section we construct a fermion-antifermion field  $\psi(x)$  (the Dirac field) which transforms under restricted Lorentz transformations as a four-component spinor. The Dirac field satisfies the Dirac equation. In addition, we describe the Lagrangian method for the system.

### 6.7.1 Definition of the Dirac field

We define the Dirac field  $\psi(x)$  by

$$\psi(x) = e^{iP \cdot x/\hbar} \psi e^{-iP \cdot x/\hbar} \quad (6.96)$$

$$\psi = a \sum_{r=1,2} \int \frac{d^3 p}{\sqrt{\epsilon(p)}} \left[ u^r(p) F^\lambda(p) + u^{r+2}(p) \bar{F}^{\lambda\dagger}(-p) \right] \quad (6.97)$$

$$u^r(p) = (u_d^r(p)) \quad d = 1, 2, 3, 4 \quad (6.98)$$

$$r + \lambda = \frac{3}{2} \quad (6.99)$$

$$a = \sqrt{\frac{mc}{(2\pi)^3 \hbar}} \quad (6.100)$$

As in Chapter 10 of Part I, the  $u^r(p)$  are Dirac spinors (four-element column matrices). The 16 functions  $u_d^r(p)$  are given explicitly in Chapter 10 of Part I.

## 6.7.2 Properties of the Dirac field

### 1. Notation

In this Section  $x$  stands for the 4-vector  $x^\mu = (x^0, x^1, x^2, x^3)$ .

We recall from the Appendix of Part I that  $P.x = P_\mu x^\mu$ .

As in Part I  $\square = \partial^\mu \partial_\mu$  and  $\partial = \gamma^\mu \partial_\mu$ .

We define  $\partial_0^{\leftrightarrow}$  by

$$A \partial_0^{\leftrightarrow} B = A \partial_0 B - (\partial_0 A) B \quad (6.101)$$

### 2. Fermion and antifermion variables

$\psi$  involves fermion and antifermion variables: (6.97) contains fermion annihilators and antifermion creators.

### 3. Heisenberg picture

$\psi(x)$  is the space-translation of the operator  $\psi$  to the point  $(x^1, x^2, x^3)$  in the Heisenberg picture.

$$\psi(x) = e^{iHt/\hbar} D(x^1, x^2, x^3) \psi D^\dagger(x^1, x^2, x^3) e^{-iHt/\hbar} \quad (6.102)$$

$$D(x^1, x^2, x^3) = D^1(x^1) D^2(x^2) D^3(x^3) \quad (6.103)$$

$$D^j(a) = e^{-iP^j a/\hbar} \quad (6.104)$$

### 4. Transformation under a homogeneous Lorentz transformation

$\psi$  transforms under a homogeneous Lorentz transformation as a Lorentz spinor.

$$U(\Lambda, 0) \psi U^\dagger(\Lambda, 0) = S^{-1}(\Lambda) \psi \quad (6.105)$$

$S(\Lambda)$  is a  $4 \times 4$  representation of the group  $SL(2, c)$ .

### 5. Transformation under a restricted Lorentz transformation

$\psi(x)$  transforms under a restricted Lorentz transformation according to

$$U(\Lambda, a)\psi(x)U^\dagger(\Lambda, a) = S^{-1}(\Lambda)\psi(\Lambda x + a) \quad (6.106)$$

## 6. Field equation

$\psi(x)$  satisfies

$$i\hbar\partial^\mu\psi(x) = [\psi(x), P^\mu] \quad (6.107)$$

## 7. Explicit expression for $\psi(x)$

$\psi(x)$  may be written in the form

$$\psi(x) = a \sum_{r=1,2} \int \frac{d^3p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} u^r(p) F^\lambda(p) + e^{ip \cdot x / \hbar} u^{r+2}(-p) \bar{F}^{\lambda\dagger}(p) \right] \quad (6.108)$$

## 8. Dirac equation

$\psi(x)$  satisfies the Dirac equation.

$$\left( -i\partial + \frac{mc}{\hbar} \right) \psi(x) = 0 \quad (6.109)$$

### 9. Klein-Gordon equation

$\psi(x)$  satisfies the Klein-Gordon equation

$$\left[ \square + \left( \frac{mc}{\hbar} \right)^2 \right] \psi(x) = 0 \quad (6.110)$$

### 10. Fermion and antifermion annihilators in terms of $\psi(x)$

(6.108) may be inverted to give

$$F^\lambda(p) = i \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} u^{r\dagger}(p) \int d^3x \sqrt{\frac{c}{2\epsilon(p)}} e^{ip \cdot x / \hbar} \partial_0^{\leftrightarrow} \psi(x) \quad (6.111)$$

$$\bar{F}^{\lambda\dagger}(-p) = i \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} u^{r+2\dagger}(p) \int d^3x \sqrt{\frac{c}{2\epsilon(p)}} e^{ip \cdot x / \hbar} \partial_0^{\leftrightarrow} \psi(x) \quad (6.112)$$

$$r = 1, 2 \quad r + \lambda = \frac{3}{2} \quad (6.113)$$

### 11. Equal-time anticommutation relation

The following anticommutation relation<sup>2</sup> for the components of  $\psi(x)$  holds

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<sup>2</sup> (6.114) determines the constant  $a$  in (6.97).

for all  $x$  and  $y$  with  $x^0 = y^0$

$$\left\{ \psi_a(x), \psi_{a'}^\dagger(y) \right\}_{x^0=y^0} = \delta^3(x-y) \delta_{aa'} \quad (6.114)$$

## 12. General anticommutation relations

The following anticommutation relations for the components of  $\psi(x)$  hold for all  $x$  and  $y$

$$\{ \psi_a(x), \psi_{a'}(y) \} = 0 \quad (6.115)$$

$$\{ \bar{\psi}_a(x), \bar{\psi}_{a'}(y) \} = 0 \quad (6.116)$$

$$\{ \psi_a(x), \bar{\psi}_{a'}(y) \} = S_{aa'}(x-y) \quad (6.117)$$

where

$$\bar{\psi}(x) = \psi^\dagger(x) \gamma^0 \quad (6.118)$$

$$S(x) = \left( -i\partial + \frac{mc}{\hbar} \right) \Delta(x) \quad (6.119)$$

$$\Delta(x) = -\frac{ic}{(2\pi)^3 \hbar} \int \frac{d^3 p}{\epsilon(p)} \sin(p \cdot x / \hbar) \quad (6.120)$$



13. Equation for  $\bar{\psi}(x)$

$\bar{\psi}(x)$  satisfies

$$\bar{\psi}(x) \left( i\partial^{\leftarrow} + \frac{mc}{\hbar} \right) = 0 \quad (6.121)$$

Comments

1. Fundamental dynamical variables

In view of (6.111) and (6.112),  $\psi(x)$  and  $\bar{\psi}(x)$  are fundamental dynamical variables for a system of spin  $\frac{1}{2}$  fermions and antifermions each with rest mass  $m$ .

(6.115) to (6.117) are a fundamental algebra for the system.

2. The function  $\Delta(x)$

(6.120) is a Lorentz scalar.

$$\Delta(\Lambda x) = \Delta(x) \quad (6.122)$$

(6.120) can be written as

$$\Delta(x) = \frac{1}{(2\pi)^3 \hbar} \int d^4 p \delta(p \cdot p - m^2 c^2) s(p^0) e^{-ip \cdot x / \hbar} \quad (6.123)$$

$$p_0 = \frac{\epsilon(p)}{c} \quad (6.124)$$

$$s(p^0) = \begin{cases} +1 & \text{if } p^0 > 0 \\ -1 & \text{if } p^0 < 0 \end{cases} \quad (6.125)$$

(6.123) makes the Lorentz invariance of  $\Delta(x)$  manifest.

It follows from (6.120) and (6.122) that

$$\Delta(x - y) = 0 \quad \text{if } x - y \text{ is space-like} \quad (6.126)$$

Proof of (6.126):

$\Delta(x - y) = 0$  when  $x^0 = y^0$  since the integrand in (6.120) is odd; in view of (6.122),  $\Delta(x - y) = 0$  for all space-like  $x - y$ .

### 3. The matrix $S(x)$

(6.119) may be written in the form

$$S(x) = \frac{1}{(2\pi)^4 \hbar^2} \int d^4 p e^{-ip \cdot x / \hbar} \frac{p \cdot \gamma + mc}{p \cdot p - m^2 c^2 + i\eta} \quad (6.127)$$

#### 4. Microcausality Condition

It follows from (6.117) and (6.126) that

$$\{\psi(x), \bar{\psi}(y)\} = 0 \quad \text{if } x - y \text{ is space-like} \quad (6.128)$$

Since every observable  $O(x)$  of the system is a function of bilinear combinations of  $\psi(x)$  and  $\bar{\psi}(x)$ , it follows using (6.128) that

$$[O(x), O(y)] = 0 \quad \text{if } x - y \text{ is space-like} \quad (6.129)$$

$O(x)$  and  $O(y)$  are compatible if  $x - y$  is space-like.

(6.129) is the quantal version of the statement that no effect travels faster than the speed of light. Measurements of any observable at any two space-like separated points do not affect each other.

(6.129) is the causality condition for a system of spin  $\frac{1}{2}$  fermions and antifermions.

(6.128) is the Microcausality Condition for a system of spin  $\frac{1}{2}$  fermions and antifermions.

### 6.7.3 The Dirac Lagrangian

(6.109) and (6.121) are the Euler-Lagrange equations

$$\frac{\partial \mathcal{L}}{\partial \bar{\psi}(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \bar{\psi}(x))} = 0 \quad (6.130)$$

$$\frac{\partial \mathcal{L}}{\partial \psi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \psi(x))} = 0 \quad (6.131)$$

for the Lagrangian density

$$\mathcal{L}(x) = \bar{\psi}(x) \left( -i\partial + \frac{mc}{\hbar} \right) \psi(x) \quad (6.132)$$

### Comments

#### 1. Normal order

Equations involving  $\psi(x)$  given by (6.108) are assumed to be written in normal order.

Creators are moved to the left of annihilators as if creators and annihilators anticommute.

#### 2. Manifest covariance

(6.132) is a Lorentz scalar.

$$U(\Lambda, a) \mathcal{L}(x) U^\dagger(\Lambda, a) = \mathcal{L}(\Lambda x + a) \quad (6.133)$$

The manifest covariance of the formalism is expressed in terms of a manifestly covariant Lagrangian.

### 3. Energy-momentum tensor and Poincare generators

The construction of a manifestly covariant Lagrangian density allows use of results derived in classical field theory. Noether's Theorem gives the energy-momentum tensor

$$T^{\mu\nu}(x) = (\partial^\mu \psi(x)) \frac{\partial \mathcal{L}(x)}{\partial (\partial_\nu \psi(x))} + (\partial^\mu \bar{\psi}(x)) \frac{\partial \mathcal{L}(x)}{\partial (\partial_\nu \bar{\psi}(x))} - g^{\mu\nu} \mathcal{L}(x) \quad (6.134)$$

and the Poincare generators

$$P^\mu = \int d^3x T^{\mu 0}(x) \quad (6.135)$$

$$M^{\mu\nu} = \int d^3x [x^\mu T^{\nu 0}(x) - x^\nu T^{\mu 0}(x)] \quad (6.136)$$

### 4. Expressions for the 4-momentum

(6.16) gives the 4-momentum in terms of the momentum/helicity creators and annihilators (6.2) to (6.5).

(6.135) gives the 4-momentum in terms of the Dirac fields (6.96) and (6.118).

(6.135) simplifies to (6.16) on calculation.

## 5. The Lagrangian method

The Lagrangian method, that is, the determination of Poincare generators from a Lorentz invariant Lagrangian density, can be used to construct theories of interacting particles.

One such theory, quantum electrodynamics (the theory of electrons, positrons and photons in interaction) is discussed very briefly in Chapter 8.

The Lagrangian method has received the very briefest of introductions here. The many excellent books on relativistic quantum field theory should be consulted for further discussion of the Lagrangian method.

## Chapter 7                      FOCK SPACE FOR BOSONS

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### Section 7.1 Introductory remarks

In this Chapter we give the Fock space description of a system of bosons.

The system of bosons may be any of those listed in *Introductory Remarks* of Chapter 3. That is, it may, for example, be a system of

- photons characterizing an electromagnetic field<sup>1</sup>
- phonons characterizing the lattice vibrations of a crystal
- pions or kaons created in collisions of nuclear projectiles
- gluons in nuclear matter.

Whatever the system, each boson has integral spin and all states of the system are symmetric under permutation of the particles.

The Fock space description of a boson system is analogous to the Fock space description of a fermion system given in Chapters 4 and 5.

Fock space and creators and annihilators for bosons are defined in Sections 7.2 and 7.3. These operators are defined in terms of a denumerable set of vectors which form an orthonormal basis for the Hilbert space for a one-boson system.

Creators and annihilators labelled by a continuous variable are defined in Sections 7.4 to 7.6.

The Poincare generators for a Lorentz invariant system of free bosons are given in Section 7.7.

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<sup>1</sup> Photons and Maxwell's equations as quantum field equations are discussed explicitly in Chapter 8.

In Section 7.8 we give a covariant description of a system of free spinless bosons. This involves construction of the scalar field  $\phi(x)$  and determination of the Poincare generators from a Lorentz invariant Lagrangian density. The construction of the scalar field is analogous to the construction of the Dirac field given in Chapter 6. The scalar field satisfies the Klein—Gordon equation.

## Section 7.2 Boson Fock space defined

1. Let

$$\psi = (\psi_0, \psi_1, \dots, \psi_n, \dots) \quad (7.1)$$

where  $\psi_n$  is a vector in the  $n$ —boson Hilbert space  ${}^b_n\mathfrak{H}^s$  (2.36).<sup>1</sup>

Each  $\psi_n$  is of the form (3.40).  $\psi_n$  is the component of  $\psi$  in  ${}^b_n\mathfrak{H}^s$ .

2. Addition of  $\psi$  and  $\chi = (\chi_0, \chi_1, \dots, \chi_n, \dots)$  is defined as

$$\psi + \chi = (\psi_0 + \chi_0, \psi_1 + \chi_1, \dots, \psi_n + \chi_n, \dots) \quad (7.2)$$

3. Multiplication of  $\psi$  by a scalar  $a$  is defined as

$$a\psi = (a\psi_0, a\psi_1, \dots, a\psi_n, \dots) \quad (7.3)$$

4. The scalar product of  $\psi$  and  $\chi$  is defined as

$$(\psi, \chi) = \sum_{n=0}^{\infty} (\psi_n, \chi_n) \quad (7.4)$$

It is required that  $(\psi, \psi) < \infty$  for all  $\psi$ .

5. The set of elements  $\psi$  is a separable Hilbert space.

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<sup>1</sup>  ${}^b_n\mathfrak{H}^s$  is defined in item 4 of the Comments list.



## Comments

### 1. Boson Fock space ${}^b\mathfrak{H}^s$

The above Hilbert space is called boson Fock space. It will be denoted by  ${}^b\mathfrak{H}^s$ .

${}^b\mathfrak{H}^s$  is the direct sum of the Hilbert spaces  ${}^b_n\mathfrak{H}^s$  (2.36) for all  $n$ .

$${}^b\mathfrak{H}^s = {}^b_0\mathfrak{H}^s \oplus {}^b_1\mathfrak{H}^s \oplus \cdots \oplus {}^b_n\mathfrak{H}^s \oplus \cdots \quad (7.5)$$

$\oplus$  denotes direct sum.

### 2. States of the system

The unit norm vectors (7.1) in  ${}^b\mathfrak{H}^s$  correspond to states of the system.

The probability  $P_n$  that the system has  $n$  particles in it is

$$P_n = \langle \psi_n | \psi_n \rangle \quad (7.6)$$

$$\sum_{n=0}^{\infty} P_n = 1 \quad (7.7)$$

### 3. Components of $\psi$

There is no conservation law for the number of bosons in a physical system.

$\psi$  can have any number of nonzero components.

4. Hilbert space  ${}^b_0\mathfrak{H}^s$

${}^b_0\mathfrak{H}^s$  is defined to be a one-dimensional space.

The unit norm vector spanning  ${}^b_0\mathfrak{H}^s$  is labelled  $|0[00\dots]\rangle_0$ .

5. Basis vectors for  ${}^b\mathfrak{H}^s$

A basis for  ${}^b\mathfrak{H}^s$  is the set of vectors

$$|n[n_1n_2\dots]\rangle \quad (7.8)$$

defined by

$$|0[00\dots]\rangle = \left( |0[00\dots]\rangle_0, 0, \dots \right) \quad (7.9)$$

$$|1[n_1n_2\dots]\rangle = \left( 0, |1[n_1n_2\dots]\rangle_1, 0, \dots \right) \quad (7.10)$$

$$|2[n_1n_2\dots]\rangle = \left( 0, 0, |2[n_1n_2\dots]\rangle_2, 0, \dots \right) \quad (7.11)$$

⋮

where

$$|n[n_1n_2\dots]\rangle_n \quad (7.13)$$

for all  $n = 1, 2, \dots$  is the symmetric determinant (3.33).

Then

$$\sum_{n_1 n_2 \dots}^b |n[n_1 n_2 \dots]\rangle \langle n[n_1 n_2 \dots]| = 1 \quad (7.14)$$

$$\sum_{n_1 n_2 \dots}^b = \sum_{n=0}^{\infty} \sum_{n_1 n_2 \dots}^b \quad (7.15)$$

$$\sum_{n_1 n_2 \dots}^b = \sum_{n_1=0}^n \sum_{n_2=0}^n \dots \delta_{n_1+n_2+\dots, n} \quad (7.16)$$

$$\langle n[n_1 n_2 \dots] | n'[n'_1 n'_2 \dots] \rangle = \delta_{nn'} \delta_{n_1 n'_1} \delta_{n_2 n'_2} \dots \quad (7.17)$$

## 6. Vacuum state

(7.9) is the vacuum state of the system. It will be denoted by  $|0\rangle$ .

$$|0\rangle = |0[00 \dots]\rangle \quad (7.18)$$

## 7. General state of the system

The general state of the system has the form

$$|\psi(t)\rangle = \sum_{n_1 n_2 \dots}^b |n[n_1 n_2 \dots]\rangle \langle n[n_1 n_2 \dots] | \psi(t)\rangle \quad (7.19)$$

$$\langle n[n_1 n_2 \dots] | \psi(t)\rangle \quad (7.20)$$

is the probability amplitude that at time  $t$  there are  $n$  bosons in the system with  $n_1$  bosons occupying the single-particle state  $|\beta_1\rangle$ , and  $n_2$  bosons occupying the single-particle state  $|\beta_2\rangle$ , etc.

### Section 7.3 Creators and annihilators

We define boson creation and annihilation operators in this Section. These operators are fundamental dynamical variables for a system of identical bosons. They obey commutation relations.

As with the fermion case, introduction of creation and annihilation operators yields intuitive and elegant expressions for observables and basis states.

### 7.3.1 Definitions

For each  $r = 1, 2, \dots$ , we define

$$B_r^\dagger = \sum_{nn_1n_2\cdots}^b |n+1[n_1n_2\cdots n_r+1\cdots] \rangle \sqrt{n_r+1} \langle n[n_1n_2\cdots n_r\cdots] | \quad (7.21)$$

$$B_r = \sum_{nn_1n_2\cdots}^b |n[n_1n_2\cdots n_r\cdots] \rangle \sqrt{n_r+1} \langle n+1[n_1n_2\cdots n_r+1\cdots] | \quad (7.22)$$

where  $|n[n_1n_2\cdots] \rangle$  is the basis vector (7.8) in boson Fock space  ${}^b\mathfrak{F}^s$ .

It follows from (7.21) and (7.22) that

$$B_r^\dagger |n[n_1n_2\cdots n_r\cdots] \rangle = \sqrt{n_r+1} |n+1\{n_1n_2\cdots n_r+1\cdots\} \rangle \quad (7.23)$$

$$B_r |n+1[n_1n_2\cdots n_r+1\cdots] \rangle = \sqrt{n_r+1} |n[n_1n_2\cdots n_r\cdots] \rangle \quad (7.24)$$

In particular,

$$B_r | 0 \rangle = 0 \quad (7.25)$$

$$\langle 0 | B_r^\dagger = 0 \quad (7.26)$$

$$B_r^\dagger | 0 \rangle = \left( 0, |\beta_r \rangle_1, 0, \dots \right) \quad (7.27)$$

### Comments

#### 1. Boson creator

$B_r^\dagger$  is a boson creation operator or boson creator.

When acting on an  $n$ —boson basis vector (7.8) with  $n_r$  particles occupying single-particle state  $|\beta_r \rangle$ ,  $B_r^\dagger$  yields an  $n + 1$ —boson basis vector (7.8) with  $n_r + 1$  particles occupying single-particle state  $|\beta_r \rangle$ .

#### 2. Boson annihilator

$B_r$  is a boson annihilation operator or boson annihilator.

When acting on an  $n + 1$ —boson basis state (7.8) with  $n_r + 1$  particles occupying single-particle state  $|\beta_r \rangle$ ,  $B_r$  yields an  $n$ —boson basis state (7.8) with  $n_r$  particles occupying single-particle state  $|\beta_r \rangle$ .

#### 3. Creating an elementary boson

When acting on the vacuum state  $| 0 \rangle$ ,  $B_r^\dagger$  creates an elementary boson with rest mass  $m$  and spin  $s$  in single-particle state  $|\beta_r \rangle$ .

#### 4. One-boson state

The general one-boson state at time  $t$  is

$$|\psi(t)\rangle = \sum_{r=1}^{\infty} \psi_r(t) B_r^\dagger |0\rangle \quad (7.28)$$

$$\psi_r(t) = \langle 0 | B_r \psi(t) \rangle \quad (7.29)$$

is the probability amplitude that the boson is in the state  $|\beta_r\rangle$  at time  $t$ .

#### 7.3.2 Commutation relations

It follows from (7.21) and (7.22) that<sup>1</sup>

$$[B_r, B_s] = 0 \quad (7.30)$$

$$[B_r^\dagger, B_s^\dagger] = 0 \quad (7.31)$$

$$[B_r, B_s^\dagger] = \delta_{rs} \quad (7.32)$$

<sup>1</sup> The proof of (7.30) to (7.32) is similar to the proof of (4.34) to (4.36) in Topic 4.3.5.

### 7.3.3 Basis vectors

It follows from (7.23) that the basis vector (7.8) may be expressed as  $n$  creators acting on the vacuum state.

$$|n[n_1 n_2 \dots]\rangle = \frac{1}{\sqrt{n_1! n_2! \dots}} (B_1^\dagger)^{n_1} (B_2^\dagger)^{n_2} \dots |0\rangle \quad (7.33)$$

$$\sum_{r=1}^{\infty} n_r = n \quad (7.34)$$

The proof of (7.33) follows on evaluating the right side using (7.23).

#### Comments

1. Form of the basis vector

(7.33) is a compact, intuitive and elegant expression for the basis vector (7.8).

2. Manifest symmetry

In view of (7.31), (7.33) is unchanged when any two particle labels are interchanged.

(7.33) is manifestly symmetric under particle interchange.

3. Fundamental dynamical variables

Each basis vector (7.8) can be expressed in terms of boson creators acting on the vacuum state. The set of creators and annihilators defined by (7.21) and (7.22) is a set of fundamental dynamical variables for a system of identical bosons.



Commutation relations (7.30) to (7.32) are a fundamental algebra for the system.

## Section 7.4 Creators and annihilators labelled by position and spin

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### 7.4.1 Definitions

The creator  $B_r^\dagger$  (7.21) and the annihilator  $B_r$  (7.22) are defined in terms of the denumerable set of vectors  $|\beta_r\rangle_\alpha$  (3.27) which form an orthonormal basis for the Hilbert space for a one-boson system with rest mass  $m$  and spin  $s$ .<sup>1</sup> The  $|\beta_r\rangle_\alpha$  are simultaneous eigenvectors of specified one-particle operators.

In Section 5.2 we defined fermion field operators  $F_{m_s}^\dagger(x)$  (5.6) and  $F_{m_s}(x)$  (5.7) in terms of the coordinate-space/spin representatives of the one-fermion basis vectors  $|\phi_r\rangle_\alpha$  (3.1) and the fermion operators  $F_r^\dagger$  and  $F_r$  given by (4.22) and (4.23).

We follow an identical procedure in this Section to define boson field operators  $B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  in terms of the coordinate-space/spin representatives of the one-boson basis vectors  $|\beta_r\rangle_\alpha$  and the boson operators  $B_r^\dagger$  and  $B_r$ .

We define

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<sup>1</sup> As in Section 5.2, the subscript  $\alpha$  serves as a reminder that the states are defined in the Hilbert space for particle  $\alpha$ .

$$B_{m_s}^\dagger(x) = \sum_{r=1}^{\infty} \beta_{rm_s}^*(x) B_r^\dagger \quad (7.35)$$

$$B_{m_s}(x) = \sum_{r=1}^{\infty} \beta_{rm_s}(x) B_r \quad (7.36)$$

The function

$$\beta_{rm_s}(x) = \langle x, m_s | \beta_r \rangle_\alpha \quad (7.37)$$

is the coordinate-space/spin representative of the vector  $|\beta_r \rangle_\alpha$  (3.27).

These functions satisfy

$$\sum_{m_s=-s}^{+s} \int d^3x \beta_{rm_s}^*(x) \beta_{um_s}(x) = \delta_{ru} \quad (7.38)$$

$$\sum_{r=1}^{\infty} \beta_{rm_s}^*(x) \beta_{rm'_s}(x') = \delta(x-x') \delta_{m_s m'_s} \quad (7.39)$$

It follows using (7.25) to (7.27) that

$$B_{m_s}(x) | 0 \rangle = 0 \quad (7.40)$$

$$\langle 0 | B_{m_s}^\dagger(x) = 0 \quad (7.41)$$

$$B_{m_s}^\dagger(x) | 0 \rangle = \left( 0, | x, m_s \rangle_1, 0, \dots \right) \quad (7.42)$$

It follows using (7.38) and (7.39) that

$$B_r^\dagger = \sum_{m_s=-s}^{+s} \int d^3x \beta_{r m_s}(x) B_{m_s}^\dagger(x) \quad (7.43)$$

$$B_r = \sum_{m_s=-s}^{+s} \int d^3x \beta_{r m_s}^*(x) B_{m_s}(x) \quad (7.44)$$

### Comments

#### 1. Fundamental dynamical variables

(7.35) and (7.36) define fundamental dynamical variables  $B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  for a system of identical bosons each with rest mass  $m$  and spin  $s$ .

#### 2. Transformation equations

(7.43) and (7.44) allow transformations from  $B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  to  $B_r^\dagger$  and  $B_r$  given by (7.21) and (7.22).

### 3. Quantum field theory

$B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  are labelled by the continuous variable  $x$ .

$B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  are quantum field operators in the Schrodinger picture.

The description of a system of identical bosons using quantum fields as fundamental dynamical variables is called quantum field theory (QFT) of bosons.

### 4. One-boson state

When acting on the vacuum state  $|0\rangle$ ,  $B_{m_s}^\dagger(x)$  creates an elementary boson at position  $x$  with rest mass  $m$ , spin  $s$  and  $z$ -component of spin  $m_s$ .

The general one-boson state at time  $t$  is

$$|\psi(t)\rangle = \sum_{m_s=-s}^{+s} \int d^3x \psi_{m_s}(x,t) B_{m_s}^\dagger(x) |0\rangle \quad (7.45)$$

$$\psi_{m_s}^*(x,t) \psi_{m_s}(x,t) d^3x \quad (7.46)$$

is the probability that the boson is in the volume  $dx dy dz$  about the point  $(x, y, z)$  with  $z$ -component of spin  $m_s$  at time  $t$

$\psi_{m_s}(x, t)$  is the coordinate-space/spin wave function of the boson.

## 7.4.2 Commutation relations

It follows from definitions (7.35) and (7.36) and commutation relations (7.30) to (7.31) that

$$[B_{m_s}(x), B_{m'_s}(x')] = 0 \quad (7.47)$$

$$[B_{m_s}^\dagger(x), B_{m'_s}^\dagger(x')] = 0 \quad (7.48)$$

$$[B_{m_s}(x), B_{m'_s}^\dagger(x')] = \delta(x - x')\delta_{m_s m'_s} \quad (7.49)$$

(7.47) to (7.49) are a fundamental algebra for the system of bosons.

## 7.4.3 One- and two-particle operators

The operators (4.58) and (4.77) on fermion Fock space  ${}^f\mathfrak{H}^s$  are equal, respectively, to the one- and two-particle operators (4.60) and (4.79) on the  $n$ -fermion Hilbert space  ${}^f_n\mathfrak{H}^s$  for all  $n$ . (5.18) and (5.21) give these operators in terms of fermion field operators. Similar equations hold for a system of bosons.

The expression on boson Fock space  ${}^b\mathfrak{H}^s$  corresponding to (5.18) is

$$A = \int d^3x d^3x' B^\dagger(x) A(x, x') B(x') \quad (7.50)$$

where

$$B(x) = \begin{pmatrix} B_s(x) \\ B_{s-1}(x) \\ \vdots \\ B_{-s}(x) \end{pmatrix} \quad (7.51)$$

$B^\dagger(x)$  is the corresponding  $2s + 1$  row matrix and  $A(x, x')$  is the  $2s + 1$  by  $2s + 1$  square matrix (5.20) of one-particle matrix elements.

The expression on boson Fock space  ${}^b\mathfrak{F}^s$  corresponding to (5.21) is

$$A = \int d^3x d^3y d^3x' d^3y' B^\dagger(x) B^\dagger(y) A(x, y, x', y') B(y') B(x') \quad (7.52)$$

where  $A(x, y, x', y')$  is the tensor (5.22) of two-particle matrix elements.

## Section 7.5 Creators and annihilators labelled by momentum and spin

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### 7.5.1 Definitions

In Section 5.3 we defined fermion momentum/spin operators  $F_{m_s}^\dagger(p)$  (5.26) and  $F_{m_s}(p)$  (5.27) in terms of the fermion field operators  $F_{m_s}^\dagger(x)$  and  $F_{m_s}(x)$  given by (5.6) and (5.7).

We follow an identical procedure in this Section to define boson momentum/spin operators  $B_{m_s}^\dagger(p)$  and  $B_{m_s}(p)$  in terms of the boson field operators  $B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  given by (7.35) and (7.36).

We define

$$B_{m_s}^\dagger(p) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3x e^{ip\cdot x/\hbar} B_{m_s}^\dagger(x) \quad (7.53)$$

$$B_{m_s}(p) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3x e^{-ip\cdot x/\hbar} B_{m_s}(x) \quad (7.54)$$

It follows using (7.40) to (7.42) that

$$B_{m_s}(p) |0\rangle = 0 \quad (7.55)$$

$$\langle 0 | B_{m_s}^\dagger(p) = 0 \quad (7.56)$$

$$B_{m_s}^\dagger(p) |0\rangle = \left(0, |p, m_s, \underset{1}{>}, 0, \dots\right) \quad (7.57)$$

It follows from (7.53) and (7.54) that

$$B_{m_s}^\dagger(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p e^{-ip\cdot x/\hbar} B_{m_s}^\dagger(p) \quad (7.58)$$

$$B_{m_s}(x) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3p e^{ip\cdot x/\hbar} B_{m_s}(p) \quad (7.59)$$

## Comments

### 1. Fundamental dynamical variables

(7.53) and (7.54) define fundamental dynamical variables  $B_{m_s}^\dagger(p)$  and  $B_{m_s}(p)$  for a system of identical bosons each with rest mass  $m$  and spin  $s$ .

### 2. Transformation equations

(7.58) and (7.59) allow transformations from  $B_{m_s}^\dagger(p)$  and  $B_{m_s}(p)$  to  $B_{m_s}^\dagger(x)$  and  $B_{m_s}(x)$  given by (7.35) and (7.36).

### 3. One-boson state

When acting on the vacuum state  $|0\rangle$ ,  $B_{m_s}^\dagger(p)$  creates an elementary boson with mass  $m$ , spin  $s$ ,  $z$ -component of spin  $m_s$  and momentum  $p$ .

The general one-boson state at time  $t$  is

$$|\psi(t)\rangle = \sum_{m_s=-s}^{+s} \int d^3p \psi_{m_s}(p, t) B_{m_s}^\dagger(p) |0\rangle \quad (7.60)$$

$$\psi_{m_s}^*(p, t) \psi_{m_s}(p, t) d^3p \quad (7.61)$$

is the probability that the boson has momentum in the volume  $d^3p$  about  $(p^1, p^2, p^3)$  with  $z$ -component of spin  $m_s$  at time  $t$ .

$\psi_{m_s}(p, t)$  is the momentum-space/spin wave function for the boson.



## 7.5.2 Commutation relations

It follows from definitions (7.53) and (7.54) and commutation relations (7.47) to (7.49) that

$$[B_{m_s}(p), B_{m'_s}(p')] = 0 \quad (7.62)$$

$$[B_{m_s}^\dagger(p), B_{m'_s}^\dagger(p')] = 0 \quad (7.63)$$

$$[B_{m_s}(p), B_{m'_s}^\dagger(p')] = \delta(p - p') \delta_{m_s m'_s} \quad (7.64)$$

It follows also that

$$[B_{m_s}(x), B_{m'_s}^\dagger(p)] = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{ip \cdot x/\hbar} \delta_{m_s m'_s} \quad (7.65)$$

(7.62) to (7.64) are a fundamental algebra for a system of identical bosons each with rest mass  $m$  and spin  $s$ .

## 7.5.3 One- and two-particle operators

It follows using (7.58) and (7.59) that (7.50) can be written as

$$A = \int d^3 p d^3 p' B^\dagger(p) A(p, p') B(p') \quad (7.66)$$

where

$$B(p) = \begin{pmatrix} B_s(p) \\ B_{s-1}(p) \\ \vdots \\ B_{-s}(p) \end{pmatrix} \quad (7.67)$$

$B^\dagger(p)$  is the corresponding  $2s + 1$  row matrix and  $A(p, p')$  is the  $2s + 1$  by  $2s + 1$  square matrix (5.41) of one-particle matrix elements.

It follows using (7.58) and (7.59) that (7.52) can be written as

$$A = \int d^3 p d^3 q d^3 p' d^3 q' B^\dagger(p) B^\dagger(q) A(p, q, p', q') B(q') B(p') \quad (7.68)$$

where  $A(p, q, p', q')$  is the tensor (5.43) of two-particle matrix elements.

## Section 7.6 Creators and annihilators labelled by momentum and helicity

### 7.6.1 Definitions

In Section 5.4 we defined fermion momentum/helicity operators  $F^{\lambda\dagger}(p)$  (5.48) and  $F^\lambda(p)$  (5.49) in terms of the fermion momentum/spin operators  $F_{m_s}^\dagger(p)$  and  $F_{m_s}(p)$  given by (5.26) and (5.27).

We follow an identical procedure in this Section to define boson momentum/helicity operators  $B^{\lambda\dagger}(p)$  and  $B^\lambda(p)$  in terms of the boson momentum/spin operators  $B_{m_s}^\dagger(p)$  and  $B_{m_s}(p)$  given by (7.53) and (7.54).

We define

$$B^{\lambda\dagger}(p) = \sum_{m_s=-s}^{+s} D_{m_s\lambda}^s(\varphi, \theta, 0) B_{m_s}^\dagger(p) \quad (7.69)$$

$$B^\lambda(p) = \sum_{m_s=-s}^{+s} D_{m_s\lambda}^{s*}(\varphi, \theta, 0) B_{m_s}(p) \quad (7.70)$$

It follows using (7.55) and (7.56) that

$$B^\lambda(p) |0\rangle = 0 \quad (7.71)$$

$$\langle 0 | B^{\lambda\dagger}(p) = 0 \quad (7.72)$$

$$B^{\lambda\dagger}(p) |0\rangle = \left( 0, |h^\lambda(p)\rangle_1, 0, \dots \right) \quad (7.73)$$

It follows from (7.69) and (7.70) that

$$B_{m_s}^\dagger(p) = \sum_{\lambda=-s}^{+s} D_{m_s\lambda}^{s*}(\varphi, \theta, 0) B^{\lambda\dagger}(p) \quad (7.74)$$

$$B_{m_s}(p) = \sum_{\lambda=-s}^{+s} D_{m_s\lambda}^s(\varphi, \theta, 0) B^\lambda(p) \quad (7.75)$$

### Comments

#### 1. Fundamental dynamical variables

(7.69) and (7.70) define fundamental dynamical variables  $B^{\lambda\dagger}(p)$  and  $B^\lambda(p)$  for a system of identical bosons each with rest mass  $m$  and spin  $s$ .

#### 2. Transformation equations

(7.74) and (7.75) allow transformations from  $B^{\lambda\dagger}(p)$  and  $B^\lambda(p)$  to  $B_{m_s}^\dagger(p)$  and  $B_{m_s}(p)$  given by (7.53) and (7.54).

#### 3. One-boson state

When acting on the vacuum state  $|0\rangle$ ,  $B^{\lambda\dagger}(p)$  creates an elementary boson with rest mass  $m$ , spin  $s$ , momentum  $p$  and helicity  $\lambda$ .

The general one-boson state at time  $t$  is

$$|\psi(t)\rangle = \sum_{\lambda=-s}^{+s} \int d^3p \psi^{\lambda\dagger}(p, t) B^{\lambda\dagger}(p) |0\rangle \quad (7.76)$$

$$\psi^{\lambda*}(p, t) \psi^\lambda(p, t) d^3p \quad (7.77)$$

is the probability that the boson has momentum in the volume  $dp^1 dp^2 dp^3$  about  $(p^1, p^2, p^3)$  with helicity  $\lambda$  at time  $t$ .

$\psi^\lambda(p, t)$  is the momentum-space/helicity wave function for the boson.

We recall from Chapter 9 of Part I that for a boson with zero rest mass which is described by a Hamiltonian which is invariant under space inversion

$$\psi^\lambda(p, t) = 0 \quad \text{unless} \quad \lambda = \pm s \quad (7.78)$$

## 7.6.2 Commutation relations

It follows from definitions (7.69) and (7.70) and commutation relations (7.62) to (7.64) that

$$\left[ B^\lambda(p), B^{\lambda'}(p') \right] = 0 \quad (7.79)$$

$$\left[ B^{\lambda\dagger}(p), B^{\lambda'\dagger}(p') \right] = 0 \quad (7.80)$$

$$\left[ B^\lambda(p), B^{\lambda'\dagger}(p') \right] = \delta(p - p') \delta_{\lambda\lambda'} \quad (7.81)$$

It follows also that

$$[B_{m_s}(x), B^{\lambda\dagger}(p)] = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{ip \cdot x/\hbar} D_{m_s\lambda}^s(\varphi, \theta, 0) \quad (7.82)$$

$$[B_{m_s}(p), B^{\lambda\dagger}(p')] = \delta(p - p') D_{m_s\lambda}^s(\varphi, \theta, 0) \quad (7.83)$$

(7.79) to (7.81) are a fundamental algebra for a system of identical bosons each with rest mass  $m$  and spin  $s$ .

### 7.6.3 One- and two-particle operators

It follows using (7.74) and (7.75) that (7.66) can be written as

$$A = \int d^3p d^3p' B^\dagger(p) \mathcal{A}(p, p') B(p') \quad (7.84)$$

where

$$B(p) = \begin{pmatrix} B^s(p) \\ B^{s-1}(p) \\ \vdots \\ B^{-s}(p) \end{pmatrix} \quad (7.85)$$

$B^\dagger(p)$  is the corresponding  $2s + 1$  row matrix and  $\mathcal{A}(p, p')$  is the  $2s + 1$  by  $2s + 1$  square matrix ( ) of one-particle matrix elements.

It follows using (7.74) and (7.75) that (7.68) can be written as

$$A = \int d^3 p d^3 q d^3 p' d^3 q' B^\dagger(p) B^\dagger(q) A(p, q, p', q') B(q') B(p') \quad (7.86)$$

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$A(p, q, p', q')$  is the tensor (5.67) of two-particle matrix elements.

## Section 7.7 Lorentz invariant system of free bosons

Equations (7.50), (7.66), (7.84) and (7.52), (7.68), (7.86) are general expressions on boson Fock space  ${}^b\mathfrak{F}^s$  which are equal, respectively, to one- and two-particle operators on  $n$ -boson Hilbert space  ${}^b_n\mathfrak{F}^s$  for all  $n$ .

These general expressions involve different sets of fundamental dynamical variables. The variables are expressed in column matrices  $B(x)$  (7.51),  $B(p)$  (7.67),  $B(p)$  (7.85) and corresponding row matrices  $B^\dagger(x)$ ,  $B^\dagger(p)$ ,  $B^\dagger(p)$ .

We consider some examples of these general expressions in this Section. In particular, we give the Poincare generators of a Lorentz invariant system of free bosons.

### 7.7.1 Poincare generators

To describe a physical system which is Lorentz invariant, one must construct the Poincare generators from the fundamental dynamical variables. That is, one must construct ten Hermitian operators  $H, P^j, J^j, K^j$  (where  $j = 1, 2, 3$ ) which satisfy the Poincare Algebra (5.68) to (5.76) or (5.77) to (5.79).

For a system of free identical bosons each with rest mass  $m$  and spin  $s$ , (5.68) to (5.76) are satisfied when

$$H = \int d^3 p \epsilon(p) B^\dagger(p) B(p) = \int d^3 p \epsilon(p) \mathcal{B}^\dagger(p) \mathcal{B}(p) \quad (7.87)$$

$$\epsilon(p) = \sqrt{p^2 c^2 + m^2 c^4} \quad (7.88)$$

$$P^j = \int d^3 p p^j B^\dagger(p) B(p) = \int d^3 p p^j \mathcal{B}^\dagger(p) \mathcal{B}(p) \quad (7.89)$$

$$= -i\hbar \int d^3 x B^\dagger(x) \frac{\partial}{\partial x^j} B(x)$$

$$J^j = -i\hbar \int d^3 x B^\dagger(x) (x \times \nabla)^j B(x) + \int d^3 x B^\dagger(x) s^j B(x) \quad (7.90)$$

$$K^j = -\frac{i\hbar}{2c^2} \int d^3 p \epsilon(p) B^\dagger(p) \frac{\partial}{\partial p^j} B(p) + \int d^3 p B^\dagger(p) \frac{(s \times p)^j}{\epsilon(p) + mc^2} B(p) \quad (7.91)$$

The  $s^j$  in (7.90) and (7.91) are  $2s + 1$  by  $2s + 1$  matrices satisfying (5.89).



The number operator  $N$  for the system is

$$N = \int d^3x B^\dagger(x)B(x) = \int d^3p B^\dagger(p)B(p) = \int d^3p B^\dagger(p)B(p) \quad (7.92)$$

The helicity  $\Lambda$  for the system is

$$\Lambda = \int d^3p B^\dagger(p)s^3B(p) \quad (7.93)$$

when  $s^3$  is diagonal.

### Comment

#### 1. Conservation of number of bosons

The generators (7.87) to (7.91) and the helicity (7.93) commute with  $N$ .

The number of bosons in the system is constant in time.

In Chapter 9 we consider a system of interacting fermions and bosons where the number of bosons in the system is not constant in time.

## Section 7.8 The scalar field and the Lagrangian method

The formalism developed so far in this Chapter is not manifestly covariant. That is, it has not been expressed in terms of quantities which transform under Lorentz transformations like scalars, vectors, tensors, etc.

The noncovariance of the formalism does not present any problem in the description of a system of free bosons. It does, however, complicate the description of a system of bosons interacting with themselves or with fermions.

With this point in mind, in this Section we construct a field  $\phi(x)$  (the scalar field) for a system of spinless bosons which transforms as a scalar under restricted Lorentz transformations. The scalar field satisfies the Klein—Gordon equation. In addition, we describe the Lagrangian method for the system.

The procedure used in this Section is similar to the procedure used in Section 6.7 for construction of the Dirac field  $\psi(x)$ .

### 7.8.1 Definition of the scalar field

We consider a system of spinless bosons each with rest mass  $m$ .

The fundamental dynamical variables of the system are momentum creators and annihilators which satisfy the fundamental algebra (7.62) to (7.64) with  $s = 0$ .

We write the fundamental dynamical variables as  $B^\dagger(p)$  and  $B(p)$ .

We define the scalar field  $\phi(x)$  by

$$\phi(x) = e^{iP \cdot x / \hbar} \phi e^{-iP \cdot x / \hbar} \quad (7.94)$$

$$\phi = a \int \frac{d^3 p}{\sqrt{\epsilon(p)}} [B(p) + B^\dagger(p)] \quad (7.95)$$

$$a = \sqrt{\frac{c}{2(2\pi)^3 \hbar}} \quad (7.96)$$

## 7.8.2 Properties of the scalar field

### 1. Notation

In this Section  $x$  stands for the 4-vector  $x^\mu = (x^0, x^1, x^2, x^3)$ .

As in Section 6.7  $P \cdot x = P_\mu x^\mu$ ,  $\square = \partial^\mu \partial_\mu$ ,  $A \partial_0^\dagger B = A \partial_0 B - (\partial_0 A) B$ .

### 2. Hermitian field

$\phi$  and  $\phi(x)$  are Hermitian.

$$\phi^\dagger = \phi \quad (7.97)$$

$$\phi^\dagger(x) = \phi(x) \quad (7.98)$$

### 3. Heisenberg picture

$\phi(x)$  is the space-translation of the operator  $\phi$  to the point  $(x^1, x^2, x^3)$  in the

Heisenberg picture.

$$\phi(x) = e^{iHt/\hbar} D(x^1, x^2, x^3) \phi D^\dagger(x^1, x^2, x^3) e^{-iHt/\hbar} \quad (7.99)$$

$$D(x^1, x^2, x^3) = D^1(x^1) D^2(x^2) D^3(x^3) \quad (7.100)$$

$$D^j(a) = e^{-iP^j a/\hbar} \quad (7.101)$$

#### 4. Transformation under a homogeneous Lorentz transformation

$\phi$  is invariant under a homogeneous Lorentz transformation.

$$[M^{\mu\nu}, \phi] = 0 \quad (7.102)$$

$$U(\Lambda, 0) \phi U^\dagger(\Lambda, 0) = \phi \quad (7.103)$$

#### 5. Transformation under a restricted Lorentz transformation

$\phi(x)$  transforms under a restricted Lorentz transformation according to

$$U(\Lambda, a) \phi(x) U^\dagger(\Lambda, a) = \phi(\Lambda x + a) \quad (7.104)$$

#### 6. Field equation

$\phi(x)$  satisfies

$$i\hbar\partial^\mu\phi(x) = [\phi(x), P^\mu] \quad (7.105)$$

7. Explicit expression for  $\phi(x)$

$\phi(x)$  may be written in the form

$$\phi(x) = a \int \frac{d^3p}{\sqrt{\epsilon(p)}} \left[ e^{ip \cdot x/\hbar} B(p) + e^{-ip \cdot x/\hbar} B^\dagger(p) \right] \quad (7.106)$$

8. Klein-Gordon equation

$\phi(x)$  satisfies the Klein-Gordon equation.

$$\left[ \square + \left( \frac{mc}{\hbar} \right)^2 \right] \phi(x) = 0 \quad (7.107)$$

9. Boson annihilator in terms of  $\phi(x)$

(7.106) may be inverted to give

$$B(p) = i \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \int d^3x \sqrt{\frac{c}{2\epsilon(p)}} e^{ip \cdot x/\hbar} \partial_0^{\leftrightarrow} \phi(x) \quad (7.108)$$

10. The field  $\pi(x)$

$\pi(x)$  is defined as

$$\pi(x) = \partial_0 \phi(x) \quad (7.109)$$

11. Equal-time commutation relation

The following commutation relation holds for all  $x$  and  $y$  with  $x^0 = y^0$

$$[\phi(x), \pi(y)]_{x^0=y^0} = i\hbar \delta^3(x - y) \quad (7.110)$$

(7.110) determines the constant  $a$  in (7.95).

12. General commutation relation

The following commutation relation holds for all  $x$  and  $y$

$$[\phi(x), \phi(y)] = \Delta(x - y) \quad (7.111)$$

$\Delta(x)$  is given by (6.120).

Comments

1. Nomenclature

$\pi(x)$  is the momentum canonically conjugate to  $\phi(x)$ .

## 2. Fundamental dynamical variable

In view of (7.108),  $\phi(x)$  is a fundamental dynamical variable for a system of spinless bosons each with rest mass  $m$ .

(7.111) is a fundamental algebra for the system.

## 3. Microcausality Condition

It follows from (7.111) and (6.126) that

$$[\phi(x), \phi(y)] = 0 \quad \text{if } x - y \text{ is space-like} \quad (7.112)$$

Every observable  $O(x)$  of the system is a function of  $\phi(x)$ . It follows from (7.112) that

$$[O(x), O(y)] = 0 \quad \text{if } x - y \text{ is space-like} \quad (7.113)$$

$O(x)$  and  $O(y)$  are compatible if  $x - y$  is space-like.

(7.113) is the quantal version of the statement that no effect travels faster than the speed of light. Measurements of any observable at any two space-like separated points do not affect each other.

(7.113) is the causality condition for a system of spinless bosons.

(7.112) is the Microcausality Condition for a system of spinless bosons.

### 7.8.3 The scalar field Lagrangian

(7.107) is the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial \phi(x)} - \partial_\mu \frac{\partial \mathcal{L}}{\partial (\partial_\mu \phi(x))} = 0 \quad (7.114)$$

for the Lagrangian density

$$\mathcal{L}(x) = \frac{1}{2} \left[ (\partial_\mu \phi(x)) (\partial^\mu \phi(x)) - \left( \frac{mc}{\hbar} \right)^2 \phi^2(x) \right] \quad (7.115)$$

#### Comments

1. Normal order

Equations involving  $\phi(x)$  given by (7.106) are assumed to be written in normal order.

Creators are moved to the left of annihilators as if creators and annihilators commute.

2. Manifest covariance

(7.115) is a Lorentz scalar.

$$U(\Lambda, a) \mathcal{L}(x) U^\dagger(\Lambda, a) = \mathcal{L}(\Lambda x + a) \quad (7.116)$$



The manifest covariance of the formalism is expressed in terms of a manifestly covariant Lagrangian.

### 3. Energy-momentum tensor and Poincare generators

The construction of a manifestly covariant Lagrangian density allows use of results derived in classical field theory. Noether's Theorem gives the energy-momentum tensor

$$T^{\mu\nu}(x) = (\partial^\mu \phi(x)) \frac{\partial \mathcal{L}(x)}{\partial (\partial_\nu \phi(x))} - g^{\mu\nu} \mathcal{L}(x) \quad (7.117)$$

and the Poincare generators

$$P^\mu = \int d^3x T^{\mu 0}(x) \quad (7.118)$$

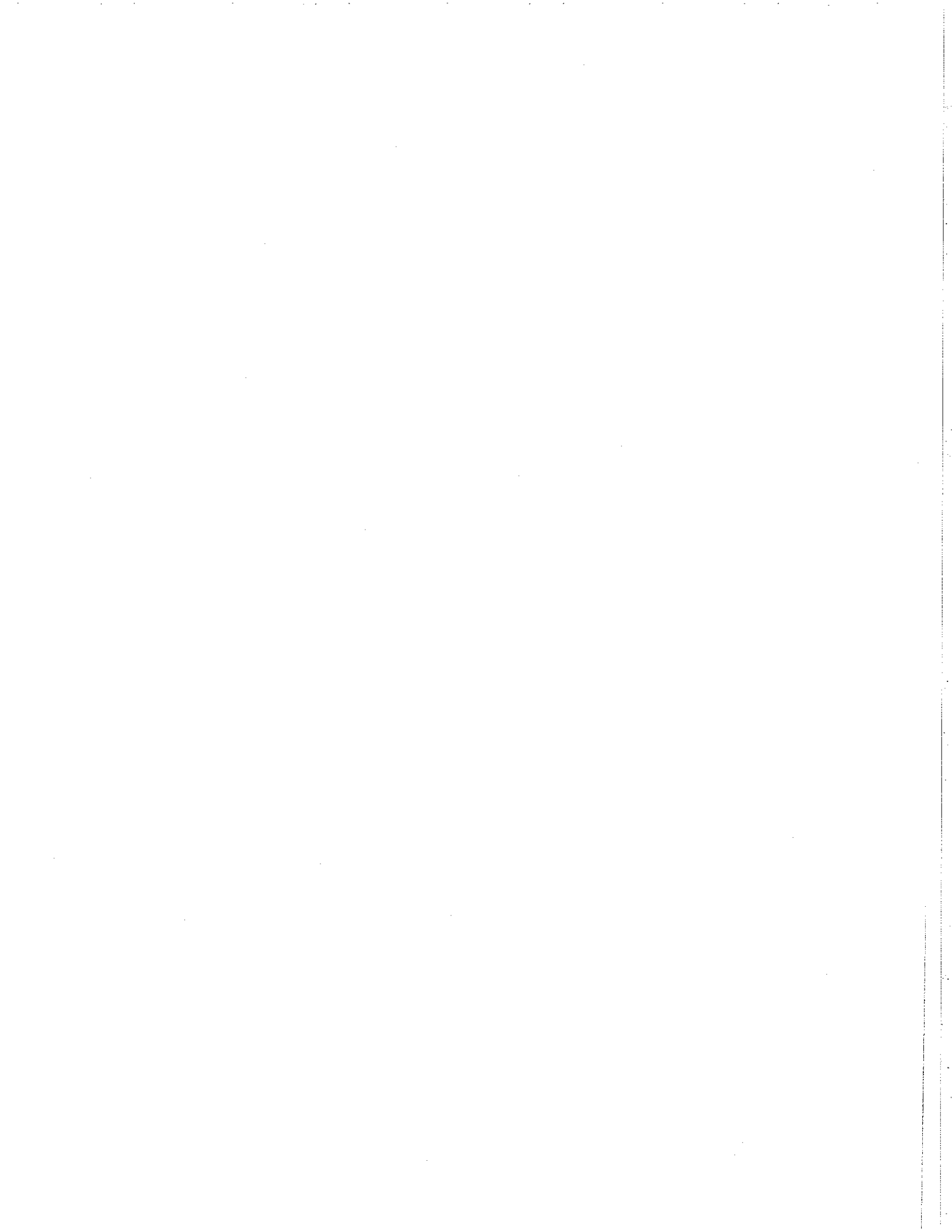
$$M^{\mu\nu} = \int d^3x [x^\mu T^{\nu 0}(x) - x^\nu T^{\mu 0}(x)] \quad (7.119)$$

### 4. Expressions for the 4-momentum

(7.87) and (7.89) give the 4-momentum in terms of creators and annihilators.

(7.118) gives the 4-momentum in terms of the scalar field.

(7.118) simplifies to (7.87) and (7.89) on calculation.



## Chapter 8      FOCK SPACE FOR PHOTONS

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### Section 8.1 Introductory remarks

The Fock space description of a system of identical bosons each with rest mass  $m$  and spin  $s$  is given in Chapter 7. In this Chapter, we specialize the formalism developed in Chapter 7 to consider a system of photons.

Photons are negative parity spin—one bosons with zero rest mass. They are described by a Hamiltonian which is invariant under space inversion.

Fundamental dynamical variables and some observables of the system are given in Section 8.2.

Transformation of the fundamental dynamical variables under space inversion, time reversal and charge conjugation is specified in Section 8.3.

Transverse photon creators and annihilators are defined in Section 8.4 and the vector field is defined in Section 8.5. Quantum electric and magnetic fields are defined in Section 8.6.

The quantum electromagnetic field satisfies Maxwell's equations in free space. Each momentum component of the field corresponds to a transverse wave moving with the speed of light in the direction of the photon momentum. The expression for the electromagnetic field provides a union of the "wave and corpuscular views of electromagnetic radiation".

The Lorentz invariant theory of electromagnetism developed in Sections 8.2 to 8.6 is not manifestly covariant or gauge invariant. A manifestly covariant and gauge invariant theory is developed in Section 8.7. This development is accomplished through the introduction of creators and annihilators for fictitious longitudinal and time-like photons.

Section 8.7 contains the Lagrangian for the free electromagnetic field.

Section 8.8 contains a very brief introduction to quantum electrodynamics.

## Section 8.2 Fundamental dynamical variables

We take the fundamental dynamical variables of the system to be momentum/helicity creators and annihilators.

Momentum/helicity creators and annihilators for a general system of identical bosons each with rest mass  $m$  and spin  $s$  are given in Section 7.6 . The fundamental algebra is given by (7.79) to (7.81). The Poincare generators are given by (7.87) to (7.91).

We recall from (7.78) that the momentum-space/helicity wave function  $\psi^\lambda(p, t)$  for a spin-one boson with zero rest mass which is described by a Hamiltonian which is invariant under space inversion satisfies

$$\psi^\lambda(p, t) = 0 \quad \text{unless} \quad \lambda = \pm 1 \quad (8.1)$$

Accordingly, we take the fundamental dynamical variables for a system of photons to be the momentum/helicity operators

$$B^{\lambda\dagger}(p) \quad \text{and} \quad B^\lambda(p) \quad (\lambda = \pm 1) \quad (8.2)$$

These operators satisfy the fundamental algebra (7.79) to (7.81).

The four-momentum for the system is

$$P^\mu = \int d^3p p^\mu B^\dagger(p) B(p) \quad (8.3)$$

$$p^\mu = \left( \frac{\epsilon(p)}{c}, p^1, p^2, p^3 \right) \quad (8.4)$$

$$\epsilon(p) = \sqrt{p^2 c^2} = pc \quad (8.5)$$

$$B(p) = \begin{pmatrix} B^{+1}(p) \\ B^{-1}(p) \end{pmatrix} \quad (8.6)$$

$B^\dagger(p)$  is the corresponding 2—row matrix.<sup>1</sup>

The number operator  $N$  for the system is

$$N = \int d^3p p B^\dagger(p) B(p) \quad (8.7)$$

<sup>1</sup> In this Chapter  $B(p)$  is the 2—column matrix (8.6). In other Chapters  $B(p)$  denotes the  $2s + 1$  column matrix (7.85).

The helicity  $\Lambda$  for the system is

$$\Lambda = \hbar \int d^3 p B^\dagger(p) \sigma^3 B(p) \quad (8.8)$$

$\sigma^3$  is the Pauli matrix.

### Comments

#### 1. Vacuum state

The vacuum state  $|0\rangle$  contains no photons.

$$B^\lambda(p) |0\rangle = 0 \quad (8.9)$$

$$\langle 0 | B^{\lambda\dagger}(p) = 0 \quad (8.10)$$

#### 2. Creating photons

When acting on the vacuum state  $|0\rangle$ ,  $B^{\lambda\dagger}(p)$  creates a photon with momentum  $(p^1, p^2, p^3)$ , energy  $\epsilon(p)$  and helicity  $\lambda$ .

#### 3. One-photon state

The general one-photon state at time  $t$  is

$$|\psi(t)\rangle = \sum_{\lambda=\pm 1} \int d^3 p \psi^\lambda(p, t) B^{\lambda\dagger}(p) |0\rangle \quad (8.11)$$

$$\psi^{\lambda*}(p, t)\psi^\lambda(p, t)d^3p \quad (8.12)$$

is the probability that the photon has momentum in the volume  $dp^1 dp^2 dp^3$  about  $(p^1, p^2, p^3)$  with helicity  $\lambda$  at time  $t$ .

$\psi^\lambda(p, t)$  is the momentum-space/helicity wave function for the photon.

### Section 8.3 Space inversion, time reversal, charge conjugation

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The space inversion and time reversal transformations for a general physical system are discussed in Chapter 6 of Part I. In Section 6.4 we specify how the fundamental dynamical variables (6.2) to (6.5) for a system of spin  $\frac{1}{2}$  fermions and antifermions transform under space inversion, time reversal and charge conjugation.

In this Section we specify how the fundamental dynamical variables (8.2) for a system of photons transform under space inversion, time reversal and charge conjugation.

#### 8.3.1 Space inversion

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The space inversion operator  $P$  on photon Fock space is a linear operator satisfying (6.32) and (6.33). The photon annihilator (8.2) transforms under space inversion as follows:

$$PB^\lambda(p)P^\dagger = -B^{-\lambda}(-p) \quad (8.13)$$

It follows from (8.3) to (8.8) and (8.13) that

$$PHP^\dagger = H \quad (8.14)$$

$$PP^jP^\dagger = -P^j \quad (8.15)$$

$$P\Lambda P^\dagger = -\Lambda \quad (8.16)$$

$$PNP^\dagger = N \quad (8.17)$$

### 8.3.2 Time reversal

The time reversal operator  $T$  on photon Fock space is an antilinear operator satisfying (6.42) and (6.43). The photon annihilator (8.2) transforms under time reversal as follows:

$$TB^\lambda(p)T^\dagger = B^\lambda(-p) \quad (8.18)$$

It follows from (8.3) to (8.8) and (8.18) that



$$THT^\dagger = H \quad (8.19)$$

$$TP^jT^\dagger = -P^j \quad (8.20)$$

$$TAT^\dagger = \Lambda \quad (8.21)$$

$$TNT^\dagger = N \quad (8.22)$$

### 8.3.3 Charge conjugation

The charge conjugation operator  $C$  on boson Fock space is a linear operator satisfying (6.52) and (6.53). The photon annihilator (8.2) transforms under charge conjugation as follows:

$$CB^\lambda(p)C^\dagger = -B^\lambda(p) \quad (8.23)$$

It follows from (8.3) to (8.8) and (8.23) that

$$CHC^\dagger = H \quad (8.24)$$

$$CP^jC^\dagger = P^j \quad (8.25)$$

$$CAC^\dagger = \Lambda \quad (8.26)$$

$$CNC^\dagger = N \quad (8.27)$$

## Comments

### 1. Invariance of the Hamiltonian

$H$  given by (8.3) satisfies (8.14), (8.19) and (8.24).

The Hamiltonian for a system of photons is invariant under space inversion, time reversal and charge conjugation.

## Section 8.4 Transverse photons

### 8.4.1 Introductory remarks

In this Section we begin the construction of quantum electric and magnetic fields in terms of the fundamental dynamical variables (8.2),

The constructed quantum electromagnetic field satisfies Maxwell's equations in free space. Each momentum component of the electromagnetic field corresponds to a transverse wave moving with speed  $c$  in the direction of the photon momentum.

The first step in the construction is to define basis vectors (spherical basis vectors) in the fixed inertial frame which are appropriate for describing a vector field whose momentum components are perpendicular to the direction of the momentum. This is done in Topic 8.4.2.

Transverse photon creators and annihilators are defined in Topic 8.4.3.

### 8.4.2 Spherical basis vectors

We define real unit vectors  $\vec{1}(p)$ ,  $\vec{2}(p)$ ,  $\vec{3}(p)$  by

$$\vec{1}(p) = \vec{1} \cos \theta \cos \varphi + \vec{2} \cos \theta \sin \varphi - \vec{3} \sin \theta \quad (8.28)$$

$$\vec{2}(p) = -\vec{1} \sin \varphi + \vec{2} \cos \varphi \quad (8.29)$$

$$\vec{3}(p) = \vec{1} \sin \theta \cos \varphi + \vec{2} \sin \theta \sin \varphi + \vec{3} \cos \theta \quad (8.30)$$

$(\theta, \varphi)$  are the spherical polar coordinates of  $\vec{p}$  in the fixed inertial frame defined by the orthonormal Cartesian vectors

$$(\vec{1}, \vec{2}, \vec{3}) = (\vec{i}, \vec{j}, \vec{k}) \quad (8.31)$$

We define spherical basis vectors  $\vec{e}_{+1}(p)$ ,  $\vec{e}_0(p)$ ,  $\vec{e}_{-1}(p)$  by

$$\vec{e}_{+1}(p) = -\frac{1}{\sqrt{2}} [\vec{1}(p) + i\vec{2}(p)] \quad (8.32)$$

$$\vec{e}_0(p) = \vec{3}(p) \quad (8.33)$$

$$\vec{e}_{-1}(p) = \frac{1}{\sqrt{2}} [\vec{1}(p) - i\vec{2}(p)] \quad (8.34)$$

### Comments

1. Properties of  $\vec{1}(p)$ ,  $\vec{2}(p)$ ,  $\vec{3}(p)$

$\vec{1}(p), \vec{2}(p), \vec{3}(p)$  are an orthonormal set of vectors for all  $\vec{p}$ .

$\vec{3}(p)$  is parallel to  $\vec{p}$ .

$\vec{2}(p)$  lies in the 12-plane.

$\vec{1}(p), \vec{2}(p), \vec{3}(p)$  are arrived at by rotating the triad  $\vec{1}, \vec{2}, \vec{3}$ .

$$\vec{j}(p) = R(\varphi, \theta, 0)\vec{j} \quad (j = 1, 2, 3) \quad (8.35)$$

$$R(\alpha, \beta, \gamma) = R^3(\alpha)R^2(\beta)R^3(\gamma) \quad (8.36)$$

$$R^j(\xi) = e^{-iS^j\xi/\hbar} \quad (8.37)$$

$$S^j\vec{a} = i\hbar\vec{j} \times \vec{a} \quad (8.38)$$

(8.38) defines  $S^j$ .  $\vec{a}$  in (8.38) is an arbitrary vector in the fixed inertial frame.

## 2. Properties of $S^1, S^2, S^3$

$S^1, S^2, S^3$  defined by (8.38) satisfy

$$[S^j, S^k] = i\hbar\epsilon_{jkl}S^l \quad (8.39)$$

$$S^2 = S \cdot S = s(s+1)\hbar^2 \quad (8.40)$$

$$s = 1 \quad (8.41)$$

$S^1, S^2, S^3$  effect a mixing of the components of a vector field under space rotations.

In view of (8.41) one says that the intrinsic angular momentum of a vector field is unity.

Further discussion is given in Section 21 of Rose [B16].

### 3. The operators $S^1(p), S^2(p), S^3(p)$

We define  $S^1(p), S^2(p), S^3(p)$  by

$$S^j(p) = R S^j R^\dagger \quad (8.42)$$

$$R = R(\varphi, \theta, 0) \quad (8.43)$$

Then,

$$S^j(p) = i\hbar \vec{j}(p) \times \vec{a} \quad (8.44)$$

$$[S^j(p), S^k(p)] = i\hbar \epsilon_{jkl} S^l(p) \quad (8.45)$$

$$S^2(p) = S(p) \cdot S(p) = s(s+1)\hbar^2 \quad (8.46)$$

$$s = 1 \quad (8.47)$$

4. Properties of  $\vec{e}_{+1}(p)$ ,  $\vec{e}_0(p)$ ,  $\vec{e}_{-1}(p)$

$\vec{e}_{+1}(p)$ ,  $\vec{e}_0(p)$ ,  $\vec{e}_{-1}(p)$  are an orthonormal set of vectors for all  $\vec{p}$ .

$$\vec{e}_\lambda^*(p) \cdot \vec{e}_{\lambda'}(p) = \delta_{\lambda\lambda'} \quad (8.48)$$

$\vec{e}_0(p)$  is parallel to  $\vec{p}$

$\vec{e}_{+1}(p)$  and  $\vec{e}_{-1}(p)$  are perpendicular to  $\vec{p}$

$\vec{e}_\lambda(p)$  is an eigenvector of  $S^3(p)$  belonging to eigenvalue  $\lambda\hbar$ .

$$S^3(p)\vec{e}_\lambda(p) = \lambda\hbar\vec{e}_\lambda(p) \quad (8.49)$$

More generally,

$$S_\lambda(p)\vec{e}_{\lambda'}(p) = -\sqrt{2}(1, 1, \lambda, \lambda' | 1, \lambda + \lambda')\vec{e}_{\lambda+\lambda'}(p) \quad (8.50)$$

$(j_1, j_2, m_1, m_2 | j_3, m_3)$  is a Clebsch-Gordan coefficient and

$$S_{+1}(p) = -\frac{1}{\sqrt{2}}[S^1(p) + iS^2(p)] \quad (8.51)$$

$$S_0(p) = S^3(p) \quad (8.52)$$

$$S_{-1}(p) = \frac{1}{\sqrt{2}}[S^1(p) - iS^2(p)] \quad (8.53)$$

$\vec{\epsilon}_\lambda(p)$  is the vector analog of the spherical harmonic  $Y_{1m}(\theta, \varphi)$ .

#### 5. Transverse vector field

$\vec{\epsilon}_{+1}(p)$  and  $\vec{\epsilon}_{-1}(p)$  are useful for describing a vector field whose momentum components are perpendicular to the direction of the momentum.

Such a field is constructed in Section 8.5.

### 8.4.3 Transverse photon creator and annihilator

We define

$$\vec{B}(p) = \sum_{\lambda=\pm 1} \vec{\epsilon}_\lambda^*(p) B^\lambda(p) \quad (8.54)$$

#### Comments

##### 1. Notation

The Cartesian components of  $\vec{B}(p)$  will be denoted by  $B^j(p)$  ( $j = 1, 2, 3$ ).

The meaning of  $B^j(p)$  and  $B^\lambda(p)$  will be clear from the context: we use a Roman superscript to refer to a component of a 3—vector and the superscript  $\lambda$  as a helicity label.

## 2. Transversality

It follows from (8.54) that

$$\vec{p} \cdot \vec{B}(p) = 0 \quad (8.55)$$

In view of (8.55), we say that  $\vec{B}^\dagger(p)$  creates a transverse photon and  $\vec{B}(p)$  annihilates a transverse photon.

## 3. Commutation relations

It follows from (8.54) that

$$\left[ B^j(p), B^{k\dagger}(p') \right] = \delta(p - p') \left( \delta_{jk} - \frac{p^j p^k}{p \cdot p} \right) \quad (8.56)$$

## Section 8.5 Vector field

In this Section we define a vector field  $\vec{A}(x)$  for photons in analogy to the definitions (6.96) and (7.94) of, respectively, the Dirac field  $\psi(x)$  for fermions and antifermions and the scalar field  $\phi(x)$  for spinless bosons.



We define  $\vec{A}(x)$  by

$$\vec{A}(x) = e^{iP \cdot x / \hbar} \vec{A} e^{-iP \cdot x / \hbar} \quad (8.57)$$

$$\vec{A} = a \int \frac{d^3 p}{\sqrt{\epsilon(p)}} [\vec{B}(p) + \vec{B}^\dagger(p)] \quad (8.58)$$

$$a = \frac{c}{\sqrt{2(2\pi)^3 \hbar}} \quad (8.59)$$

### Properties of the vector field

#### 1. Notation

In this Chapter,  $x$  stands for the 4-vector  $x^\mu = (x^0, x^1, x^2, x^3)$ .

As in Section 6.7  $P \cdot x = P_\mu x^\mu$ ,  $\square = \partial^\mu \partial_\mu$ ,  $A \partial_0^\leftrightarrow B = A \partial_0 B - (\partial_0 A) B$ .

#### 2. Units

The Gaussian unit of  $\vec{A}$  is the statvolt.

#### 3. Hermitian field

$\vec{A}$  and  $\vec{A}(x)$  are Hermitian.

$$\vec{A}^\dagger = \vec{A} \quad (8.60)$$

$$\vec{A}^\dagger(x) = \vec{A}(x) \quad (8.61)$$

#### 4. Heisenberg picture

$\vec{A}(x)$  is the space-translation of the operator  $\vec{A}$  to the point  $(x^1, x^2, x^3)$  in the Heisenberg picture.

$$\vec{A}(x) = e^{iHt/\hbar} D(x^1, x^2, x^3) \vec{A} D^\dagger(x^1, x^2, x^3) e^{-iHt/\hbar} \quad (8.62)$$

$$D(x^1, x^2, x^3) = D^1(x^1) D^2(x^2) D^3(x^3) \quad (8.63)$$

$$D^j(a) = e^{-iP^j a/\hbar} \quad (8.64)$$

#### 5. Field equation

$\vec{A}(x)$  satisfies

$$i\hbar \partial^\mu \vec{A}(x) = [\vec{A}(x), P^\mu] \quad (8.65)$$

6. Explicit expression for  $\vec{A}(x)$

$\vec{A}(x)$  may be written in the form

$$\begin{aligned} \vec{A}(x) &= a \int \frac{d^3 p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} \vec{B}(p) + e^{ip \cdot x / \hbar} \vec{B}^\dagger(p) \right] \\ &= a \sum_{\lambda=\pm 1} \int \frac{d^3 p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} \vec{\epsilon}_\lambda^*(p) B^\lambda(p) + e^{ip \cdot x / \hbar} \vec{\epsilon}_\lambda(p) B^{\lambda\dagger}(p) \right] \end{aligned} \quad (8.66)$$

7. Zero divergence; wave equation

$\vec{A}(x)$  has zero divergence and satisfies the wave equation.

$$\nabla \cdot \vec{A}(x) = 0 \quad (8.67)$$

$$\square \vec{A}(x) = 0 \quad (8.68)$$

8. Photon annihilator in terms of  $\vec{A}(x)$

(8.66) may be inverted to give

$$B^\lambda(p) = i \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \vec{\epsilon}_\lambda(p) \cdot \int d^3 x \sqrt{\frac{c}{2\epsilon(p)}} e^{ip \cdot x / \hbar} \partial_0^{\leftrightarrow} \vec{A}(x) \quad (8.69)$$

9. The field  $\vec{\pi}(x)$

$\vec{\pi}(x)$  is defined as

$$\vec{\pi}(x) = \partial_0 \vec{A}(x) \quad (8.70)$$

10. Equal-time commutation relation

The following commutation relation holds for all  $x$  and  $y$  with  $x^0 = y^0$

$$\left[ A^j(x), \pi^k(y) \right]_{x^0=y^0} = i\hbar \delta_{jk}(x-y) \quad (8.71)$$

$$\delta_{jk}(x) = \left( \frac{1}{2\pi} \right)^3 \int d^3 p e^{ip \cdot x} \left( \delta_{jk} - \frac{p^j p^k}{p \cdot p} \right) \quad (8.72)$$

(7.110) determines the constant  $a$  in (8.58).

11. General commutation relation

The following commutation relation holds for all  $x$  and  $y$

$$\left[ A^j(x), A^k(y) \right] = \delta_{jk} D(x-y) \quad (8.73)$$

$$D(x) = \lim_{m \rightarrow 0} \Delta(x) \quad (8.74)$$

$\Delta(x)$  is given by (6.120).

### Comments

#### 1. Nomenclature

$\vec{\pi}(x)$  is the momentum canonically conjugate to  $\vec{A}(x)$ .

#### 2. Fundamental dynamical variable

In view of (8.69),  $\vec{A}(x)$  is a fundamental dynamical variable for a system of photons.

(8.73) is a fundamental algebra for the system.

## Section 8.6 Quantum electric and magnetic fields

### 8.6.1 Definitions

In view of the fact that we are dealing with a system of free photons and also that  $\vec{A}(x)$  is a zero-divergence vector field satisfying the wave equation with propagation speed  $c$ , we interpret  $\vec{A}(x)$  as the quantum electromagnetic vector field in the radiation gauge.

That is, we define the quantum electric field  $\vec{E}(x)$  and the quantum magnetic field  $\vec{B}(x)$  by

$$\vec{E}(x) = -\frac{1}{c} \frac{\partial \vec{A}(x)}{\partial t} \quad (8.75)$$

$$\vec{B}(x) = \nabla \times \vec{A}(x) \quad (8.76)$$

### Properties of the quantum electromagnetic field

#### 1. Explicit expressions

It follows from (8.66) that

$$\vec{E}(x) = \frac{ia}{\hbar c} \sum_{\lambda=\pm 1} \int d^3p \sqrt{\epsilon(p)} \left[ e^{-ip \cdot x / \hbar} \vec{\epsilon}_\lambda^*(p) B^\lambda(p) - e^{ip \cdot x / \hbar} \vec{\epsilon}_\lambda(p) B^{\lambda\dagger}(p) \right] \quad (8.77)$$

$$\vec{B}(x) = \frac{ia}{\hbar} \sum_{\lambda=\pm 1} \int \frac{d^3p}{\sqrt{\epsilon(p)}} \vec{p} \times \left[ e^{-ip \cdot x / \hbar} \vec{\epsilon}_\lambda^*(p) B^\lambda(p) - e^{ip \cdot x / \hbar} \vec{\epsilon}_\lambda(p) B^{\lambda\dagger}(p) \right] \quad (8.78)$$

#### 2. Maxwell's equations

$\vec{E}(x)$  and  $\vec{B}(x)$  satisfy Maxwell's equations in free space.

$$\nabla \cdot \vec{E}(x) = 0 \quad (8.79)$$

$$\nabla \cdot \vec{B}(x) = 0 \quad (8.80)$$

$$\nabla \times \vec{E}(x) = -\frac{1}{c} \frac{\partial \vec{B}(x)}{\partial t} \quad (8.81)$$

$$\nabla \times \vec{B}(x) = \frac{1}{c} \frac{\partial \vec{E}(x)}{\partial t} \quad (8.82)$$

### 3. Electromagnetic waves

$\vec{E}(x)$  and  $\vec{B}(x)$  satisfy the wave equation.

$$\square \vec{E}(x) = 0 \quad (8.83)$$

$$\square \vec{B}(x) = 0 \quad (8.84)$$

Each momentum component of  $\vec{E}(x)$  and  $\vec{B}(x)$  corresponds to a plane sine wave moving with phase and group speed  $c$  in the direction of the photon momentum.

The electromagnetic wave is transverse: the electric and magnetic components are mutually perpendicular and both are perpendicular to the direction of the photon momentum.

#### 4. Union of the wave and corpuscular views of electromagnetic radiation

$\vec{E}(x)$  and  $\vec{B}(x)$  satisfy the wave equation and involve photon operators.

(8.77) and (8.78) thus provide a union of the “wave and corpuscular views of electromagnetic radiation”.

I expect students in lectures to be on their feet cheering on seeing (8.77) and (8.78). After years of hearing about the union of the wave and corpuscular views of electromagnetic radiation, they finally have a mathematical expression for it.

#### 8.6.2 Gauge transformations; the radiation gauge

We recall from classical electromagnetic theory that Maxwell’s equations for the electric field  $\vec{E}(x)$  and the magnetic field  $\vec{B}(x)$  produced by a source with charge density  $\rho(x)$  and current density  $\vec{J}(x)$  are

$$\nabla \cdot \vec{E}(x) = 4\pi\rho(x) \quad (8.85)$$

$$\nabla \cdot \vec{B}(x) = 0 \quad (8.86)$$

$$\nabla \times \vec{E}(x) = -\frac{1}{c} \frac{\partial \vec{B}(x)}{\partial t} \quad (8.87)$$

$$\nabla \times \vec{B}(x) = \frac{4\pi}{c} \vec{J}(x) + \frac{1}{c} \frac{\partial \vec{E}(x)}{\partial t} \quad (8.88)$$

To satisfy (8.86) and (8.87) one writes



$$\vec{B}(x) = \nabla \times \vec{A}(x) \quad (8.89)$$

$$\vec{E}(x) = -\nabla\varphi(x) - \frac{1}{c} \frac{\partial \vec{A}(x)}{\partial t} \quad (8.90)$$

substitution of which into (8.85) and (8.88) yields

$$\square\varphi(x) = 4\pi\rho(x) \quad (8.91)$$

$$\square\vec{A}(x) = \frac{4\pi}{c} \vec{J}(x) \quad (8.92)$$

provided

$$\nabla \cdot \vec{A}(x) + \frac{1}{c} \frac{\partial\varphi(x)}{\partial t} = 0 \quad (8.93)$$

(8.93) is the Lorentz condition.

The Lorentz condition can be imposed because  $\vec{E}(x)$  and  $\vec{B}(x)$  are unchanged when the following replacements are made

$$\varphi(x) \rightarrow \varphi(x) - \frac{1}{c} \frac{\partial\chi(x)}{\partial t} \quad (8.94)$$

$$\vec{A}(x) \rightarrow \vec{A}(x) + \nabla\chi(x) \quad (8.95)$$

where  $\chi(x)$  is arbitrary.

Transformations (8.94) and (8.95) are a gauge transformation.

$\vec{E}(x)$  and  $\vec{B}(x)$  are gauge invariant, that is, invariant under the gauge transformation (8.94) and (8.95).

The choice

$$\varphi(x) = 0 \quad (8.96)$$

which from (8.93) yields

$$\nabla \cdot \vec{A}(x) = 0 \quad (8.97)$$

is possible when  $\rho(x) = 0$ .

The choice (8.96) is called the radiation gauge.

## Section 8.7 Manifest covariance and gauge invariance

### 8.7.1 Introductory remarks

The Lorentz invariant quantum theory of electromagnetism given in Sections 8.2 to 8.6 is not manifestly covariant or gauge invariant. That is, it has not been expressed in terms of quantities which transform under Lorentz transformations like scalars, vectors, tensors, etc. and it has been developed in a particular gauge (the radiation gauge).

The noncovariant and gauge-specific formalism does not present any problem in the description of a system of free photons. It does, however, complicate the description of a system of photons interacting with matter.

With this point in mind, in this Section we extend the formalism developed in Sections 8.2 to 8.6 to produce a manifestly covariant and gauge invariant quantum theory of electromagnetism.

The first steps in this extension are to construct polarization 4-vectors and to introduce creators and annihilators for fictitious longitudinal and time-like photons. This is done in Topic 8.7.2.

The 4-vector field and the electromagnetic field tensor are constructed in Topics 8.7.4 and 8.7.5.

Gauge transformations are discussed in Topic 8.7.6.

The Lagrangian for the free electromagnetic field is given in Topic 8.7.7.

### 8.7.2 Fictitious photons

We introduce four orthonormal polarization 4-vectors  $\varepsilon_\lambda^\mu(p)$  ( $\lambda = \pm 1, 0, 4$ ) defined by

$$\varepsilon_\lambda^\mu(p) = (0, \varepsilon_\lambda^1(p), \varepsilon_\lambda^2(p), \varepsilon_\lambda^3(p)) \quad (\lambda = \pm 1, 0) \quad (8.98)$$

$$\varepsilon_\lambda^\mu(p) = (1, 0, 0, 0) \quad (\lambda = 4) \quad (8.99)$$

where  $\varepsilon_\lambda^j(p)$  is the  $j$ -component of the spherical basis vector  $\vec{e}_\lambda(p)$  defined by (8.32) to 8.34).

We introduce two new annihilation and creation operators

$$B^\lambda(p) \quad \text{and} \quad B^{\lambda\dagger}(p) \quad (\lambda = 0, 4) \quad (8.100)$$

satisfying (7.79) to (7.81) with  $\lambda, \lambda' = \pm 1, 0, 4$ .

### Comments

#### 1. Larger Hilbert space

We consider a larger Hilbert space in this Section.

The  $\lambda = 0, 4$  annihilators and creators are mathematical objects.

States of the system are written in terms of the  $\lambda = \pm 1$  creators acting on the vacuum state.

Every state  $|\psi\rangle$  of the system satisfies

$$B^\lambda(p) |\psi\rangle = 0 \quad (\lambda = 0, 4) \quad (8.101)$$

#### 2. Fictitious photons

One says that

$$B^{\lambda\dagger}(p) (\lambda = 0) \text{ creates a fictitious longitudinal photon} \quad (8.102)$$

$$B^{\lambda\dagger}(p) (\lambda = 4) \text{ creates a fictitious time-like photon} \quad (8.103)$$

The terms “longitudinal” and “time-like” arise from (8.104).

### 8.7.3 Photon annihilator $B^\mu(p)$

In analogy with (8.54) we define

$$B^\mu(p) = \sum_{\lambda=\pm 1,0,4} \varepsilon_\lambda^{\mu*}(p) B^\lambda(p) \quad (8.104)$$

#### Comments

##### 1. Notation

The meaning of  $B^\mu(p)$  and  $B^\lambda(p)$  is always clear from the context: the superscript  $\mu = (0, 1, 2, 3)$  refers to the component of a 4-vector; the superscript  $\lambda = (\pm 1, 0, 4)$  is a (generalized) helicity label.

##### 2. Generalized transversality

It follows from (8.101) and (8.104) and

$$p_\mu \varepsilon_0^\mu = \frac{\varepsilon(p)}{c} = p = -p_\mu \varepsilon_4^\mu \quad (8.105)$$

$$p_\mu \varepsilon_\lambda^\mu = 0 \quad (\lambda = \pm 1) \quad (8.106)$$

that

$$p_\mu B^\mu(p) | \psi \rangle = 0 \quad (8.107)$$

for every state  $| \psi \rangle$  of the system.

(8.107) is the generalized transversality condition.

#### 8.7.4 4-vector field

In analogy with (8.57) we define the 4-vector field  $A^\mu(x)$  by

$$A^\mu(x) = e^{iP \cdot x / \hbar} A^\mu e^{-iP \cdot x / \hbar} \quad (8.108)$$

$$A^\mu = a \int \frac{d^3 p}{\sqrt{\epsilon(p)}} [B^\mu(p) + B^{\mu\dagger}(p)] \quad (8.109)$$

$$a = \frac{c}{\sqrt{2(2\pi)^3 \hbar}} \quad (8.110)$$

#### Properties of the 4-vector field

##### 1. Explicit expression

$A^\mu(x)$  may be written in the form

$$\begin{aligned}
 A^\mu(x) &= a \int \frac{d^3p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} B^\mu(p) + e^{ip \cdot x / \hbar} B^{\mu\dagger}(p) \right] \\
 &= a \sum_{\lambda=\pm 1,0,4} \int \frac{d^3p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} \epsilon_\lambda^{\mu*}(p) B^\lambda(p) + e^{ip \cdot x / \hbar} \epsilon_\lambda^\mu(p) B^{\lambda\dagger}(p) \right]
 \end{aligned}
 \tag{8.111}$$

## 2. Wave equation

$A^\mu(x)$  satisfies the wave equation.

$$\square A^\mu(x) = 0
 \tag{8.112}$$

## 3. Lorentz condition

For every state  $|\psi\rangle$  of the system,

$$\partial_\mu A^\mu(x) |\psi\rangle = 0
 \tag{8.113}$$

(8.113) is the quantal version of the Lorentz condition.

## 4. Manifest covariance

(8.112) and (8.113) are a manifestly covariant form of Maxwell's equations for quantum electromagnetism.

5. Photon annihilator in terms of  $A^\mu(x)$

(8.111) may be inverted to give

$$B^\lambda(p) = i \left( \frac{1}{2\pi\hbar} \right)^{\frac{3}{2}} \varepsilon_\lambda^\mu(p) \int d^3x \sqrt{\frac{c}{2\varepsilon(p)}} e^{ip \cdot x / \hbar} \partial_0^\mu A^\mu(x) \quad (8.114)$$

6. General commutation relation

The following commutation relation holds for all  $x$  and  $y$

$$[A^\mu(x), A^\nu(y)] = -g^{\mu\nu} D(x - y) \quad (8.115)$$

$D(x)$  is given by (8.74).

Comments

1. Fundamental dynamical variable

In view of (8.114),  $A^\mu(x)$  is a fundamental dynamical variable for a system of photons.

(8.115) is a fundamental algebra for the system.



### 8.7.5 Electromagnetic field-strength tensor

The electromagnetic field—strength tensor  $F^{\mu\nu}(x)$  is defined by

$$F^{\mu\nu}(x) = \partial^\mu A^\nu(x) - \partial^\nu A^\mu(x) \quad (8.116)$$

The Cartesian components of the electric and magnetic fields are defined in terms of  $F^{\mu\nu}(x)$  by

$$E^j(x) = -F^{0j}(x) \quad (8.117)$$

$$B^j(x) = -F^{kl}(x) \quad (\text{cyclically}) \quad (8.118)$$

#### Comment

##### 1. Manifest covariance

It follows from (8.80) and (8.81) that

$$\partial^\alpha F^{\beta\gamma}(x) + \partial^\beta F^{\gamma\alpha}(x) + \partial^\gamma F^{\alpha\beta}(x) = 0 \quad (8.119)$$

where  $\alpha, \beta, \gamma$  are any three of 0, 1, 2, 3.

It follows from (8.112) and (8.113) that

$$\partial_\mu F^{\mu\nu}(x) | \psi \rangle = 0 \quad (8.120)$$

for every state  $| \psi \rangle$  of the system.

(8.119) and (8.120) are a manifestly covariant form of Maxwell's equations for quantum electromagnetism.

### 8.7.6 Gauge invariance

Let

$$\hat{B}^\mu(p) = B^\mu(p) + p^\mu f(p) \quad (8.121)$$

where  $B^\mu(p)$  is given by (8.104) and  $f(p)$  is some function.

Let

$$\hat{A}^\mu(x) = b \int \frac{d^3p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} \hat{B}^\mu(p) + e^{ip \cdot x / \hbar} \hat{B}^{\mu\dagger}(p) \right] \quad (8.122)$$

$$\hat{F}^{\mu\nu}(x) = \partial^\mu \hat{A}^\nu(x) - \partial^\nu \hat{A}^\mu(x) \quad (8.123)$$

$$\widehat{E}^j(x) = -\widehat{F}^{0j}(x) \quad (8.124)$$

$$\widehat{B}^j(x) = -\widehat{F}^{kl}(x) \quad (\text{cyclically}) \quad (8.125)$$

Then

$$\widehat{A}^\mu(x) = A^\mu(x) + \partial^\mu \chi(x) \quad (8.126)$$

$$\chi(x) = a \int \frac{d^3 p}{\sqrt{\epsilon(p)}} \left[ e^{-ip \cdot x / \hbar} f(p) + e^{ip \cdot x / \hbar} f^*(p) \right] \quad (8.127)$$

$$\square \widehat{A}^\mu(x) = 0 \quad (8.128)$$

$$\partial_\mu \widehat{A}^\mu(x) | \psi \rangle = 0 \quad \text{for every state } | \psi \rangle \quad (8.129)$$

$$\widehat{F}^{\mu\nu}(x) = F^{\mu\nu}(x) \quad (8.130)$$

$$\widehat{E}^j(x) = E^j(x) \quad (8.131)$$

$$\widehat{B}^j(x) = B^j(x) \quad (8.132)$$

## Comments

### 1. Gauge transformation

The transformation

$$A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu \chi(x) \quad (8.133)$$

where  $\chi(x)$  given by (8.127) is a gauge transformation.

(8.133) is generated by the transformation

$$B^\mu(p) \rightarrow B^\mu(p) + p^\mu f(p) \quad (8.134)$$

### 2. Gauge invariance

It follows from (8.128) and (8.129) that Maxwell's equations (8.112) and (8.113), or (8.119) and (8.120), are invariant under the gauge transformation (8.133).

It follows from (8.130) to (8.132) that the electromagnetic field—strength tensor and the electric and magnetic fields are invariant under the gauge transformation (8.133).

### 8.7.7 Lagrangian for the electromagnetic field

(8.112) is the Euler-Lagrange equation

$$\frac{\partial \mathcal{L}}{\partial A^\mu(x)} - \partial_\nu \frac{\partial \mathcal{L}}{\partial (\partial_\nu A^\mu(x))} = 0 \quad (8.135)$$

for the Lagrangian density

$$\mathcal{L}(x) = -\frac{1}{16\pi} F^{\mu\nu}(x) F_{\mu\nu}(x) \quad (8.136)$$

#### Comments

1. Normal order

Equations involving  $A^\mu(x)$  given by (8.111) are assumed to be written in normal order.

Creators are moved to the left of annihilators as if creators and annihilators commute.

2. Manifest covariance

(8.136) is a Lorentz scalar.

$$U(\Lambda, a) \mathcal{L}(x) U^\dagger(\Lambda, a) = \mathcal{L}(\Lambda x + a) \quad (8.137)$$

The manifest covariance of the formalism is expressed in terms of a manifestly covariant Lagrangian.

### 3. Gauge invariance

(8.136) is invariant under the gauge transformation (8.133).

## Section 8.8 Introduction to quantum electrodynamics

In this Section we give a very brief introduction to quantum electrodynamics (QED). The many excellent books on QFT should be consulted for further discussion.<sup>2</sup> Schwinger [B18] is a collection of a number of important contributions to QED.

Quantum electrodynamics is a Lorentz invariant theory of interacting electrons, positrons and photons which combines the theory of electrons and positrons given in Section 6.7 with the theory of electromagnetism given in Section 8.7.

The fundamental dynamical variables of QED are spinor fields  $\psi(x)$  and  $\bar{\psi}(x) = \psi^\dagger(x)\gamma^0$  and a vector field  $A^\mu(x)$ .

The fundamental algebra is given by (6.115) to (6.117) and (8.115). In addition,  $\psi(x)$  and  $\bar{\psi}(x)$  commute with  $A^\mu(y)$  for all  $x$  and  $y$ .

$A^\mu(x)$  satisfies the Lorentz condition (8.113).

The electromagnetic field strength tensor  $F^{\mu\nu}(x)$  is defined by (8.116).

The Cartesian components of the electric and magnetic fields are defined in terms of  $F^{\mu\nu}(x)$  by (8.117) and (8.118).

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<sup>2</sup> A list of selected reference books, journal articles and theses follows Chapter 12 of Part I.

The field equations of quantum electrodynamics are

$$\left(-i\partial - \frac{e}{c}\gamma \cdot A(x) + \frac{mc}{\hbar}\right)\psi(x) = 0 \quad (8.138)$$

$$\square A^\mu(x) = \frac{4\pi}{c}J^\mu(x) \quad (8.139)$$

$$J^\mu(x) = -\frac{e}{c}\bar{\psi}(x)\gamma^\mu\psi(x) \quad (8.140)$$

### Comments

#### 1. Form of the field equations

(8.138) and (8.139) are coupled equations for  $\psi(x)$  and  $A^\mu(x)$ .

The spinor field is “minimally” coupled to the vector field.

The spinor field provides the source of an electromagnetic field.

#### 2. Special case: free fields

(8.138) and (8.139) reduce to (6.109) and (8.112) [that is, the Dirac equation for free electrons and positrons and the wave equation for free photons] when the coupling constant  $e$  is set equal to zero.

QED reduces to the covariant quantum field theory of electrons and positrons given in Section 6.7 and the covariant quantum field theory of electromagnetism given in Section 8.7 when the fields are uncoupled.

### 3. Current 4-vector

$J^\mu(x)$  defined by (8.140) is conserved.

$$\partial_\mu J^\mu(x) = 0 \quad (8.141)$$

$J^\mu(x)$  is the charge current which is the source of the electromagnetic field  $A^\mu(x)$ .

### 4. Lagrangian

(8.138) and (8.139) are the Euler-Lagrange equations for the Lagrangian density

$$\mathcal{L}(x) = \mathcal{L}_{\text{dirac}}(x) + \mathcal{L}_{\text{em}}(x) + \mathcal{L}_{\text{int}}(x) \quad (8.142)$$

$\mathcal{L}_{\text{dirac}}(x)$  and  $\mathcal{L}_{\text{em}}(x)$  are given by (6.132) and (8.136) and

$$\mathcal{L}_{\text{int}}(x) = -J(x) \cdot A(x) \quad (8.143)$$

### 5. Gauge invariance

(8.142) is invariant under the gauge transformation



$$\psi(x) \rightarrow e^{-ie\chi(x)}\psi(x) \quad (8.144)$$

$$A^\mu(x) \rightarrow A^\mu(x) + \partial^\mu\chi(x) \quad (8.145)$$

where  $\chi(x)$  is an arbitrary function.

## 6. Bare mass, bare charge

The parameters  $m$  and  $e$  appearing in (8.138) and (8.139) are not the mass and charge of an electron.

$m$  and  $e$  are the bare mass and bare charge of the electron.

The mass and charge of the electron are related to  $m$  and  $e$  by the equations of QED.

As stated by Schwinger<sup>1</sup>

"A free electron is accompanied by an electromagnetic field which effectively alters the inertia of the system, and an electromagnetic field is accompanied by a current of electron-positron pairs which effectively alters the strength of the field and of all its charges. Hence a process of renormalization must be carried out, in which the initial parameters are eliminated in favor of those with immediate physical significance."

The renormalization procedure for QED is beyond the scope of these notes.

The relationship between bare mass and physical mass is illustrated for a system of interacting fermions and bosons in Chapter 9.

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<sup>1</sup> page xi, [B18]



## Chapter 9

# FOCK SPACE FOR FERMIONS AND BOSONS

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### Section 9.1 Introductory remarks

The Fock space descriptions of a system of fermions is given in Chapters 4 and 5 and a system of bosons in Chapter 7. In this Chapter we give the Fock space description of a system of fermions and bosons in interaction.

The Lagrangian method is usually used for the description of a system of interacting fermions and bosons. We do not follow this approach. We consider the interacting fermion-boson system using the instant and point forms of dynamics discussed by Dirac [J5].

The many excellent books on relativistic quantum field theory should be consulted for discussion of the Lagrangian method.<sup>1</sup>

The fermion-boson system considered in this Chapter is a prototype for the physically interesting systems listed in *Introductory Remarks* of Chapter 3. That is, it is a prototype for a system of

- electrons and photons
- electrons and phonons
- nucleons and pions
- quarks and gluons

The Hilbert space for the system is a direct product of a fermion Fock space and a boson Fock space. Fundamental dynamical variables for the system are given in Section 9.2.

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<sup>1</sup> A list of selected reference books, journal articles and theses follows Chapter 12 of Part I.

The Poincare generators for a Lorentz invariant system of free fermions and bosons are given in Section 9.3.

The instant form of dynamics is considered in Section 9.4. The point form of dynamics is considered in Section 9.5.

The fermion-boson trilinear interaction in the instant form of dynamics is discussed in Section 9.6.

The dressing transformation method for expressing the Hamiltonian in terms of creators and annihilators for physical particles is given in Section 9.7.

The dressing transformation for the trilinear interaction is given in Section 9.8.

The Yukawa potential for interacting physical fermions is derived using the dressing transformation method in Topic 9.8.4.

## Section 9.2 Fundamental dynamical variables

We consider a system of fermions and bosons. Each fermion has spin  $s_f$  where  $s_f$  is a half-odd integer. Each boson has spin  $s_b$  where  $s_b$  is a non-negative integer.

The Hilbert space  ${}^{fb}\mathfrak{H}^{s_f s_b}$  for the system is the direct product of the fermion Fock space  ${}^f\mathfrak{H}^{s_f}$  discussed in Chapters 4 and 5 and the boson Fock space  ${}^b\mathfrak{H}^{s_b}$  discussed in Chapter 7.

$${}^{fb}\mathfrak{H}^{s_f s_b} = {}^f\mathfrak{H}^{s_f} \otimes {}^b\mathfrak{H}^{s_b} \quad (9.1)$$

Fundamental dynamical variables for the system are creators and annihilators

for fermions

$$F_{m_{s_f}}^\dagger(p) \quad (9.2)$$

$$F_{m_{s_f}}(p) \quad (9.3)$$

as given in Section 5.3 and creators and annihilators for bosons

$$B_{m_{s_b}}^\dagger(p) \quad (9.4)$$

$$B_{m_{s_b}}(p) \quad (9.5)$$

as given in Section 7.5.

The spin labels in (9.2) to (9.5) take the values

$$m_{s_f} = s_f, s_f - 1, \dots, -s_f \quad (9.6)$$

$$m_{s_b} = s_b, s_b - 1, \dots, -s_b \quad (9.7)$$

The fundamental dynamical variables (9.2) to (9.5) satisfy

$$\left\{ F_{m_{s_f}}(p), F_{m'_{s_f}}(p') \right\} = 0 \quad (9.8)$$

$$\left\{ F_{m_{s_f}}^\dagger(p), F_{m'_{s_f}}^\dagger(p') \right\} = 0 \quad (9.9)$$

$$\left\{ F_{m_{s_f}}(p), F_{m'_{s_f}}^\dagger(p') \right\} = \delta(p - p') \delta_{m_{s_f} m'_{s_f}} \quad (9.10)$$

$$\left[ B_{m_{s_b}}(p), B_{m'_{s_b}}(p') \right] = 0 \quad (9.11)$$

$$\left[ B_{m_{s_b}}^\dagger(p), B_{m'_{s_b}}^\dagger(p') \right] = 0 \quad (9.12)$$

$$\left[ B_{m_{s_b}}(p), B_{m'_{s_b}}^\dagger(p') \right] = \delta(p - p') \delta_{m_{s_b} m'_{s_b}} \quad (9.13)$$

$$[F, B] = 0 \quad (9.14)$$

$$F = F_{m_{s_f}}(p) \quad \text{or} \quad F_{m_{s_f}}^\dagger(p) \quad (9.15)$$

$$B = B_{m_{s_b}}(p) \quad \text{or} \quad B_{m_{s_b}}^\dagger(p) \quad (9.16)$$

### Section 9.3 Free fermions and bosons

To describe a Lorentz invariant system of fermions and bosons, one must construct the Poincare generators  $H, P^j, J^j, K^j$  in terms of the fundamental dynamical variables (9.2) to (9.5). The Poincare generators satisfy the Poincare Algebra (5.68) to (5.76) or (5.77) to (5.79).

The Poincare generators for a system of free fermions are given by (5.84) to (5.88). The Poincare generators for a system of free bosons are given by (7.87) to (7.91). The Poincare generators for a system of free fermions and bosons is the sum of the right sides of the corresponding equations (5.84) to (5.88) and (7.87) to (7.91).

We add a subscript 0 to the generators for the free system to distinguish them from the generators for the interacting system.

The  $j$ —component of momentum  $P_0^j$  and the Hamiltonian  $H_0$  are

$$P_0^j = \int d^3p \left[ F^\dagger(p) p^j F(p) + B^\dagger(p) p^j B(p) \right] \quad (9.17)$$

$$H_0 = \int d^3p \left[ F^\dagger(p) \epsilon_f(p) F(p) + B^\dagger(p) \epsilon_b(p) B(p) \right] \quad (9.18)$$

$$\epsilon_f(p) = \sqrt{p^2 c^2 + m_f^2 c^4} \quad (9.19)$$

$$\epsilon_b(p) = \sqrt{p^2 c^2 + m_b^2 c^4} \quad (9.20)$$

$$F(p) = \begin{pmatrix} F_{s_f}(p) \\ F_{s_f-1}(p) \\ \vdots \\ F_{-s_f}(p) \end{pmatrix} \quad (9.21)$$

$$B(p) = \begin{pmatrix} B_{s_b}(p) \\ B_{s_b-1}(p) \\ \vdots \\ B_{-s_b}(p) \end{pmatrix} \quad (9.22)$$

$F^\dagger(p)$  and  $B^\dagger(p)$  are row matrices corresponding to (9.21) and (9.22).

The  $j$ -component of the spin angular momentum  $S_0^j$  is

$$S_0^j = \int d^3p \left[ F^\dagger(p) s_f^j F(p) + B^\dagger(p) s_b^j B(p) \right] \quad (9.23)$$

$s_f^j$  in (9.23) is a  $2s_f + 1$  by  $2s_f + 1$  matrix satisfying

$$[s_f^j, s_f^k] = i\hbar \epsilon_{jkl} s_f^l \quad (9.24)$$

$$s_f \cdot s_f = s_f(s_f + 1)\hbar^2 \quad (9.25)$$



$s_b^j$  in (9.23) is a  $2s_b + 1$  by  $2s_b + 1$  matrix satisfying

$$\left[ s_b^j, s_b^k \right] = i\hbar \epsilon_{jkl} s_b^l \quad (9.26)$$

$$s_b \cdot s_b = s_b(s_b + 1)\hbar^2 \quad (9.27)$$

The fermion number operator is

$$N_f = \int d^3p F^\dagger(p) F(p) \quad (9.28)$$

The boson number operator is

$$N_b = \int d^3p B^\dagger(p) B(p) \quad (9.29)$$

### Comments

#### 1. Vacuum state

The vacuum state  $|0\rangle$  contains no fermions or bosons.

$$F_{m_{s_f}}(p) |0\rangle = B_{m_{s_b}}(p) |0\rangle = 0 \quad (9.30)$$

$$\langle 0 | F_{m_{s_f}}^\dagger(p) = \langle 0 | B_{m_{s_b}}^\dagger(p) = 0 \quad (9.31)$$

## 2. Creating an elementary fermion

When acting on the vacuum state  $|0\rangle$ ,  $F_{m_{s_f}}^\dagger(p)$  creates an elementary fermion with rest mass  $m_f$ ,  $z$ -component of spin  $m_{s_f}$ ,  $j$ -component of momentum  $p^j$  and energy  $\epsilon_f(p)$ .

## 3. Creating an elementary boson

When acting on the vacuum state,  $B_{m_{s_b}}^\dagger(p)$  creates an elementary boson with rest mass  $m_b$ ,  $z$ -component of spin  $m_{s_b}$ ,  $j$ -component of momentum  $p^j$  and energy  $\epsilon_b(p)$ .

## Section 9.4 Instant form of dynamics

In this Section we consider a system of fermions and bosons in interaction.

We include interaction using the instant form of dynamics. That is, we take the  $j$ -component of momentum and angular momentum equal to the free system forms and we modify the Hamiltonian and the  $j$ -component of the booster.

For the interacting system,

$$H = H_0 + H_1 \quad (9.32)$$

$$P^j = P_0^j \quad (9.33)$$

$$J^j = J_0^j \quad (9.34)$$

$$K^j = K_0^j + K_1^j \quad (9.35)$$

Using (9.32) to (9.35) in the Poincare Algebra (5.68) to (5.76) yields

$$[P_0^j, H_1] = 0 \quad (9.36)$$

$$[J_0^j, H_1] = 0 \quad (9.37)$$

$$[K_1^j, J_0^k] = i\hbar\epsilon_{jkl}K_1^l \quad (9.38)$$

$$[K_1^j, P_0^k] = -i\hbar\delta_{jk}H_1/c^2 \quad (9.39)$$

$$[K_0^j, H_1] + [K_1^j, H_0] + [K_1^j, H_1] = 0 \quad (9.40)$$

$$[K_0^j, K_1^k] + [K_1^j, K_0^k] + [K_1^j, K_1^k] = 0 \quad (9.41)$$

## Comments

### 1. Equations for $H_1$ and $K_1^j$

(9.36) to (9.41) are the instant form of dynamics equations for  $H_1$  and  $K_1^j$ .

### 2. Free system

(9.36) to (9.41) are solved by

$$H_1 = 0 \quad (9.42)$$

$$K_1^j = 0 \quad (9.43)$$

(9.42) and (9.43) yield a system of free fermions and bosons.

### 3. Linear and nonlinear equations for $H_1$ and $K_1^j$

(9.36) to (9.39) are linear equations for  $H_1$  and  $K_1^j$ .

(9.40) and (9.41) are nonlinear equations for  $H_1$  and  $K_1^j$ .

Dirac [J5] states that “ [(9.40) and (9.41)] are not easily fulfilled and provide the real difficulty in the problem of constructing a theory of a relativistic dynamical system in the instant form.”

## Section 9.5 Point form of dynamics

In this Section we include interaction using the instant form of dynamics. That is, we take the  $j$ —component of the angular momentum and the booster equal to the free system forms and we modify the Hamiltonian and the  $j$ —component of the momentum.

For the interacting system,

$$P^\mu = P_0^\mu + P_1^\mu \quad (9.44)$$

$$M^{\mu\nu} = M_0^{\mu\nu} \quad (9.45)$$

Equivalently, using (5.80) to (5.82),

$$H = H_0 + H_1 \quad (9.46)$$

$$P^j = P_0^j + P_1^j \quad (9.47)$$

$$J^j = J_0^j \quad (9.48)$$

$$K^j = K_0^j \quad (9.49)$$

Using (9.44) and (9.45) in the Poincare Algebra (5.77) to (5.79) yields

$$[M_0^{\mu\nu}, P_1^\sigma] = i\hbar(g^{\nu\sigma} P_1^\mu - g^{\mu\sigma} P_1^\nu) \quad (9.50)$$

$$[P_0^\mu, P_1^\nu] + [P_1^\mu, P_0^\nu] + [P_1^\mu, P_1^\nu] = 0 \quad (9.51)$$

Equivalently, using (5.80) to (5.82),

$$\left[ J_0^j, H_1 \right] = 0 \quad (9.52)$$

$$\left[ J_0^j, P_1^k \right] = i\hbar \epsilon_{jkl} P_1^l \quad (9.53)$$

$$\left[ K_0^j, H_1 \right] = -i\hbar P_1^j \quad (9.54)$$

$$\left[ K_0^j, P_1^k \right] = -i\hbar \delta_{jk} H_1 / c^2 \quad (9.55)$$

$$\left[ P_0^j, P_1^k \right] + \left[ P_1^j, P_0^k \right] + \left[ P_1^j, P_1^k \right] = 0 \quad (9.56)$$

$$\left[ P_0^j, H_1 \right] + \left[ P_1^j, H_0 \right] + \left[ P_1^j, H_1 \right] = 0 \quad (9.57)$$

### Comments

1. Equations for  $P_1^\mu$

(9.50) and (9.51) are the point form of dynamics equations for  $P_1^\mu$ .

2. Free system

(9.50) and (9.51) are solved by

$$P_1^\mu = 0 \quad (9.58)$$

(9.58) yields a system of free fermions and bosons.

3. **Linear and nonlinear equations for  $P_1^\mu$**

(9.50) are linear equations for  $P_1^\mu$ .

(9.51) are nonlinear equations for  $P_1^\mu$ .

Dirac [J5] states that “[9.51] are not easily fulfilled and provide the real difficulty in the problem of constructing a theory of a relativistic dynamical system in the point form.”

4. **Comparison of the instant and point forms of dynamics**

The instant form has the advantage that the momentum is unchanged from its noninteracting form.

The point form has the advantage that the interaction terms are grouped as the the components of a 4-vector.

## **Section 9.6 Trilinear interaction**

In this Section we consider the instant form of dynamics discussed in Section 9.4 and we take the simplest form of  $H_1$  which involves both fermion and boson operators and which is Hermitian and conserves fermion number.

$$H_1 = \int d^3p d^3q F^\dagger(p-q)h(p,q)F(p)B^\dagger(q) + \text{adjoint} \quad (9.59)$$

### **Comments**

1. **Notation**

$h(p, q)$  is a matrix whose elements are functions of momenta and 3-components of spins.

$$(h_{m_s b}(p, q))_{m_s f m'_s f} = h_{m_s b m_s f m'_s f}(p, q) \quad (9.60)$$

## 2. Pictorial representation

The integrand in (9.59) can be represented pictorially.

## 3. Nomenclature

(9.59) is the trilinear interaction.

$h_{m_s b m_s f m'_s f}(p, q)$  is the vertex function for the trilinear interaction.

## 4. Vertex functions

$h(p, q)$  specifies the trilinear interaction.

For a Lorentz invariant system,  $h(p, q)$  is determined by requiring that (9.36) to (9.41) hold.

## 5. Conservation of fermion number

(9.59) conserves fermion number.

$$[H_1, N_f] = 0 \quad (9.61)$$



6. Nonconservation of boson number

(9.59) does not conserve boson number.

$$[H_1, N_b] \neq 0 \quad (9.62)$$

7. Prototype Hamiltonian

(9.59) is the simplest interaction which allows creation and annihilation of bosons through interaction with fermions.

(9.59) is the prototype interaction Hamiltonian for a number of systems of interacting fermions and bosons.

8. Creating a bare fermion

It follows from (9.18) that

$$H_0 F_{m_s_f}^\dagger(p) |0\rangle = \epsilon_f(p) F_{m_s_f}^\dagger(p) |0\rangle \quad (9.63)$$

It follows from (9.18), (9.32) and (9.59) that

$$H F_{m_s_f}^\dagger(p) |0\rangle \neq \tilde{\epsilon}_f(p) F_{m_s_f}^\dagger(p) |0\rangle \quad (9.64)$$

for any function  $\tilde{\epsilon}_f(p)$ .

In view of (9.64),  $F_{m_s_f}^\dagger(p)$  does not create a physical fermion when acting

on the vacuum state.

In view of (9.63),  $F_{m_{s_f}}^\dagger(p)$  is said to create a bare fermion when acting on the vacuum state.

$m_f$  in (9.19) is the rest mass of the bare fermion.

### 9. Creating a physical fermion

The creation operator  $\tilde{F}_{m_{s_f}}^\dagger(p)$  for a physical fermion satisfies

$$N_f \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle = \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle \quad (9.65)$$

$$S^3 \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle = \hbar m_{s_f} \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle \quad (9.66)$$

$$P^j \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle = p^j \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle \quad (9.67)$$

$$H \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle = \tilde{\epsilon}_f(p) \tilde{F}_{m_{s_f}}^\dagger(p) |0\rangle \quad (9.68)$$

$$\tilde{\epsilon}_f(p) = \sqrt{p^2 c^2 + \tilde{m}_f^2 c^4} \quad (9.69)$$

$\tilde{m}_f$  is the rest mass of the physical fermion.

## 10. Form of the physical fermion eigenket

The physical fermion eigenket  $\tilde{F}_{m_s f}^\dagger(p) |0\rangle$  has the form

$$\tilde{F}_{m_s f}^\dagger(p) |0\rangle = F_{m_s f}^\dagger(p) |0\rangle + \sum_{n=1}^{\infty} \left[ (B^\dagger)^n F^\dagger \right]_n |0\rangle \quad (9.70)$$

$$\left[ (B^\dagger)^n F^\dagger \right]_n |0\rangle$$

$$= \sum_{m_1, \dots, m_n = -s_b}^{+s_b} \sum_{m_{n+1} = -s_f}^{+s_f} \int_{-\infty}^{+\infty} d^3 p_1 \cdots d^3 p_{n+1} \quad (9.71)$$

$$\cdot c_{m_1 \dots m_{n+1}}(p_1 \cdots p_{n+1}) B_{m_1}^\dagger(p_1) \cdots B_{m_n}^\dagger(p_n) F_{m_{n+1}}^\dagger(p_{n+1}) |0\rangle$$

(9.70) satisfies (9.65).

The functions  $c_{m_1 \dots m_{n+1}}(p_1 \cdots p_{n+1})$  in (9.71) are determined by satisfying (9.66) to (9.68).

## 11. Physical fermion, bare fermion, cloud of bosons

(9.70) is interpreted by the statement:

The physical fermion is a bare fermion surrounded by a cloud of bosons.

## 12. Satisfying the Poincare Algebra

(5.68) to (5.76) are satisfied in the instant form of dynamics when (9.36) to (9.41) hold.

### i. The linear equations (9.36) to (9.39)

(9.36) holds with no restrictions on  $h(p, q)$ .

(9.39) holds if  $K_1^j$  has the form

$$K_1^j = \int d^3p d^3q F^\dagger(p - q) k^j(p, q) F(p) B^\dagger(q) + \text{adjoint} \quad (9.72)$$

$k^j(p, q)$  in (9.72) is related to  $h(p, q)$  through (9.39).

(9.37) and (9.38) restrict the form of  $h(p, q)$  under rotations.

### ii. The nonlinear equations (9.40) and (9.41)

(9.40) and (9.41) yield nonlinear equations for  $h(p, q)$ .

Hearn, McMillan and Raskin [J14] found for some special cases that the only solution to these nonlinear equations is

$$h(p, q) = 0 \quad (9.73)$$

(9.73) yields (9.42) and (9.43).

(9.73) yields a system of free fermions and bosons.

### 13. Unanswered questions

(9.73) raises a number of unanswered questions regarding the fermion-boson interaction for a Lorentz invariant system of fermions and bosons.

Can interaction be included in the instant form of dynamics?

Can interaction be included in the point form of dynamics?

Is it necessary to modify all Poincare generators in order to include interaction?

Is a nonzero trilinear interaction (9.59) possible if all generators are modified?

Is a nonzero trilinear interaction (9.59) an approximation to a correct fermion-boson interaction?

## Section 9.7 Dressing transformation

So far in this Chapter we have considered the fundamental dynamical variables to be the bare creators and annihilators  $F_{m_s f}^\dagger(p)$  and  $B_{m_s b}^\dagger(p)$  and their adjoints given by (9.2) to (9.5).

In this Section we introduce a unitary transformation of  $F_{m_s f}^\dagger(p)$  and  $B_{m_s b}^\dagger(p)$  which yields creators  $\tilde{F}_{m_s f}^\dagger(p)$  and  $\tilde{B}_{m_s b}^\dagger(p)$  for physical particles. The transformation “dresses” bare variables to yield physical variables.

The creator  $\tilde{F}_{m_s f}^\dagger(p)$  for a physical fermion satisfies (9.65) to (9.68).

The creator  $\tilde{B}_{m_s b}^\dagger(p)$  for a physical boson satisfies equations analogous to (9.65) to (9.68).

For the trilinear interaction (9.59), the physical fermion eigenket is given by (9.70). The physical fermion is a bare fermion surrounded by a cloud of bosons.

Further details of the dressing transformation method are given in Greenberg [J12], Schweber [B17], Hearn [T1], James [T3] and Hearn, McMillan and Raskin [J14].

We write

$$\tilde{F}_{m_s f}^\dagger(p) = U F_{m_s f}^\dagger(p) U^\dagger \quad (9.74)$$

$$\tilde{B}_{m_s b}^\dagger(p) = U B_{m_s b}^\dagger(p) U^\dagger \quad (9.75)$$

$$U = e^D \quad (9.76)$$

$$D^\dagger = -D \quad (9.77)$$

### Comments

#### 1. Dressing transformation

$U$  is unitary.

$$U U^\dagger = U^\dagger U = 1 \quad (9.78)$$

$U$  is the dressing transformation.

$D$  is the dressing operator.

The strategy for determining  $D$  is given in Item 6.

2. Fundamental dynamical variables

$\tilde{F}_{m_s f}^\dagger(p)$  and  $\tilde{B}_{m_s b}^\dagger(p)$  and their adjoints obey (9.8) to (9.14).

$\tilde{F}_{m_s f}^\dagger(p)$  and  $\tilde{B}_{m_s b}^\dagger(p)$  and their adjoints are fundamental dynamical variables for a system of fermions and bosons.

3. Expressions for operators.

Every operator  $A$  may be expressed in terms of the variables (9.2) to (9.5).

We write

$$A = A(F, B) \quad (9.79)$$

In view of (9.74), (9.75) and (9.78),

$$UA(F, B)U^\dagger = A(\tilde{F}, \tilde{B}) \quad (9.80)$$

$A(\tilde{F}, \tilde{B})$  is obtained from  $A(F, B)$  by replacing  $F$  by  $\tilde{F}$  and  $B$  by  $\tilde{B}$ .

4. Invariance of the dressing operator

It follows from (9.76) and (9.80) that

$$D(\tilde{F}, \tilde{B}) = D(F, B) \quad (9.81)$$

Proof:

$$D(\tilde{F}, \tilde{B}) = UD(F, B)U^\dagger = e^{D(F, B)}D(F, B)e^{-D(F, B)} = D(F, B) \quad (9.82)$$

### 5. Dressed Hamiltonian

It follows from (9.80) and (9.81) that

$$H(F, B) = \mathcal{H}(\tilde{F}, \tilde{B}) \quad (9.83)$$

where

$$\mathcal{H}(\tilde{F}, \tilde{B}) = e^{-D(\tilde{F}, \tilde{B})}H(\tilde{F}, \tilde{B})e^{D(\tilde{F}, \tilde{B})} \quad (9.84)$$

Proof:

$$H(F, B) = U^\dagger H(\tilde{F}, \tilde{B})U = e^{-D(\tilde{F}, \tilde{B})}H(\tilde{F}, \tilde{B})e^{D(\tilde{F}, \tilde{B})} = \mathcal{H}(\tilde{F}, \tilde{B}) \quad (9.85)$$

(9.84) defines  $\mathcal{H}$  in terms of  $H$  and  $D$ .

$\mathcal{H}(\tilde{F}, \tilde{B})$  is the Hamiltonian in terms of dressed creators and annihilators.

$H(F, B)$  is the dressed Hamiltonian.



6. Determining the dressing operator

$D(\tilde{F}, \tilde{B})$  is chosen in order that  $\tilde{F}_{m_s f}^\dagger(p) | 0 \rangle$  and  $\tilde{B}_{m_s b}^\dagger(p) | 0 \rangle$  are eigenkets of the dressed Hamiltonian.

$$\mathcal{H}(\tilde{F}, \tilde{B}) \tilde{F}_{m_s f}^\dagger(p) | 0 \rangle = \tilde{\epsilon}_f(p) \tilde{F}_{m_s f}^\dagger(p) | 0 \rangle \quad (9.86)$$

$$\mathcal{H}(\tilde{F}, \tilde{B}) \tilde{B}_{m_s b}^\dagger(p) | 0 \rangle = \tilde{\epsilon}_b(p) \tilde{B}_{m_s b}^\dagger(p) | 0 \rangle \quad (9.87)$$

for some functions  $\tilde{\epsilon}_f(p)$  and  $\tilde{\epsilon}_b(p)$ .

(9.86) and (9.87) hold if, apart from the terms

$$\tilde{F}^\dagger \tilde{F} \quad \text{and} \quad \tilde{B}^\dagger \tilde{B} \quad (9.88)$$

there are no terms in (9.84) which contain only one annihilation operator.

That is, (9.86) and (9.87) hold if (9.84) does not contain terms of the form

$$\tilde{F}^\dagger \tilde{F} \tilde{B}^\dagger, \quad \tilde{F}^\dagger \tilde{F} \tilde{B}^\dagger \tilde{B}^\dagger, \quad \text{etc.} \quad (9.89)$$

$D(\tilde{F}, \tilde{B})$  is chosen in order to eliminate terms of the form (9.89) from  $\mathcal{H}(\tilde{F}, \tilde{B})$ .

## Section 9.8 Dressing the trilinear interaction

### 9.8.1 General remarks

The operator  $H(\tilde{F}, \tilde{B})$  for a system of fermions and bosons interacting via the trilinear interaction is

$$H(\tilde{F}, \tilde{B}) = H_0(\tilde{F}, \tilde{B}) + \lambda H_1(\tilde{F}, \tilde{B}) \quad (9.90)$$

$H_0(\tilde{F}, \tilde{B})$  and  $H_1(\tilde{F}, \tilde{B})$  in (9.90) are given by (9.18) and (9.59) with  $F$  replaced by  $\tilde{F}$  and  $B$  replaced by  $\tilde{B}$ .

The real parameter  $\lambda$  in (9.90) is an order-counting parameter which can be set equal to unity.

We write the dressing operator  $D(\tilde{F}, \tilde{B})$  in the form

$$D = \sum_{n=1}^{\infty} \lambda^n D_n \quad (9.91)$$

Substituting (9.91) into (9.84) yields

$$\begin{aligned}
\mathcal{H} = & H_0 + \lambda\{H_1 + [H_0, D_1]\} \\
& + \lambda^2 \left\{ [H_1, D_1] + \frac{1}{2}[[H_0, D_1], D_1] + [H_0, D_2] \right\} \\
& + \dots
\end{aligned} \tag{9.92}$$

### Comments

1. Dressed Hamiltonian

(9.92) gives the dressed Hamiltonian in terms of the known operators  $H_0$  and  $H_1$  and operators  $D_1, D_2, \dots$  to be determined.

2. Determining the  $D_n$

The  $D_n(\tilde{F}, \tilde{B})$  are chosen in order to eliminate terms of the form (9.89) from (9.92).

This can be done order by order in  $\lambda$ .

3. Expression for  $D_1$

$H_1(\tilde{F}, \tilde{B})$  given by (9.59) contains a term of the form (9.89).

It follows from (9.92) that  $D_1(\tilde{F}, \tilde{B})$  is chosen in order that

$$[H_0, D_1] = -H_1 \tag{9.93}$$

It follows from (9.18) and (9.59) that (9.93) holds if

$$D_1 = \int d^3 p d^3 q \tilde{F}^\dagger(p - q) d_1(p, q) \tilde{F}(p) \tilde{B}^\dagger(q) - \text{adjoint} \quad (9.94)$$

$$d_1(p, q) = \frac{-h(p, q)}{\Delta(p, q)} \quad (9.95)$$

$$\Delta(p, q) = \epsilon_f(p - q) + \epsilon_b(q) - \epsilon_f(p) \quad (9.96)$$

#### 4. Second equation for the dressed Hamiltonian

Substitution of (9.93) into (9.92) yields

$$\begin{aligned} \mathcal{H} = & H_0 + \lambda^2 \left\{ \frac{1}{2} [H_1, D_1] + [H_0, D_2] \right\} \\ & + \lambda^3 \left\{ \frac{1}{3} [[H_1, D_1], D_1] + \frac{1}{2} [[H_0, D_2], D_1] + \frac{1}{2} [H_1, D_2] + [H_0, D_3] \right\} \quad (9.97) \\ & + \dots \end{aligned}$$

(9.97) gives the dressed Hamiltonian with  $D_1(\tilde{F}, \tilde{B})$  given by (9.94).

#### 5. Choice of $D_2, D_3, \dots$

$D_2$  is chosen in order to eliminate terms of the form (9.89) in the  $\lambda^2$  term in (9.97).

Substitution of  $D_2$  in the  $\lambda^3$  term in (9.97) yields a new equation for the dressed Hamiltonian.

$D_3$  is chosen in order to eliminate terms of the form (9.89) in the  $\lambda^3$  term in the new equation for the dressed Hamiltonian, and so on.

### 9.8.2 Hamiltonian dressed to second-order

For simplicity, in the remainder of this Chapter we take the vertex function (9.60) to depend only on the boson momentum and to be independent of spin.<sup>1</sup>

to be real

$$h_{m_s_b m_s_f m'_s_f}(p, q) = h(q) \quad (9.98)$$

Here  $h(q)$  is a real function. Invariance of the Hamiltonian under rotations requires that  $h(q)$  be a function of  $|q|$ .

$$h(q) = h^*(q) \quad (9.99)$$

$$h(q) = h(|q|) \quad (9.100)$$

Retaining only the  $\lambda^2$  term in (9.97) after the above choice of  $D_2$  yields an approximation  $\mathcal{H}_2(\tilde{F}, \tilde{B})$  to the dressed Hamiltonian  $\mathcal{H}(\tilde{F}, \tilde{B})$ .

<sup>1</sup> Spin has been included by Hearn [T] and Hearn, McMillan and Raskin [J14].

$\mathcal{H}_2(\tilde{F}, \tilde{B})$  is the Hamiltonian dressed to second order.

It follows on substituting (9.94) in the  $\lambda^2$  term in (9.97) that

$$\mathcal{H}_2 = T + V_{ff} + V_{fb} \quad (9.101)$$

$$T = \int dp \left[ \tilde{F}^\dagger(p) \tilde{\epsilon}_f(p) \tilde{F}(p) + \tilde{B}^\dagger(p) \tilde{\epsilon}_b(p) \tilde{B}(p) \right] \quad (9.102)$$

$$\tilde{\epsilon}_f(p) = \epsilon_f(p) - \lambda^2 \int d^3 q \frac{h^2(q)}{\Delta(p, q)} \quad (9.103)$$

$$\tilde{\epsilon}_b(p) = \epsilon_b(p) \quad (9.104)$$

$$V_{ff} = \frac{1}{2} \int d^3 k d^3 k' d^3 K$$

$$\tilde{F}^\dagger\left(\frac{1}{2}K + k\right) \tilde{F}^\dagger\left(\frac{1}{2}K - k\right) V_{ff}(k, k', K) \tilde{F}\left(\frac{1}{2}K - k'\right) \tilde{F}\left(\frac{1}{2}K + k'\right) \quad (9.105)$$

$$V_{ff}(k, k', K) = \frac{-\lambda^2 h^2(k - k')}{\Delta\left(\frac{1}{2}K - k', k - k'\right)} \quad (9.106)$$

$$+k \leftrightarrow k'$$

$$V_{fb} = \frac{1}{2} \int d^3 k d^3 k' d^3 K$$

$$\tilde{F}^\dagger\left(\frac{1}{2}K + k\right) \tilde{B}^\dagger\left(\frac{1}{2}K - k\right) V_{fb}(k, k', K) \tilde{B}\left(\frac{1}{2}K - k'\right) \tilde{F}\left(\frac{1}{2}K + k'\right) \quad (9.107)$$

$$V_{fb}(k, k', K) = \lambda^2 h \left(\frac{1}{2}K - k\right) h \left(\frac{1}{2}K - k'\right) \cdot$$

$$\left[ \frac{1}{\Delta(K, \frac{1}{2}K - k)} - \frac{1}{\Delta(\frac{1}{2}K + k', \frac{1}{2}K - k)} \right] \quad (9.108)$$

$$+ k \leftrightarrow k'$$

### Comments

#### 1. Physical processes

(9.101) contains an interaction  $V_{ff}$  between two physical fermions and an interaction  $V_{fb}$  between a physical fermion and a physical boson.

#### 2. Fermion-fermion potential

(9.105) is the second-order fermion—fermion potential.

(9.106) gives the form of this potential in terms of the vertex function (9.98).

The right side of (9.106) is negative. The fermion—fermion force is attractive.

The right side (9.106) can be represented pictorially.

The fermion—fermion potential arises because of “boson exchange” between the pair of interacting fermions.

### 3. Fermion-boson potential

(9.107) is the second-order fermion—boson potential.

(9.108) gives the form of this potential in terms of the vertex function (9.98).

The two terms in the second line of (9.108) have different signs.

One term gives an attractive force and the other gives a repulsive force.

The right side (9.108) can be represented pictorially.

The fermion—boson potential arises because of “fermion exchange” between the interacting fermion-boson pair.

### 4. Bare mass in terms of physical mass

$\tilde{m}_f$  is the known mass of the physical fermion.

$\tilde{m}_f$  is related to the energy  $\tilde{\epsilon}_f(p)$  of the physical fermion by

$$\tilde{m}_f = \tilde{\epsilon}_f(0)/c^2 \quad (9.109)$$

The mass  $m_f$  of the bare fermion is not observed.



$m_f$  is given to second order by (9.103) and (9.109).

$$m_f = \tilde{m}_f + \frac{\lambda^2}{c^2} \int d^3q \frac{h^2(q)}{\Delta(0, q)} \quad (9.110)$$

$$\Delta(0, q) = \epsilon_f(q) + \epsilon_b(q) - m_f c^2 \quad (9.111)$$

### 5. Higher order terms

The  $\lambda^3$  term in (9.97) contains terms of the form

$$\tilde{F}^\dagger \tilde{F}^\dagger \tilde{B}^\dagger \tilde{F} \tilde{F} \quad \text{and} \quad \tilde{F}^\dagger \tilde{F}^\dagger \tilde{B} \tilde{F} \tilde{F} \quad (9.112)$$

These terms correspond to boson production and annihilation in a fermion—fermion interaction.

### 9.8.3 Terminating the dressing operator series

(9.91) terminates at the first term, that is,

$$D_n = 0 \quad n = 2, 3, \dots \quad (9.113)$$

when the approximation

$$\Delta(p, q) = \epsilon_b(q) \quad (9.114)$$

is made in (9.95).

### Comments

#### 1. Fermion-boson potential

Approximation (9.114) yields

$$V_{fb}(k, k', K) = 0 \quad (9.115)$$

The fermion-boson potential vanishes when approximation (9.114) is used.

#### 2. Fermion-fermion potential in momentum space

Approximation (9.114) yields the following simplification to (9.106):

$$\begin{aligned} V_{ff}(k, k', K) &= \frac{-2\lambda^2 h^2 (k - k')}{\epsilon_b(k - k')} \\ &= V_{ff}(|k - k'|) \end{aligned} \quad (9.116)$$

(9.116) is a momentum-space fermion-fermion potential with depends only on the momentum transferred between the interacting pair.

#### 3. Fermion-fermion potential in coordinate space

It follows from (5.106) that the fermion-fermion coordinate-space potential is

$$V_{ff}(r) = \int d^3q e^{iq \cdot r / \hbar} V_{ff}(q) \quad (9.117)$$

### 9.8.4 Yukawa potential

It follows from (9.117) that

$$V_{ff}(r) = -g \left( \frac{e^{-r/r_b}}{r/r_b} \right) \quad (9.118)$$

$$g = g_0 m_b c^2 \quad (9.119)$$

$$r_b = \frac{\hbar}{m_b c} \quad (9.120)$$

when

$$h(q) = \frac{1}{2\pi} \sqrt{\frac{g_0 c^3}{\epsilon_b(q)}} \quad (9.121)$$

## Comments

### 1. Yukawa potential

(9.118) is the Yukawa potential for interacting fermions.

(9.118) was derived by H. Yukawa in 1935.

### 2. Range of the Yukawa potential

(9.120) is the range of the Yukawa potential.

The range is equal to the Compton wavelength of the exchanged boson.

Yukawa used the known range of the nucleon-nucleon potential to predict the existence of a boson with rest mass  $140 \text{ MeV}/c^2$ .

The pion was discovered in 1947.

### 3. Form of the vertex function (9.121)

The factor  $1/\sqrt{\epsilon_b(q)}$  in (9.121) appears in the expression (7.106) for the scalar field  $\phi(x)$ .

### 4. Infinite bare mass

The integral in (9.110) diverges when approximation (9.114) is used with (9.121).

The bare mass  $m_f$  in this approximation is infinite.

## Appendix: Some Commutators

In this Appendix we give some commutators of products of fermion and boson creators and annihilators.

The right sides of all equations are written in normal order. That is, all creators are written to the left of all annihilators.

### Section A.1 Commutators for fermions

The following commutators result from (4.34) to (4.36) and from the identities given in the Appendix of Part I.

$$\left[ F_r, F_s^\dagger F_t \right] = F_t \delta_{rs} \quad (\text{A.1})$$

$$\left[ F_r^\dagger, F_s^\dagger F_t \right] = -F_s^\dagger \delta_{rt} \quad (\text{A.2})$$

$$\left[ F_r, F_s^\dagger F_t^\dagger \right] = F_t^\dagger \delta_{rs} - F_s^\dagger \delta_{rt} \quad (\text{A.3})$$

$$\left[ F_r^\dagger, F_s F_t \right] = F_t \delta_{rs} - F_s \delta_{rt} \quad (\text{A.4})$$

$$\left[ F_r^\dagger F_s, F_t^\dagger F_u \right] = F_r^\dagger F_u \delta_{st} - F_t^\dagger F_s \delta_{ru} \quad (\text{A.5})$$

$$\left[ F_r^\dagger F_s, F_t^\dagger F_u^\dagger \right] = F_r^\dagger \left( F_t^\dagger \delta_{su} + F_u^\dagger \delta_{st} \right) \quad (\text{A.6})$$

$$\begin{aligned} \left[ F_r^\dagger F_s^\dagger, F_t F_u \right] &= \\ &= F_r^\dagger (F_u \delta_{st} - F_t \delta_{su}) + F_s^\dagger (F_t \delta_{ru} - F_u \delta_{rt}) - \delta_{ru} \delta_{st} + \delta_{su} \delta_{rt} \end{aligned} \quad (\text{A.7})$$

$$\left[ F_r, F_s^\dagger F_t^\dagger F_u F_v \right] = \left( F_t^\dagger \delta_{rs} - F_s^\dagger \delta_{rt} \right) F_u F_v \quad (\text{A.8})$$

$$\left[ F_r^\dagger, F_s^\dagger F_t^\dagger F_u F_v \right] = F_s^\dagger F_t^\dagger (F_v \delta_{ru} - F_u \delta_{rv}) \quad (\text{A.9})$$

$$\begin{aligned} \left[ F_r^\dagger F_s, F_t^\dagger F_u^\dagger F_v F_w \right] &= \\ &= F_r^\dagger \left( F_u^\dagger \delta_{st} - F_t^\dagger \delta_{su} \right) F_v F_w + F_t^\dagger F_u^\dagger (F_w \delta_{rv} - F_v \delta_{rw}) F_s \end{aligned} \quad (\text{A.10})$$

### Comments

#### 1. Commutators involving field operators

The commutators (5.15) to (5.17) and the identities given in the Appendix of Part I yield a set of identities similar to the above.

For example, corresponding to (A.1) is

$$\left[ F_r(x_r), F_s^\dagger(x_s) F_t(x_t) \right] = F_t(x_t) \delta(x_r - x_s) \delta_{rs} \quad (\text{A.11})$$

## 2. Commutators involving momentum and spin

The commutators (5.35) to (5.37) and the identities given in the Appendix of Part I yield a set of identities similar to the above.

For example, corresponding to (A.1) is

$$\left[ F_r(p_r), F_s^\dagger(p_s) F_t(p_t) \right] = F_t(p_t) \delta(p_r - p_s) \delta_{rs} \quad (\text{A.12})$$

## 3. Commutators involving momentum and helicity

The commutators (5.58) to (5.60) and the identities given in the Appendix of Part I yield a set of identities similar to the above.

For example, corresponding to (A.1) is

$$\left[ F^r(p_r), F^{s\dagger}(p_s) F^t(p_t) \right] = F^t(p_t) \delta(p_r - p_s) \delta_{rs} \quad (\text{A.13})$$

## Section A.2 Commutators for bosons

The following commutators result from (7.30) to (7.32) and from the identities given in the Appendix of Part I.

$$\left[ B_r, B_s^\dagger B_t \right] = B_t \delta_{rs} \quad (\text{A.14})$$

$$\left[ B_r^\dagger, B_s^\dagger B_t \right] = -B_s^\dagger \delta_{rt} \quad (\text{A.15})$$

$$\left[ B_r, B_s^\dagger B_t^\dagger \right] = B_t^\dagger \delta_{rs} + B_s^\dagger \delta_{rt} \quad (\text{A.16})$$

$$\left[ B_r^\dagger, B_s B_t \right] = -B_t \delta_{rs} - B_s \delta_{rt} \quad (\text{A.17})$$

$$\left[ B_r^\dagger B_s, B_t^\dagger B_u \right] = B_r^\dagger B_u \delta_{st} - B_t^\dagger B_s \delta_{ru} \quad (\text{A.18})$$

### Comments

#### 1. Commutators involving field operators

The commutators (7.47) to (7.49) and the identities given in the Appendix of Part I yield a set of identities similar to the above.



For example, corresponding to (A.14) is

$$\left[ B_r(x_r), B_s^\dagger(x_s) B_t(x_t) \right] = B_t(x_t) \delta(x_r - x_s) \delta_{rs} \quad (\text{A.19})$$

## 2. Commutators involving momentum and spin

The commutators (7.62) to (7.64) and the identities given in the Appendix of Part I yield a set of identities similar to the above.

For example, corresponding to (A.14) is

$$\left[ B_r(p_r), B_s^\dagger(p_s) B_t(p_t) \right] = B_t(p_t) \delta(p_r - p_s) \delta_{rs} \quad (\text{A.20})$$

## 3. Commutators involving momentum and helicity

The commutators (7.79) to (7.81) and the identities given in the Appendix of Part I yield a set of identities similar to the above.

For example, corresponding to (A.14) is

$$\left[ B^r(p_r), B^{s\dagger}(p_s) B^t(p_t) \right] = B^t(p_t) \delta(p_r - p_s) \delta_{rs} \quad (\text{A.21})$$

## Section A.3 Commutators for fermions and bosons

Commutators for fermions and bosons are derived from (4.34) to (4.36), (7.30) to (7.32) and from the results given in the previous Sections.

The following commutator arises in the calculations in Chapter 9.

$$\begin{aligned} & \left[ F_r^\dagger F_s B_t, F_u^\dagger F_v B_w^\dagger \right] = \\ & -F_r^\dagger F_u^\dagger F_s F_v \delta_{tw} + F_r^\dagger B_w^\dagger F_v B_t \delta_{su} - F_u^\dagger B_w^\dagger F_s B_t \delta_{rv} + F_r^\dagger F_v \delta_{su} \delta_{tw} \end{aligned} \tag{A.22}$$

### Comments

#### 1. Commutators involving other variables

Commutators involving fermion and boson field operators, or momentum and spin operators or momentum and helicity operators are similar to the above.

The examples in the previous Sections illustrate the appropriate correspondence.

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