QUANTUM LEAPS AND BOUNDS

Quantum Mechanics in Fock Space

Malcolm McMillan

Department of Physics

The University of British Columbia

Contents

Preface

The six volumes of notes *Quantum Leaps and Bounds (QLB)* form the basis of the introductory graduate quantum mechanics course I have given in the Department of Physics at the University of British Columbia at various times since 1973.

The six volumes of *QLB* are

- *Introductory Topics*: a collection of miscellaneous topics in introductory quantum mechanics
- *Scattering Theory*: an introduction to the basic ideas of quantum scattering theory by considering the scattering of a relativistic spinless particle from a fixed target
- *Quantum Mechanics in Fock Space*: an introduction to the second-quantization description of nonrelativistic many-body systems
- *Relativistic Quantum Mechanics*: an introduction to incorporating special relativity in quantum mechanics
- *Some Lorentz Invariant Systems*: some examples of systems incorporating special relativity in quantum mechanics
- *Relativistic Quantum Field Theory*: an elementary introduction to the relativistic quantum field theory of spinless bosons, spin $\frac{1}{2}$ fermions and antifermions and to quantum electrodynamics, the relativistic quantum field theory of electrons, positrons and photons

QLB assumes no familiarity with relativistic quantum mechanics. It does assume that students have taken undergraduate courses in nonrelativistic quantum mechanics which include discussion of the nonrelativistic Schrodinger equation and the solutions of some standard problems (e.g., the one-dimensional harmonic oscillator and the hydrogen atom) and perturbation theory and other approximation methods.

QLB assumes also that students will take other graduate courses in condensed matter physics, nuclear and particle physics and relativistic quantum field theory. Accordingly, our purpose in *QLB* is to introduce some basic ideas and formalism and thereby give students sufficient background to read the many excellent texts on these subjects.

I am happy to have this opportunity to thank my friends and colleagues R. Barrie, M. Bloom, J. Feldman, D.H. Hearn, W.W. Hsieh, R.I.G. Hughes, F.A. Kaempffer, P.A. Kalyniak, R.H. Landau, E.L. Lomon, A.H. Monahan, W. Opechowski, M.H.L. Pryce, A. Raskin, P. Rastall, L. Rosen, L. Sobrino, F. Tabakin, A.W. Thomas, E.W. Vogt and G.M. Volkoff for sharing their knowledge of quantum mechanics with me.

I also thank my wife, Henrietta, for suggesting the title for these volumes of notes. Quite correctly, she found my working title *Elements of Intermediate Quantum Mechanics* a bore.

QUANTUM MECHANICS IN FOCK SPACE

Chapter 1 INTRODUCTORY REMARKS

This volume of *QLB* gives an introduction to the second-quantization description of nonrelativistic many-body systems. This description is the standard language of condensed matter physics and of low-energy nuclear physics.

The quantum mechanics of a system of identical particles is considered in Chapter 2 and occupation number representation for systems of fermions and bosons is discussed in Chapter 3.

The Fock space description of a system of identical fermions is given in Chapters 4 and 5. Chapter 4 contains a derivation of the Hartree-Fock potential and a description of correlated particle-hole states of an n-fermion system. Quantum fields for fermions are introduced in Chapter 5.

The Fock space description of the quantum mechanics of a system of identical bosons is given in Chapter 6. Quantum fields for bosons are introduced in this chapter.

A system of fermions and bosons in interaction is considered in Chapter 7. This chapter includes the dressing transformation method for expressing the Hamiltonian of the system in terms of operators for physical particles. The Yukawa potential for interacting physical fermions is derived using the dressing transformation method.

Appendices A and B contain a description of a system of two and three distinguishable particles, respectively, and Appendix C contains some useful commutation relations for fermion and boson variables.

The volume concludes with lists of selected reference books, journal articles and theses.

Chapter 2 n-PARTICLE SYSTEM

In this chapter we describe some aspects of the quantum mechanics of a system of n particles.[1]

A system of n distinguishable particles is considered in Section 2.1 and system of n indistinguishable (or identical) particles is consided in Section 2.2. An important feature of indistinguishabilty of particles is that it places restrictions on the form of the observables of the system and on symmetry properties of states of the system.

2.1 n distinguishable particles

Fundamental dynamical variables

We consider a physical system of n distinguishable particles with rest masses

$$\boxed{m_1, m_2, ..., m_n \qquad (2.1)}$$

and spins

$$\boxed{s_1, s_2, ..., s_n \qquad (2.2)}$$

The Hilbert space $\mathfrak{H}_n^{s_1 s_2 \cdots s_n}$ for the system is the direct product of n one-particle Hilbert spaces. That is,

[1] The quantum mechanics of a Lorentz invariant system of n distinguishable particles in interaction is discussed in more detail in *QLB: Some Lorentz Invariant Systems.*

$$\maltese_n^{s_1 s_2 \cdots s_n} = \maltese_1^{s_1} \otimes \maltese_1^{s_2} \otimes \cdots \otimes \maltese_1^{s_n} \tag{2.3}$$

$\maltese_1^{s_\alpha}$ denotes the Hilbert space for particle α and $\otimes$ denotes direct product.

The fundamental dynamical variables of the system are the Cartesian coordinates, momenta and spin of the individual particles

$$X_1^j, P_1^j, S_1^j, \quad X_2^j, P_2^j, S_2^j, \quad \ldots, \quad X_n^j, P_n^j, S_n^j \tag{2.4}$$

where $j = 1, 2, 3$. These variables satisfy

$$\left[X_\alpha^j, X_\beta^k\right] = 0 \tag{2.5}$$

$$\left[P_\alpha^j, P_\beta^k\right] = 0 \tag{2.6}$$

$$\left[X_\alpha^j, P_\beta^k\right] = i\hbar\delta_{\alpha\beta}\delta_{jk} \tag{2.7}$$

$$\left[S_\alpha^j, S_\beta^k\right] = i\hbar\delta_{\alpha\beta}\epsilon_{jkl}S_\alpha^l \tag{2.8}$$

$$\left(\vec{S}_\alpha\right)^2 = s_\alpha(s_\alpha + 1)\hbar^2 \tag{2.9}$$

$$\left[X^j_\alpha, S^k_\beta\right] = \left[P^j_\alpha, S^k_\beta\right] = 0 \tag{2.10}$$

where $\alpha, \beta = 1, 2, \cdots, n$

Components of a state

The set of $n!$ permutations of the n particles in the system form the symmetric group S_n. Each permutation can be regarded as the product of one or more transpositions (or interchanges) of two particles.[1]

We list some properties of permutation operators for the system. Details for two- and three-body systems are given in Appendices A and B.

1. Transposition operators

One can construct transposition operators

$$\Pi_{\alpha\beta} \qquad (\alpha, \beta = 1, 2, \cdots, n; \alpha \neq \beta) \tag{2.11}$$

which correspond to transpositions of particles in the system. $\Pi_{\alpha\beta}$ corresponds to the replacement

$$\xi_\alpha \leftrightarrow \xi_\beta \tag{2.12}$$

in any function of the fundamental dynamical variables where

1 See, for example, Tung (1985) for details about S_n.

$$\xi_\alpha = \left\{ \vec{X}_\alpha, \vec{P}_\alpha, \vec{S}_\alpha \right\} \tag{2.13}$$

Thus, for example,

$$\Pi_{\alpha\beta} \vec{X}_\alpha \Pi^\dagger_{\alpha\beta} = \vec{X}_\beta \tag{2.14}$$

2. **Permutation operators**

One can construct permutation operators $\Pi_1, \Pi_2, \cdots, \Pi_{n!}$ corresponding to permutations of the particles in the system.

Each permutation operator can be written as a product of one or more transposition operators.

3. **Symmetric, antisymmetric and mixed components of a state**

It follows from properties of S_n that every state $\mid \psi >$ of the system can be written as

$$\mid \psi > = \mid \psi_s > + \mid \psi_a > + \mid \psi_m > \tag{2.15}$$

where

$$\Pi \mid \psi_s >= + \mid \psi_s > \tag{2.16}$$

$$\Pi_{\alpha\beta} \mid \psi_a >= - \mid \psi_a > \tag{2.17}$$

where Π is any permutation operator and $\Pi_{\alpha\beta}$ is any transposition operator.

$\mid \psi_s >, \mid \psi_a >$ and $\mid \psi_m >$ are mutually orthogonal and are, respectively, the symmetric, antisymmetric and mixed components of $\mid \psi >$.

2.2 n identical particles

In this section we consider a system of n indistinguishable (identical) particles. For such a system there is no observable change when particles are interchanged.

There are two consequences of indistinguishability which are treated separately below.

Invariance of observables

The first consequence of indistinguishability (Consequence 1) is:

Invariance of observables

Every observable of the system is invariant under all permutations of the particles.

Proof of Consequence 1

Indistinguishability of all particles of the system implies that

$$< \psi_{perm} \mid A \mid \psi_{perm} >=< \psi \mid A \mid \psi > \tag{2.18}$$

for all observables A and states $\mid \psi >$ where

$$\mid \psi_{perm} >= \Pi \mid \psi > \tag{2.19}$$

where Π is any permutation operator for the system.

It follows from (2.18) and (2.19) that

$$\Pi A \Pi^{\dagger} = A \tag{2.20}$$

for all A; (2.20) is Consequence 1.

Consequences of Consequence 1

1. **Values of rest mass and spin**

   It follows from Consequence 1 that all n particles have the same rest mass m and same spin s.

2. **Hilbert space**

It follows from Consequence 1 that the Hilbert space $\maltese_n^s$ for the system consists of n copies of a one-particle Hilbert space.

$$\boxed{\maltese_n^s = \maltese_1^s \otimes \maltese_1^s \otimes \cdots \otimes \maltese_1^s \qquad (2.21)}$$

We write (2.21) more compactly as

$$\boxed{\maltese_n^s = \otimes^n \maltese_1^s \qquad (2.22)}$$

Symmetry of states

The second consequence of indistinguishability (Consequence 2) is:

> **Symmetry of states**
>
> The states of the system are either all symmetrical or all antisymmetrical with respect to interchange of any two particles, this property depending upon the species of particle.

Proof of Consequence 2

It follows from (2.18) that no experiment can determine whether the system is in the state $\mid \psi >$ or the state $\mid \psi_{trans} >$ where

$$| \psi_{trans} >= \Pi_{\alpha\beta} | \psi > \tag{2.23}$$

where $\Pi_{\alpha\beta}$ is any transposition operator for the system. The vectors $| \psi_{trans} >$ and $| \psi >$ therefore correspond to the same ray in $\mathcal{H}_n^s$. That is,

$$| \psi_{trans} >= e^{i\delta} | \psi > \tag{2.24}$$

where δ is a real number.

It follows from (2.15) to (2.17) that

$$| \psi_{trans} >=| \psi_s > - | \psi_a > + \Pi_{\alpha\beta} | \psi_m > \tag{2.25}$$

Therefore, in order to satisfy (2.24), either

$$| \psi >=| \psi_s > \tag{2.26}$$

in which case $e^{i\delta} = +1$ and $| \psi_{trans} >= + | \psi >$, or

$$| \psi >=| \psi_a > \tag{2.27}$$

in which case $e^{i\delta} = -1$ and $| \psi_{trans} >= - | \psi >$.

That is, the states of the system are either all symmetrical or all antisymmetrical, which is Consequence 2.

Comments about Consequence 2

1. **Classical Mechanics**

   Consequence 2 has no analog in Classical Mechanics.

2. **Bosons and fermions**

   Particles whose many-particle states are symmetric are called bosons.

   Particles whose many-particle states are antisymmetric are called fermions.

3. **Statistics and spin**

   Experiment shows that

Bosons have integral spin.
Fermions have half-odd integral spin.

   The correspondence between the symmetry of many-particle states and the intrinsic spin of the constituent particles is a remarkable experimental fact.

4. **Spin-Statistics Theorem**

   The above correspondence has been proven in the context of Relativistic Quantum Field Theory (Spin-Statistics Theorem; Pauli (1940)).

5. Boson and fermion Hilbert spaces

Consequence 2 restricts the Hilbert space of the system.

The n-boson Hilbert space ${}^{b}\mathfrak{H}_n^s$ is the subspace of $\mathfrak{H}_n^s$ spanned by symmetric basis vectors.

The n-fermion Hilbert space ${}^{f}\mathfrak{H}_n^s$ is the subspace of $\mathfrak{H}_n^s$ spanned by antisymmetric basis vectors.

We write symbolically

$$\boxed{\begin{aligned} {}^{b}\mathfrak{H}_n^s &= \mathbb{S}\,\mathfrak{H}_n^s = \mathbb{S} \otimes^n \mathfrak{H}_1^s && (2.28) \\ {}^{f}\mathfrak{H}_n^s &= \mathbb{A}\,\mathfrak{H}_n^s = \mathbb{A} \otimes^n \mathfrak{H}_1^s && (2.29) \end{aligned}}$$

Chapter 3 OCCUPATION NUMBER REPRESENTATION

In this chapter we construct basis vectors for a system of identical fermions and for a system of identical bosons. We need not specify details of the fermion or boson system under consideration.

The fermion system may, for example, be:

- electrons bound to a single atom
- conduction electrons in a metal
- nucleons in an atomic nucleus
- quarks in a nucleon

Whatever the system, each fermion has half-odd integral spin and all states of the system are antisymmetric under interchange of any two particles.

The boson system may, for example, be:

- photons characterizing an electromagnetic field
- phonons characterizing the lattice vibrations of a crystal
- pions or kaons created in collisions of nuclear projectiles
- gluons in nuclear matter

Whatever the system, each boson has integral spin and all states of the system are symmetric under under interchange of any two particles.

In view of the symmetry requirement on the states of the system, the basis vectors we construct will be labelled by specifying which single-particle states are occupied. We thus construct the occupation number representation for fermion and boson systems.

Sections 3.1, 3.2 and 3.3 give, respectively, basis vectors for a system of n identical fermions, n identical bosons and n identical fermions with n' identical bosons.

3.1 System of identical fermions

In this section we construct a set of basis vectors for a system of n identical fermions.

Basis vectors for the one-fermion system

Let

$$\mid \phi_r \underset{\alpha}{>} \tag{3.1}$$

where $r = 1, 2, \cdots, \infty$ be a complete orthonormal set of vectors spanning the Hilbert space $\maltese_1^s$ for fermion α. That is,

$$\sum_{r=1}^{\infty} \mid \phi_r \underset{\alpha}{>}\underset{\alpha}{<} \phi_r \mid = 1_\alpha \tag{3.2}$$

$$\underset{\alpha}{<} \phi_r \mid \phi_s \underset{\alpha}{>} = \delta_{rs} \tag{3.3}$$

where 1_α is the unit operator in the one-particle space for fermion number α.

Comments

1. **Notation: subscript α**

The subscript α on $| \phi_r \underset{\alpha}{>}$ and 1_α has been added to serve as a reminder that the vectors and operators are in the Hilbert space $\mathfrak{H}_1^s$ for fermion number α.

2. **Denumerable set of basis vectors**

It is convenient to use a denumerable set of vectors to span the one-fermion Hilbert space.

We use coordinate/spin kets $| \vec{x} m_s >$, momentum/spin kets $| \vec{p} m_s >$ and momentum/helicity kets $| h^\lambda(\vec{p}) >$ in Chapter 5 to span the one-fermion Hilbert space. These kets, which are also are discussed in *QLB: Some Lorentz Invariant Systems*, are each labelled by a continuous variable.

3. **Example of basis vectors**

The $| \phi_r >$ may, for example, be chosen to be the simultaneous eigenvectors of the Hamiltonian for a three-dimensional nonrelativistic harmonic oscillator and of $\left(\vec{J}\right)^2$ and J^3 where $\vec{J} = \vec{X} \times \vec{P} + \vec{S}$ where $\vec{X}$, $\vec{P}$ and $\vec{S}$ are the Cartesian position, momentum and spin of the particle.[1]

The eigenvalue of $\left(\vec{S}\right)^2$ is $s(s+1)\hbar^2$ where s is a half-odd integer. s is the intrinsic spin of the elementary fermion.

4. **General one-fermion state**

The general one-fermion state at time t is

---

[1] The $| \phi_r >$ may also be simultaneous eigenvectors of the internal variables charge, baryon number, lepton number, isospin, strangeness and charm. We need not specify these variables here.

$$| \psi(t) \underset{\alpha}{>} = \sum_{r=1}^{\infty} \psi_r(t) \mid \phi_r \underset{\alpha}{>} \tag{3.4}$$

and

$$\psi_r(t) = \underset{\alpha}{<} \phi_r \mid \psi(t) \underset{\alpha}{>} \tag{3.5}$$

is the probability amplitude that the fermion is in the state $\mid \phi_r \underset{\alpha}{>}$ at time t.

Basis vectors for the n-fermion system

The n-particle Hilbert space $\maltese_n^s$ is a tensor product (2.22) of n identical spaces. When s is a half-odd integer, it is spanned by vectors of the form

$$\mid \phi_r \underset{1}{>} \mid \phi_s \underset{2}{>} \cdots \mid \phi_t \underset{n}{>} \tag{3.6}$$

where particle 1 is in single-particle state $\mid \phi_r >$, particle 2 is in single-particle state $\mid \phi_s >$ and particle n is in single-particle state $\mid \phi_t >$.

The n-fermion Hilbert space ${}^f\maltese_n^s$ (2.29) is spanned by antisymmetric combinations of vectors of the form (3.6). That is, ${}^f\maltese_n^s$ is spanned by vectors of the form

$$
\mid n\{n_1 n_2 \cdots\} \underset{n}{>} = \left(\frac{1}{n!}\right)^{\frac{1}{2}} \det \begin{vmatrix} \mid \phi_r \underset{1}{>} & \mid \phi_r \underset{2}{>} & \cdots & \mid \phi_r \underset{n}{>} \\ \mid \phi_s \underset{1}{>} & \mid \phi_s \underset{2}{>} & \cdots & \mid \phi_s \underset{n}{>} \\ \vdots & \vdots & \vdots & \vdots \\ \mid \phi_t \underset{1}{>} & \mid \phi_t \underset{2}{>} & \cdots & \mid \phi_t \underset{n}{>} \end{vmatrix} \tag{3.7}
$$

where

$$
r < s < \cdots < t \tag{3.8}
$$

and where det denotes determinant.

Comments

1. **Notation: subscript n**

   The subscript n on $\mid n\{n_1 n_2 \cdots\} \underset{n}{>}$ in (3.7) has been added to serve as a reminder that the vector is in ${}^f\mathfrak{H}^s_n$.

2. **Slater determinant**

   (3.7) is a Slater determinant.

3. **Manifest antisymmetry**

   (3.7) is manifestly antisymmetric under particle interchange because the value of a determinant changes sign when any two columns are interchanged.

4. **Occupied states**

The set of single-particle labels $r, s, \cdots, t$ tells which single-particle states are occupied.

Because of the antisymmetrizing, one cannot specify which particle occupies which state.

5. **Pauli Exclusion Principle**

Since a determinant vanishes if all elements of one row are equal to all elements of another row, that is, if two or more of $r, s, \cdots, t$ are equal, it follows that no two fermions can occupy the same single-particle state.

This is the Pauli Exclusion Principle.

6. **Occupation numbers**

Let n_r be the number of particles occupying the single-particle state $\mid \phi_r >$. Then

$$\boxed{n_r = 0 \text{ or } 1 \qquad (3.9)}$$

and

$$\boxed{\sum_{r=1}^{\infty} n_r = n \qquad (3.10)}$$

n_r is the occupation number for the single-particle state $\mid \phi_r >$.

(3.7) is labelled by the occupation numbers $n_1, n_2, \cdots$.

7. Basis for the n-fermion Hilbert space

The set of vectors (3.7) is an orthonormal basis for ${}^f\mathfrak{H}_n^s$. That is,[1]

$$\sum_{n_1 n_2 \cdots}^{f} \mid n\{n_1 n_2 \cdots\} \underset{n}{>} \underset{n}{<} n\{n_1 n_2 \cdots\} \mid = 1_n \tag{3.11}$$

where

$$\sum_{n_1 n_2 \cdots}^{f} = \sum_{n_1=0}^{1} \sum_{n_2=0}^{1} \cdots \delta_{nn'} \tag{3.12}$$

where

$$n' = \sum_{j=1}^{\infty} n_j \tag{3.13}$$

Furthermore,

$$\underset{n}{<} n\{n_1 n_2 \cdots\} \mid n\{n_1' n_2' \cdots\} \underset{n}{>} = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \cdots \tag{3.14}$$

1 The subscript n on 1_n has be added to serve as a reminder that the unit operator is in ${}^f\mathfrak{H}_n^s$.

The representation provided by the set of vectors (3.7) is called the occupation number representation for the n-fermion system.

8. **General n-fermion state**

The general n-fermion state has the form[2]

$$\boxed{\mid \psi(t) \underset{n}{>} = \sum_{n_1 n_2 \cdots}^{f} \mid n\{n_1 n_2 \cdots\} \underset{n}{>} \underset{n}{<} n\{n_1 n_2 \cdots\} \mid \psi(t) \underset{n}{>} \qquad (3.15)}$$

and

$$\boxed{\underset{n}{<} n\{n_1 n_2 \cdots\} \mid \psi(t) \underset{n}{>} \qquad (3.16)}$$

is the probability amplitude at time t that n_1 fermions occupy the single-particle state $\mid \phi_1 >$, and n_2 fermions occupy the single-particle state $\mid \phi_2 >$, and so on.

3.2 System of identical bosons

In this section we construct a set of basis vectors for a system of n identical bosons. The boson system is treated analogously to the treatment of the fermion system in Section 3.1.

2 The subscript n on $\mid \psi(t) \underset{n}{>}$ has be added to serve as a reminder that the vector is in ${}^f\mathfrak{H}^s_n$.

Basis vectors for the one-boson system

Let

$$
\boxed{\mid \beta_r \underset{\alpha}{>} \qquad (3.17)}
$$

where $r = 1, 2, \cdots$ be a complete orthonormal set of vectors spanning the Hilbert space $\mathcal{H}_1^s$ for boson α. That is,

$$
\sum_{r=1}^{\infty} \mid \beta_r \underset{\alpha}{>} \underset{\alpha}{<} \beta_r \mid = 1_\alpha \qquad (3.18)
$$

$$
\underset{\alpha}{<} \beta_r \mid \beta_s \underset{\alpha}{>} = \delta_{rs} \qquad (3.19)
$$

where 1_α is the unit operator in the one-particle space for boson number α.

Comments

1. **Example of basis vectors**

The $\mid \beta_r >$ may, for example, be chosen to be the simultaneous eigenvectors of the Hamiltonian for a three-dimensional nonrelativistic harmonic oscillator and of $\left(\vec{J}\right)^2$ and J^3 where the eigenvalue of $\left(\vec{S}\right)^2$ is $s(s+1)\hbar^2$ where s is an integer. s is the intrinsic spin of the elementary boson.[1]

[1] The $\mid \beta_r >$ may also be simultaneous eigenvectors of the internal variables charge, baryon number, lepton number, isospin, strangeness and charm. As in the fermion case in Section 3.1, we need not specify these variables here.

2. **General one-boson state**

The general one-boson state at time t is

$$\mid \psi(t) \underset{\alpha}{>} = \sum_{r=1}^{\infty} \psi_r(t) \mid \beta_r \underset{\alpha}{>} \tag{3.20}$$

where

$$\psi_r(t) = \underset{\alpha}{<} \beta_r \mid \psi(t) \underset{\alpha}{>} \tag{3.21}$$

is the probability amplitude that the boson is in the state $\mid \beta_r \underset{\alpha}{>}$ at time t.

Basis vectors for the n-boson system

The n-particle Hilbert space $\maltese_n^s$ is a tensor product (2.22) of n identical spaces. When s is an integer, it is spanned by vectors of the form

$$\mid \beta_r \underset{1}{>} \mid \beta_s \underset{2}{>} \cdots \mid \beta_t \underset{n}{>} \tag{3.22}$$

where particle 1 is in single-particle state $\mid \beta_r >$, particle 2 is in single-particle state $\mid \beta_s >$ and particle n is in single-particle state $\mid \beta_t >$.

The n-boson Hilbert space ${}^b\maltese_n^s$ (2.28) is spanned by symmetric combinations of vectors of the form (3.22). That is, ${}^b\maltese_n^s$ is spanned by vectors of the form

$$
| n[n_1 n_2 \cdots] \underset{n}{>} = \left(\frac{n_1! n_2! \cdots}{n!} \right)^{\frac{1}{2}} sym \det \begin{vmatrix} | \beta_r \underset{1}{>} & | \beta_r \underset{2}{>} & \cdots & | \beta_r \underset{n}{>} \\ | \beta_s \underset{1}{>} & | \beta_s \underset{2}{>} & \cdots & | \beta_s \underset{n}{>} \\ \vdots & \vdots & \vdots & \vdots \\ | \beta_t \underset{1}{>} & | \beta_t \underset{2}{>} & \cdots & | \beta_t \underset{n}{>} \end{vmatrix} \tag{3.23}
$$

where

$$
r \leq s \leq \cdots \leq t \tag{3.24}
$$

and where $sym \det$ denotes a determinant which has plus signs in its definition rather than minus signs.

Comments

1. **Notation**

   We use a slight difference in notation to denote n-fermion and n-boson basis vectors:

   The left side of (3.7) has $\{\cdots\}$ and the left side of (3.23) has $[\cdots]$.

   This notation anticipates anticommutators for a fermion system in Chapters 4 and 5 and commutators for a boson system in Chapter 6.

2. **Manifest symmetry**

   (3.23) is manifestly symmetric under particle interchange.

3. **Occupied states**

The set of single-particle labels $r, s, \cdots, t$ tells which single-particle states are occupied.

Because of the symmetrizing, one cannot specify which particle occupies which state.

4. **No Pauli Exclusion Principle**

There is no Pauli Exclusion Principle for a system of identical bosons because sym det does not vanish if all elements of one row are equal to all elements of another row, that is, if two or more of $r, s, \cdots, t$ are equal.

5. **Occupation numbers**

Let n_r be the number of particles occupying the single-particle state $| \beta_r >$. Then

$$\boxed{0 \leq n_r \leq n \qquad (3.25)}$$

and

$$\boxed{\sum_{r=1}^{\infty} n_r = n \qquad (3.26)}$$

n_r is the occupation number for the single-particle state $| \beta_r >$.

(3.23) is labelled by the occupation numbers $n_1, n_2, \cdots$.

In contradistinction to the fermion case, all bosons may occupy one single-particle state.

6. Basis for the n-boson Hilbert space

The set of vectors (3.23) is orthonormal and spans ${}^b\mathcal{H}^s_n$. That is,

$$\sum_{n_1 n_2 \cdots}^{b} \mid n[n_1 n_2 \cdots] \underset{n}{>} \underset{n}{<} n[n_1 n_2 \cdots] \mid = 1_n \tag{3.27}$$

where

$$\sum_{n_1 n_2 \cdots}^{b} = \sum_{n_1=0}^{n} \sum_{n_2=0}^{n} \cdots \; \delta_{nn'} \tag{3.28}$$

where

$$n' = \sum_{j=1}^{\infty} n_j \tag{3.29}$$

Furthermore,

$$\underset{n}{<} n[n_1 n_2 \cdots] \mid n[n_1 n_2 \cdots] \underset{n}{>} = \delta_{n_1 n_1'} \delta_{n_2 n_2'} \cdots \tag{3.30}$$

The representation provided by the set of vectors (3.23) is called the occupation number representation for the n-boson system.

7. **General n-boson state**

The general n-boson state has the form

$$\mid \psi(t) \underset{n}{>} = \sum_{n_1 n_2 \cdots}^{b} \mid n[n_1 n_2 \cdots] \underset{n}{>}\underset{n}{<} n[n_1 n_2 \cdots] \mid \psi(t) \underset{n}{>} \tag{3.31}$$

where

$$\underset{n}{<} n[n_1 n_2 \cdots] \mid \psi(t) \underset{n}{>} \tag{3.32}$$

is the probability amplitude at time t that n_1 bosons occupy the single-particle state $\mid \beta_1 >$, and n_2 bosons occupy the single-particle state $\mid \beta_2 >$, and so on.

3.3 System of identical fermions and bosons

The Hilbert space

$$^{fb}\mathfrak{H}^{ss'}_{nn'} \tag{3.33}$$

for a system of n identical fermions each with spin s and n' identical bosons each with spin s' is the direct product of the n-fermion and n'-boson Hilbert spaces. That is,

$$ {}^{fb}\maltese^{ss'}_{nn'} = {}^{f}\maltese^{s}_{n} \otimes {}^{b}\maltese^{s'}_{n'} \tag{3.34} $$

${}^{fb}\maltese^{ss'}_{nn'}$ is spanned by vectors of the form

$$ \mid n\{n_1 n_2 \cdots\} \underset{n}{>} \mid n'\left[n'_1 n'_2 \cdots\right] \underset{n'}{>} \tag{3.35} $$

where $\mid n\{n_1 n_2 \cdots\} \underset{n}{>}$ is the Slater determinant (3.7) and $\mid n'[n'_1 n'_2 \cdots] \underset{n'}{>}$ is the symmetric determinant (3.23).

Chapter 4 FOCK SPACE FOR FERMIONS: PART 1

So far in *QLB: Quantum Mechanics in Fock Space* we have considered the total number of particles in a system to be fixed. We now drop this restriction and consider a larger system with an unspecified number of particles. This is handled mathematically by considering the Hilbert space for the system to be Fock space.

Fock space is the natural mathematical arena for accommodating particle creation and annihilation, that is, for allowing the conversion of energy to mass and *vice versa* which is allowed by relativistic quantum mechanics.

It is not necessary to use Fock space to describe a fermion system because it is an experimental fact that every fermion system has a definite number of fermions. We will see, however, that using Fock space allows an elegant and intuitive reformulation of the quantum mechanics of an n-fermion system. This reformulation (second quantization) is the standard language of nonrelativistic condensed matter physics and low-energy nuclear physics. Fock space for a system of fermions is the subject of this and the next chapter.

Fock space for fermions is defined in Section 4.1 and creation and annihilation operators for fermions are defined in Section 4.2. These operators are defined in terms of a denumerable set of vectors which form an orthonormal basis for the one-fermion system. Creation and annihilation operators which are labelled by a continuous variable are defined in Chapter 5.

General expressions for observables in terms of creation and annihilation operators are given in Sections 4.3 and 4.4.

The Fock space expression for the Hamiltonian for a system of fermions with two-body interactions is given in Section 4.5. The independent-particle model and the Hartree-Fock potential are discussed in Section 4.6.

Hole creation and annihilation operators and particle-hole states of an n-fermion system are discussed in Section 4.7. Correlated particle-hole pairs and a correlated-pair model of an n-fermion system are discussed in Section 4.8.

Derivations of some results are given in Section 4.9.

4.1 Fermion Fock space defined

1. Let

$$\psi = (\psi_0, \psi_1, \cdots, \psi_n, \cdots) \tag{4.1}$$

where ψ_n is a vector in n-fermion Hilbert space ${}^f\mathcal{H}^s_n$ (2.29).[1]

Each ψ_n is of the form (3.15). ψ_n is the component of ψ in ${}^f\mathcal{H}^s_n$.

2. Addition of ψ and $\chi = (\chi_0, \chi_1, \cdots, \chi_n, \cdots)$ is defined as

$$\psi + \chi = (\psi_0 + \chi_0, \psi_1 + \chi_1, \cdots, \psi_n + \chi_n, \cdots) \tag{4.2}$$

3. Multiplication of ψ by a scalar a is defined as

$$a\psi = (a\psi_0, a\psi_1, \cdots, a\psi_n, \cdots) \tag{4.3}$$

4. The scalar product of ψ and χ is defined as

$$(\psi, \chi) = \sum_{n=0}^{\infty} (\psi_n, \chi_n) \tag{4.4}$$

It is required that $(\psi, \psi) < \infty$ for all ψ.

5. The set of elements ψ is a separable Hilbert space.

---

[1] ${}^f\mathcal{H}^s_0$ is defined in item 5 of the Comments list.

Comments

1. **Fermion Fock space ${}^f\mathfrak{H}^s$**

The above Hilbert space is called fermion Fock space. It is denoted by ${}^f\mathfrak{H}^s$.

${}^f\mathfrak{H}^s$ is the direct sum of the Hilbert spaces ${}^f\mathfrak{H}^s_n$ (2.29) for all n. That is,

$$\boxed{{}^f\mathfrak{H}^s = {}^f\mathfrak{H}^s_0 \oplus {}^f\mathfrak{H}^s_1 \oplus \cdots \oplus {}^f\mathfrak{H}^s_n \oplus \cdots} \tag{4.5}$$

where $\oplus$ denotes direct sum.

2. **States of the system**

The unit norm vectors (4.1) in ${}^f\mathfrak{H}^s$ correspond to states of the system.

The probability P_n that the system has n fermions in it is

$$\boxed{P_n =< \psi_n \mid \psi_n >} \tag{4.6}$$

and

$$\boxed{\sum_{n=0}^{\infty} P_n = 1} \tag{4.7}$$

3. **Components of ψ**

It is an experimental fact that every fermion system has a definite number of fermions. Accordingly, ψ can have only one nonzero component. For example,

$$\boxed{\psi = (0, 0, \cdots, 0, \psi_n, 0, \cdots) \qquad (4.8)}$$

Boson Fock space ${}^b\maltese^s$ is constructed in Chapter 6 analogously to the construction of ${}^f\maltese^s$. States in ${}^b\maltese^s$ can have more than one nonzero component.

4. **Hilbert space $\maltese_{2\leftrightarrow 3}$**

The Hilbert space $\maltese_{2\leftrightarrow 3}$ defined in *QLB: Some Lorentz Invariant Systems* Chapter 6 to describe the $2 \leftrightarrow 3$ particle system is the analog of Fock space for that system.

5. **Hilbert space ${}^f\maltese^s_0$**

${}^f\maltese^s_0$ is defined to be a one-dimensional space.

The unit norm vector spanning ${}^f\maltese^s_0$ is labelled $| \, 0\{00\cdots\} \underset{0}{>}$.

6. **Basis vectors for ${}^f\maltese^s$**

A basis for ${}^f\maltese^s$ is the set of vectors

$$\boxed{| \, n\{n_1 n_2 \cdots\} > \qquad (4.9)}$$

defined by

$$\boxed{\begin{aligned} &| 0\{00\cdots\} >= \left(| 0\{00\cdots\} \underset{0}{>}, 0, \cdots \right) && (4.10) \\ &| 1\{n_1 n_2\cdots\} >= \left(0, | 1\{n_1 n_2\cdots\} \underset{1}{>}, 0, \cdots \right) && (4.11) \\ &| 2\{n_1 n_2\cdots\} >= \left(0, 0, | 2\{n_1 n_2\cdots\} \underset{2}{>}, 0, \cdots \right) && (4.12) \\ &\vdots \end{aligned}}$$

where

$$\boxed{| n\{n_1 n_2 \cdots\} \underset{n}{>} \qquad (4.14)}$$

for all $n = 1, 2, \cdots$ is the Slater determinant (3.7).

Then

$$\boxed{\sum_{n n_1 n_2 \cdots}^{f} | n\{n_1 n_2 \cdots\} >< n\{n_1 n_2 \cdots\} |= 1 \qquad (4.15)}$$

where

$$\sum_{nn_1n_2\cdots}^{f} = \sum_{n=0}^{\infty} \sum_{n_1n_2\cdots}^{f} \tag{4.16}$$

where the second summation on the right side is defined by (3.12).

Furthermore,

$$< n\{n_1n_2\cdots\} \mid n'\{n_1'n_2'\cdots\} >= \delta_{nn'}\delta_{n_1n_1'}\delta_{n_2n_2'}\cdots \tag{4.17}$$

7. **Vacuum state**

(4.10) is the vacuum state of the system. It will be denoted by $\mid 0 >$. That is,

$$\mid 0 >=\mid 0\{00\cdots\} > \tag{4.18}$$

8. **General state of the system**

The general state of the system has the form

$$\mid \psi(t) >= \sum_{nn_1n_2\cdots}^{f} \mid n\{n_1n_2\cdots\} >< n\{n_1n_2\cdots\} \mid \psi(t) > \tag{4.19}$$

and

$$< n\{n_1 n_2 \cdots\} \mid \psi(t) > \tag{4.20}$$

is the probability amplitude that at time t there are n fermions in the system with n_1 fermions occupying the single-particle state $\mid \phi_1 >$, and n_2 fermions occupying the single-particle state $\mid \phi_2 >$, and so on.

4.2 Creators and annihilators

We define fermion creation and annihilation operators in this section. These operators are fundamental dynamical variables for a system of identical fermions. They obey anticommutation relations.

Introduction of creation and annihilation operators yields intuitive and elegant expressions for observables and basis states.

Matrix elements of observables are expressed as vacuum expectation values of products of creation and annihilation operators. These vacuum expectation values are calculated through a simple stategy.

Definitions

For each $r = 1, 2, \cdots$, we define

$$F_r^\dagger = \sum_{n n_1 n_2 \cdots}^{f} \mid n+1\{n_1 n_2 \cdots n_r + 1 \cdots\} > (-)^{m_r}(1 - n_r) < n\{n_1 n_2 \cdots n_r \cdots\} \mid \tag{4.21}$$

from which

$$
\boxed{F_r = \sum_{nn_1n_2\cdots}^{f} \mid n\{n_1n_2\cdots n_r-1\cdots\} > (-)^{m_r} n_r < n+1\{n_1n_2\cdots n_r\cdots\} \mid \qquad (4.22)}
$$

where

$$
\boxed{m_r = \sum_{s=1}^{r-1} n_s \qquad (4.23)}
$$

and where $\mid n\{n_1n_2\cdots\} >$ is the basis vector (4.9) in ${}^f\text{✠}^s$.

It follows from (4.21) and (4.22) that[1]

$$
F_r^\dagger \mid n\{n_1n_2\cdots 0\cdots\} >= (-)^{m_r} \mid n+1\{n_1n_2\cdots 1\cdots\} > \qquad (4.24)
$$

$$
F_r \mid n+1\{n_1n_2\cdots 1\cdots\} >= (-)^{m_r} \mid n\{n_1n_2\cdots 0\cdots\} > \qquad (4.25)
$$

[1] The 0 and 1 in the basis states in (4.24) to (4.27) occur in the rth place.

and

$$F_r^\dagger \mid n\{n_1 n_2 \cdots 1 \cdots\} >= 0 \tag{4.26}$$

$$F_r \mid n\{n_1 n_2 \cdots 0 \cdots\} >= 0 \tag{4.27}$$

$$F_r^\dagger \mid 0 >= (0, \mid \phi_r >, 0, \cdots) \tag{4.28}$$

$$F_r \mid 0 >= 0 \tag{4.29}$$

Comments

1. **Fermion creator**

   $F_r^\dagger$ is a fermion creation operator or fermion creator.

   When acting on an n-fermion basis vector (4.9) with $n_r = 0$, $F_r^\dagger$ yields an $n+1$-fermion basis vector with $n_r = 1$.

2. **Fermion annihilator**

   F_r is a fermion annihilation operator or fermion annihilator.

   When acting on an $n+1$-fermion basis vector (4.9) with $n_r = 1$, F_r yields an n-fermion basis vector with $n_r = 0$.

3. **Value of m_r**

   m_r is the number of occupied single-particle states in $\mid n\{n_1 n_2 \cdots n_r \cdots\} >$ up to single-particle state number r.

4. **Pauli Exclusion Principle**

(4.26) is the Pauli Exclusion Principle: a fermion cannot be put in an occupied single-particle state; no state can have occupation number greater than one.

5. **Creating an elementary fermion**

When acting on the vacuum state, $F_r^\dagger$ creates an elementary fermion with rest mass m and spin s in single-particle state $\mid \phi_r >$.

6. **One-fermion state**

The general one-fermion state at time t is

$$\boxed{\mid \psi(t) >= \sum_{r=1}^{\infty} \psi_r(t) F_r^\dagger \mid 0 > \qquad (4.30)}$$

where

$$\boxed{\psi_r(t) =< 0 \mid F_r \psi(t) > \qquad (4.31)}$$

is the probability amplitude that the fermion is in the state $\mid \phi_r >$ at time t.

Anticommutation relations

We show in Section 4.9 that

$$\{F_r, F_s\} = 0 \tag{4.32}$$

$$\left\{F_r, F_s^\dagger\right\} = \delta_{rs} \tag{4.33}$$

where $\{A, B\}$ is the anticommutator of A and B.[1]

Comment

1. **Pauli Exclusion Principle**

It follows from (4.32) that

$$\left(F_r^\dagger\right)^2 = 0 \tag{4.34}$$

(4.34) is the operator form of the Pauli Exclusion Principle.

Basis vectors

We show in Section 4.9 that the basis vector (4.9) may be expressed as n creators acting on the vacuum state.

[1] Some commutators of products of fermion creators and annihilators are given in the Appendix.

In particular, if (4.9) corresponds to the n single-particle levels $r, s, \cdots, t$ occupied with $r < s < \cdots < t$, then

$$\boxed{\mid n\{n_1 n_2 \cdots\} >= F_r^\dagger F_s^\dagger \cdots F_t^\dagger \mid 0 > \qquad (4.35)}$$

Comments

1. **Form of the basis vector**

   (4.35) is a compact, intuitive and elegant expression for the basis vector (4.9).

2. **Manifest antisymmetry**

   It follows from (4.32) that (4.35) changes sign when any two of $r, s, \cdots, t$ are interchanged.

   (4.35) is manifestly antisymmetric under particle interchange.

3. **Fundamental dynamical variables**

   Each basis vector (4.9) can be expressed as fermion creators acting on the vacuum state. The set of creators and annihilators defined by (4.21) and (4.22) is a set of fundamental dynamical variables for a system of identical fermions.

   Anticommutation relations (4.32) and (4.33) are a fundamental algebra for the system.

Evaluating vacuum expectation values

Evaluation of matrix elements of observables involves evaluating matrix elements of combinations of products of fermion creators and annihilators between states of the form (4.35).

That is, it involves calculating the average value in the vacuum state of products of fermion creators and annihilators. These vacuum expectation values may be evaluated using the following stategy:

> Express products of creators and annihilators in normal order using (4.32) and (4.33) and (C.1) to (C.10), then use (4.29).

Example

The norm of the vector $F_r^\dagger F_s^\dagger \mid 0>$ is

$$< F_r^\dagger F_s^\dagger 0 \mid F_r^\dagger F_s^\dagger \mid 0 >=< 0 \mid F_s F_r F_r^\dagger F_s^\dagger \mid 0 >$$

$$=< 0 \mid \left(-\left[F_r^\dagger F_s^\dagger, F_s F_r\right] + F_r^\dagger F_s^\dagger F_s F_r\right) \mid 0 >= (\delta_{rr}\delta_{ss} - \delta_{rs}\delta_{rs}) < 0 \mid 0 >$$

$$= 1 - \delta_{rs} \tag{4.36}$$

4.3 Number operators

We define

$$\boxed{N_r = F_r^\dagger F_r \qquad (4.37)}$$

for each $r = 1, 2, \cdots$, and

$$\boxed{N = \sum_{r=1}^{\infty} N_r \qquad (4.38)}$$

It follows that

$$N_r^\dagger = N_r \qquad (4.39)$$

$$(N_r)^2 = N_r \qquad (4.40)$$

$$[N_r, N_s] = 0 \qquad (4.41)$$

for all $r, s = 1, 2, \cdots$. Moreover,

$$N_r = \sum_{nn_1n_2\cdots}^{f} | \, n\{n_1n_2\cdots\} > n_r < n\{n_1n_2\cdots\} \, | \qquad (4.42)$$

$$N = \sum_{nn_1n_2\cdots}^{f} | \, n\{n_1n_2\cdots\} > n < n\{n_1n_2\cdots\} \, | \qquad (4.43)$$

Comments

1. **Compatible observables**

It follows from (4.39) and (4.40) that N_r is an observable with eigenvalues 0 and 1.

It follows from (4.41) that the N_r are an infinite set of compatible observables.

2. **Eigenvalue decomposition**

(4.42) and (4.43) are the eigenvalue decompositions of N_r and N.

The basis vector (4.9) is a simultaneous eigenvector of $N, N_1, N_2, \cdots$ belonging to eigenvalues $n, n_1, n_2, \cdots$.

$N, N_1, N_2, \cdots$ are a complete set of compatible observables.

3. **Nomenclature**

When operating on the basis vector (4.35), N_r gives zero if the single-particle state $\mid \phi_r >$ is unoccupied and one if it is occupied.

N_r is the number operator for the single-particle state $\mid \phi_r >$.

N is the number operator for the system.

4.4 Observables

An observable on ${}^f\maltese_n^s$ is a Hermitian operator[2]

$$A_n(\xi_1, \xi_2, \cdots, \xi_n) \tag{4.44}$$

[2] The subscript n on A_n has been added to serve as a reminder that the observable is on ${}^f\maltese_n^s$.

where ξ_α denotes the fundamental dynamical variables (2.13) for particle α. (4.44) is invariant under $\xi_\alpha \leftrightarrow \xi_\beta$ for all α and β.

In this section we construct operators on fermion Fock space ${}^f\maltese^s$ which are equal to (4.44) for all n. Of particular interest are one-particle operators and two-particle operators on ${}^f\maltese^s_n$.

One-particle operators

We show in Section 4.9 that the operator

$$\boxed{A = \sum_{r,s=1}^{\infty} < r \mid A \mid s > F_r^\dagger F_s} \tag{4.45}$$

on ${}^f\maltese^s$ where

$$\boxed{< r \mid A \mid s > = < \phi_r \mid A(\xi) \mid \phi_s >} \tag{4.46}$$

is equal to

$$\boxed{A_n(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{\alpha=1}^{n} A(\xi_\alpha)} \tag{4.47}$$

on ${}^f\maltese^s_n$ for every $n = 1, 2, \cdots$.

Comments

1. **One-particle operator**

(4.47) is a one-particle operator on ${}^f\maltese^s_n$.

Matrix element (4.46) is a one-particle matrix element.

2. **Total kinetic energy on ${}^f\maltese^s_n$**

The total kinetic energy on ${}^f\maltese^s_n$ of a system of n fermions each with rest mass m is

$$\boxed{T_n(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{\alpha=1}^{n} T(\xi_\alpha) \qquad (4.48)}$$

where

$$\boxed{T(\xi_\alpha) = \frac{P_\alpha^2}{2m} \qquad (4.49)}$$

$T_n(\xi_1, \xi_2, \cdots, \xi_n)$ is a one-particle operator on ${}^f\maltese^s_n$.

3. **Total kinetic energy on ${}^f\maltese^s$**

It follows from (4.45) that the operator on ${}^f\maltese^s$ which is equal to (4.48) on ${}^f\maltese^s_n$ for every $n = 1, 2, \cdots$ is

$$T = \sum_{r,u=1}^{\infty} < r \mid T \mid u > F_r^\dagger F_u \tag{4.50}$$

where

$$< r \mid T \mid u > = < \phi_r \mid \frac{p^2}{2m} \mid \phi_u >$$

$$= \sum_{m_s=-s}^{+s} \int d^3p \phi^*_{rm_s}(\vec{p}) \frac{p^2}{2m} \phi_{um_s}(\vec{p}) \tag{4.51}$$

$$= -\frac{\hbar^2}{2m} \sum_{m_s=-s}^{+s} \int d^3x \phi^*_{rm_s}(\vec{x}) \nabla^2 \phi_{um_s}(\vec{x})$$

where $\phi_{rm_s}(\vec{p})$ and $\phi_{rm_s}(\vec{x})$ are, respectively, the momentum-space/spin and coordinate-space/spin representatives of the single-particle vector $\mid \phi_r >$. That is,

$$\phi_{rm_s}(\vec{p}) = < \vec{p} m_s \mid \phi_r > \tag{4.52}$$

$$\phi_{rm_s}(\vec{x}) = < \vec{x} m_s \mid \phi_r > \tag{4.53}$$

where $\mid \vec{p} m_s >$ and $\mid \vec{x} m_s >$ are the one-fermion simultaneous eigenket of z-component of spin and, respectively, momentum and position.

Two-particle operators

It can be proved similarly to the proof of (4.45) given in Section 4.9 that the operator

$$A = \sum_{r,s,t,u=1}^{\infty} < rs \mid A \mid ut > F_r^{\dagger} F_s^{\dagger} F_t F_u \tag{4.54}$$

on ${}^{f}\mathfrak{H}^{s}$ where

$$< rs \mid A \mid ut > = \underset{\alpha}{<} \phi_r \mid \underset{\beta}{<} \phi_s \mid A(\xi_\alpha, \xi_\beta) \mid \phi_u \underset{\alpha}{>} \mid \phi_t \underset{\beta}{>} \tag{4.55}$$

is equal to[1]

$$A_n(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{\alpha,\beta=1}^{n} A(\xi_\alpha, \xi_\beta) \tag{4.56}$$

on ${}^{f}\mathfrak{H}_n^{s}$ where

$$A(\xi_\alpha, \xi_\beta) = A(\xi_\beta, \xi_\alpha) \tag{4.57}$$

for every $n = 2, 3, \cdots$.

[1] The double summation in (4.56) is restricted to $\alpha \neq \beta$.

Comments

1. **No misprint in (4.54)**

   (4.54) is not misprinted: the subscripts on the creators and annihilators are in alphabetic order whereas the labels in the matrix element (4.55) are not.

2. **Two-particle operator**

   (4.56) is a two-particle operator on $^{f}\mathfrak{F}_n^s$.

   (4.55) is a two-particle matrix element. It follows from (4.57) that

$$< rs \mid A \mid ut >=< sr \mid A \mid tu > \tag{4.58}$$

3. **Total potential energy on $^{f}\mathfrak{F}_n^s$**

   The total potential energy on $^{f}\mathfrak{F}_n^s$ of a system of n identical fermions interacting via two-body potentials[2] is

$$V_n(\xi_1, \xi_2, \cdots, \xi_n) = \frac{1}{2} \sum_{\alpha,\beta=1}^{n} V(\xi_\alpha, \xi_\beta) \tag{4.59}$$

   where

$$V(\xi_\alpha, \xi_\beta) = V(\xi_\beta, \xi_\alpha) \tag{4.60}$$

[2] The double summation in (4.59) is restricted to $\alpha \neq \beta$.

$V_n(\xi_1, \xi_2, \cdots, \xi_n)$ is a two-particle operator on ${}^f\maltese^s_n$.

When the two-body potential is a function only of the position of the particles,

$$V(\xi_\alpha, \xi_\beta) = V\left(\vec{X}_\alpha, \vec{X}_\beta\right) \tag{4.61}$$

4. **Total potential energy on ${}^f\maltese^s$**

It follows from (4.54) that the operator on ${}^f\maltese^s$ which is equal to (4.59) on ${}^f\maltese^s_n$ for every $n = 2, 3, \cdots$ is

$$V = \frac{1}{2} \sum_{r,t,u,v=1}^{\infty} < rt \mid V \mid vu > F_r^\dagger F_t^\dagger F_u F_v \tag{4.62}$$

When (4.61) holds,

$$\begin{aligned} &< rt \mid V \mid vu > \\ &= \sum_{m_1, m_2 = -s}^{+s} \int d^3x d^3y \phi^*_{rm_1}(\vec{x}) \phi^*_{tm_2}(\vec{y}) V(\vec{x}, \vec{y}) \phi_{vm_1}(\vec{x}) \phi_{um_2}(\vec{y}) \end{aligned} \tag{4.63}$$

where $\phi_{rm_s}(\vec{x})$ is given by (4.53).

5. **Proof of (4.54)**

The proof of (4.54) and is similar to the proof of (4.45) and is not given here.

We give three examples in Section 4.9 which check (4.54) using the one-particle equations (4.45) and (4.46).

4.5 Hamiltonian

We consider a system of identical nonrelativistic fermions each with rest mass m and spin s interacting with each other via two-body potentials. Each fermion is also be subjected to an external potential.

Hamiltonian on n-fermion Hilbert space

The Hamiltonian for the system on ${}^f\mathfrak{H}_n^s$ is[1]

$$H_n = \sum_{\alpha=1}^{n} [T(\xi_\alpha) + U(\xi_\alpha)] + \frac{1}{2} \sum_{\alpha,\beta=1}^{n} V(\xi_\alpha, \xi_\beta) \tag{4.64}$$

where $T(\xi_\alpha)$ is the kinetic energy of particle α, $U(\xi_\alpha)$ is the external potential experienced by particle α and $V(\xi_\alpha, \xi_\beta)$ is the interaction potential between particles α and β . We need not specify the details of the external potential $U(\xi_\alpha)$ or the two-body potential $V(\xi_\alpha, \xi_\beta)$.

We assume that the one-particle vectors $| \ \phi_r >$ (3.1) are the eigenvectors of

$$T(\xi_\alpha) + U(\xi_\alpha) + \mathcal{V}(\xi_\alpha) \tag{4.65}$$

for some choice of $\mathcal{V}(\xi_\alpha)$. That is, we assume that

[1] The double summation in (4.64) is restricted to $\alpha \neq \beta$.

$$\boxed{[T(\xi_\alpha) + U(\xi_\alpha) + \mathcal{V}(\xi_\alpha)] \mid \phi_r >= \epsilon_r \mid \phi_r > \qquad (4.66)}$$

can be solved for $\mid \phi_r >$ and ϵ_r $(r = 1, 2, \cdots)$.

We write (4.64) in the form

$$\boxed{H_n = H_{0n} + H_{1n} \qquad (4.67)}$$

where

$$\boxed{H_{0n} = \sum_{\alpha=1}^{n} [T(\xi_\alpha) + U(\xi_\alpha) + \mathcal{V}(\xi_\alpha)] \qquad (4.68)}$$

and

$$\boxed{H_{1n} = \frac{1}{2} \sum_{\alpha,\beta=1}^{n} V(\xi_\alpha, \xi_\beta) - \sum_{\alpha=1}^{n} \mathcal{V}(\xi_\alpha) \qquad (4.69)}$$

Comments

1. **Central potential and residual interactions**

   (4.67) describes n identical fermions moving independently in a central potential with residual interactions.

   The central potential experienced by particle α is

$$\boxed{U(\xi_\alpha) + \mathcal{V}(\xi_\alpha) \qquad (4.70)}$$

   H_{1n} is the total residual interaction.

   In practice, $\mathcal{V}(\xi_\alpha)$ is chosen such that H_{0n} is a good approximation to H_n.

2. **Nucleons in an atomic nucleus**

   (4.67) describes n nucleons in an atomic nucleus.

   $V(\xi_\alpha, \xi_\beta)$ is the sum of the strong interaction potential and the repulsive Coulomb potential between nucleon α and nucleon β.

   There is no external potential experienced by nucleon α. That is, $U(\xi_\alpha) = 0$

   $\mathcal{V}(\xi_\alpha)$ is the nuclear shell model potential.

3. **Conduction electrons in a metal**

   (4.67) describes n conduction electrons in a metal.

   $V(\xi_\alpha, \xi_\beta)$ is the repulsive Coulomb potential between electron α and electron β.

   $U(\xi_\alpha)$ is the attractive Coulomb potential experienced by electron α due to the positive ions in the lattice. $U(\xi_\alpha)$ is invariant under a space displacement equal to the distance between the ions.

$\mathcal{V}(\xi_\alpha)$ may be taken to be zero. The single-particle states are Bloch states.

Hamiltonian on Fock space

It follows from (4.45) and (4.54) that the operator on ${}^f\mathfrak{F}^s$ which is equal to (4.67) on ${}^f\mathfrak{F}^s_n$ for all $n = 2, 3, \cdots$ is

$$\boxed{H = H_0 + H_1 \qquad (4.71)}$$

where

$$\boxed{H_0 = \sum_{r=1}^{\infty} \epsilon_r F_r^\dagger F_r \qquad (4.72)}$$

and

$$\boxed{H_1 = \frac{1}{2} \sum_{r,s,t,u=1}^{\infty} < rs \mid V \mid ut > F_r^\dagger F_s^\dagger F_t F_u - \sum_{r,s=1}^{\infty} < r \mid \mathcal{V} \mid s > F_r^\dagger F_s \quad (4.73)}$$

Comments

1. Fock space description

In going from (4.67) to (4.71) the description of n fermions in interaction has been carried over from a description in ${}^f\maltese_n^s$ to a description in ${}^f\maltese^s$.

(4.71) contains no reference to the number of particles. It can be used for every system of fermions interacting via two-body potentials and with an external potential.

2. Two-fermion system

To describe the deuteron, for example, or two electrons, one prepares

$$\boxed{\mid \psi >= \sum_{r,s=1}^{\infty} \psi_{rs} F_r^\dagger F_s^\dagger \mid 0 > \qquad (4.74)}$$

at time zero where

$$\boxed{\psi_{rs} =< 0 \mid F_r F_s \mid \psi > \qquad (4.75)}$$

is the probability amplitude that the system is in the state

$$F_r^\dagger F_s^\dagger \mid 0 > \tag{4.76}$$

at time zero. $\mid \psi >$ is an eigenvector of the number operator (4.38) belonging to eigenvalue 2.

The state at time t is

$$
\boxed{\begin{aligned} | \psi(t) >&= U(t) \mid \psi >= e^{-iHt/\hbar} \mid \psi > \\ &= \sum_{r,s=1}^{\infty} \psi_{rs} e^{-iHt/\hbar} F_r^\dagger F_s^\dagger \mid 0 > \end{aligned} \qquad (4.77)}
$$

where H is given by (4.71). Since

$$
[N, H] = 0 \tag{4.78}
$$

it follows that $\mid \psi(t) >$ is an eigenvector of (4.38) belonging to eigenvalue 2 for all time; the two-particle state initially prepared remains a two-particle state for all time.

3. An advantage of the Fock space description

One advantage of the Fock space description of the n-fermion system [that is, in using (4.71) instead of (4.67)] is that (4.71) is expressed explicitly in terms of the eigenvectors and eigenvalues of (4.68).

That is, (4.71) incorporates an approximate solution of its eigenvalue problem.

4. Goldstone diagrams

The potential terms in (4.71) can be represented by diagrams.

J. Goldstone (1957) was the first to develop and apply diagrammatic methods with the Fock space description of the n-fermion system.

Goldstone gave the first proof of the unlinked cluster expansion for the ground state wave function and energy of an n-fermion system.

4.6 Independent-particle model

The Hamiltonian (4.72) describes fermions moving independently. When the single-particle energies in (4.66) are labelled $\epsilon_1 \leq \epsilon_2 \leq \cdots$ then

$$\boxed{\mid F >= F_1^\dagger \cdots F_n^\dagger \mid 0 > \qquad (4.79)}$$

is an n-fermion state with the lowest n single-particle states occupied.

Comments

1. **Independent-particle approximation to the ground-state**

   $\mid F >$ is an independent-particle approximation to the n-fermion ground-state of the Hamiltonian (4.71).

2. **Fermi level and Fermi energy**

   The highest occupied single-particle level in $\mid F >$ is the Fermi level for the system.

   ϵ_n is the Fermi energy of the system.

3. **Eigenvalue of H_0**

   It follows from (4.72) and (4.79) that

$$\boxed{H_0 \mid F >= \epsilon_F \mid F > \qquad (4.80)}$$

where

$$\epsilon_F = \sum_{r=1}^{n} \epsilon_r \tag{4.81}$$

Hartree-Fock potential

The operator F_r (4.22) is defined in terms of basis vectors in ${}^f\maltese^s$ which are in turn constructed from Slater determinants (3.7) in ${}^f\maltese^s_n$. The Slater determinants are constructed from the eigenvectors $\mid \phi_r >$ of the single-particle Hamiltonian (4.65) for some single-particle potential $\mathcal{V}(\xi_\alpha)$.

Different choices of $\mathcal{V}(\xi_\alpha)$ lead to different F_r.

We show in Section 4.9 that the choice of $\mathcal{V}(\xi_\alpha)$ which gives the best independent-particle approximation to the n-particle ground state of H is

$$< r \mid \mathcal{V} \mid s >= \sum_{t=1}^{n} \{< rt \mid V \mid st > - < rt \mid V \mid ts >\} \tag{4.82}$$

(4.82) defines the the Hartree-Fock potential.

The Hartree-Fock appoximation to the ground state energy of the n-particle system is

$$< F \mid H \mid F >$$

$$= \sum_{r=1}^{n} \left\{ < r \mid (T+U) \mid r > + \frac{1}{2} \sum_{s=1}^{n} [< rs \mid V \mid rs > - < rs \mid V \mid sr >] \right\} \tag{4.83}$$

Comments

1. **Ground-state energy**

   The energy E_{ground} of the n-particle ground-state of H is less than or equal to every n-particle expectation value of H.

   (4.83) is the best approximation to E_{ground} for independent-particle states.

2. **Determining the Hartree-Fock potential**

   $\mathcal{V}$ is determined by a self-consistent process. The steps in the process are

   i. choose $\mathcal{V}$

   ii. determine the $\mid \phi_r >$ by solving (4.66)

   iii. determine a new $\mathcal{V}$ using (4.82)

   iv. repeat the process until output $\mathcal{V}$ equals input $\mathcal{V}$

4.7 Particle-hole states

The state $\mid F >$ (4.79) is a independent-particle approximation to the ground state of an n-particle system. $\mid F >$ has the lowest n single-particle states occupied; the single-particle states are determined by solving the eigenvalue problem (4.66) with some choice of $\mathcal{V}$ or, in the Hartree-Fock approximation, with $\mathcal{V}$ given by (4.82).

The single-particle states occupied in $\mid F >$ are called unexcited states. The single-particle states not occupied in $\mid F >$ are called excited states.

An improved approximation to the ground state includes the component

$$\boxed{\sum_{r=1}^{n} \sum_{r'=n+1}^{\infty} a_{rr'} F_{r'}^{\dagger} F_r \mid F > \qquad (4.84)}$$

(4.84) is an $n-$particle state which is a linear combination of states with one particle created in the excited state $\mid \phi_{r'} >$ and one particle annihilated in the unexcited state $\mid \phi_r >$.

It is convenient to express states like (4.84) in terms of particles and holes. This is accomplished by defining

$$\boxed{H_r = F_{n+1-r}^{\dagger} \qquad (4.85)}$$

where $r = 1, 2, \cdots, n$, in which case (4.24) and (4.25) are[1]

$$H_r \mid n\{n_1 n_2 \cdots 0 \cdots\} >= (-)^{m_{n+1-r}} \mid n+1\{n_1 n_2 \cdots 1 \cdots\} > \qquad (4.86)$$

$$H_r^\dagger \mid n+1\{n_1 n_2 \cdots 1 \cdots\} >= (-)^{m_{n+1-r}} \mid n\{n_1 n_2 \cdots 0 \cdots\} > \qquad (4.87)$$

Comments

1. **Hole annihilator and hole creator**

   H_r annihilates a hole in the state $\mid n\{\cdots 0 \cdots\} >$.

   $H_r^\dagger$ creates a hole in the state $\mid n+1\{\cdots 1 \cdots\} >$.

   H_r is a hole annihilation operator or hole annihilator.

   $H_r^\dagger$ is a hole creation operator or hole creator.

2. **Labelling of hole operators**

   The labelling in (4.85) labels holes starting from the Fermi level and running down to the the lowest single-particle level. That is,

   $H_1^\dagger \mid F >$ is an $n-1$ particle state with a particle removed from the Fermi level, and

   $H_n^\dagger \mid F >$ is an $n-1$ particle state with a particle removed from the lowest single-particle level.

[1] The 0 and 1 in the basis states in (4.86) and (4.87) occur in the $n-1-r$ place.

3. Fundamental dynamical variables

Fundamental dynamical variables appropriate for describing an $n-$particle system are creators and annihilators for particles in excited states

$$\boxed{F_r^\dagger \quad \text{and} \quad F_r \qquad r = n+1, n+2, \cdots, \infty \tag{4.88}}$$

and creators and annihilators for holes in unexcited states

$$\boxed{H_r^\dagger \quad \text{and} \quad H_r \qquad r = 1, 2, \cdots, n \tag{4.89}}$$

The fundamental dynamical variables obey the anticommutation relations

$$\{F_r, F_s\} = 0 \tag{4.90}$$

$$\left\{F_r, F_s^\dagger\right\} = \delta_{rs} \tag{4.91}$$

$$r, s = n+1, n+2, \cdots, \infty \tag{4.92}$$

$$\{H_r, H_s\} = 0 \tag{4.93}$$

$$\left\{H_r, H_s^\dagger\right\} = \delta_{rs} \tag{4.94}$$

$$r, s = 1, 2, \cdots, n \tag{4.95}$$

$$\{F, H\} = 0 \tag{4.96}$$

$$F = F_r \quad \text{or} \quad F_r^\dagger \tag{4.97}$$

$$H = H_s \quad \text{or} \quad H_s^\dagger \tag{4.98}$$

$$r = n+1, n+2, \cdots, \infty \tag{4.99}$$

$$s = 1, 2, \cdots, n \tag{4.100}$$

4. **Symmetry in the concepts of creation and annihilation**

(4.93) and (4.94) have the same form as (4.90) and (4.91).

The theory of fermions is symmetric in the concepts of creation and annihilation.

The theory of fermions deals with particles and holes on equal footing.

5. **Number operators**

We define particle and hole number operators

$$N_{pr} = F_{n+r}^\dagger F_{n+r} \tag{4.101}$$

$$N_{hr} = H_r^\dagger H_r \tag{4.102}$$

and

$$N_p = \sum_{r=1}^{\infty} N_{pr} \tag{4.103}$$

$$N_h = \sum_{r=1}^{n} N_{hr} \tag{4.104}$$

N_p is the excited-particle number operator. N_h is the hole number operator.

The number operator N (4.38) is expressed in terms of particle- and hole-number operators as follows:

$$N = n + N_p - N_h \tag{4.105}$$

6. **Hamiltonian H_0**

The independent-particle Hamiltonian H_0 (4.72) is expressed in terms of particle and hole number operators as follows:

$$H_0 = \epsilon_F + \sum_{r=1}^{\infty} \epsilon_{pr} N_{pr} - \sum_{r=1}^{n} \epsilon_{hr} N_{hr} \tag{4.106}$$

where

$$\epsilon_{pr} = \epsilon_{n+r} \tag{4.107}$$

$$\epsilon_{hr} = \epsilon_{n+1-r} \tag{4.108}$$

ϵ_{pr} is the energy of a particle in an excited state.

ϵ_{hr} is the energy of a hole in an unexcited state.

7. **Fermi vacuum and Fermi sea**

The independent-particle state $\mid F >$ satisfies

$$N \mid F >= n \mid F > \tag{4.109}$$

$$H_0 \mid F >= \epsilon_F \mid F > \tag{4.110}$$

$$F_r \mid F >= 0 \tag{4.111}$$

$$r = n+1, n+2, \cdots, \infty \tag{4.112}$$

$$H_r \mid F >= 0 \tag{4.113}$$

$$r = 1, 2, \cdots, n \tag{4.114}$$

$$N_p \mid F >= N_h \mid F >= 0 \tag{4.115}$$

$\mid F >$ is an $n-$particle state which is an eigenvector of the independent-particle Hamiltonian H_0 belonging to eigenvalue ϵ_F.

$\mid F >$ contains no excited particles and no holes.

We call $\mid F >$ the Fermi vacuum.

The unexcited states constitute the Fermi sea. $F_r^\dagger$ creates a particle above the Fermi sea and $H_r^\dagger$ creates a hole in the Fermi sea.

8. Particle-hole creators

The operators

$$F_r^\dagger H_s^\dagger \tag{4.116}$$

$$r = n+1, n+2, \cdots, \infty \tag{4.117}$$

$$s = 1, 2, \cdots, n \tag{4.118}$$

are particle-hole creators.

When acting on $\mid F >$, $F_r^\dagger H_s^\dagger$ yields an $n-$particle basis vector with one particle in an excited state and one particle removed from an unexcited state.

That is, when acting on $\mid F >$,$F_r^\dagger H_s^\dagger$ creates a particle above the Fermi sea and a hole in the Fermi sea.

9. **Particle-hole states**

The n-particle state

$$F_r^\dagger H_s^\dagger \mid F> \tag{4.119}$$

$$r = n+1, n+2, \cdots, \infty \tag{4.120}$$

$$s = 1, 2, \cdots, n \tag{4.121}$$

is a one-particle-one-hole state.

One can similarly construct two-particle-two-hole states, and so on.

The general n-particle state is a linear combination of $\mid F>$ and of many-particle-many-hole states.

4.8 Correlated particle-hole states

The state (4.119) of an n-particle system has a hole in single-particle state $\mid \phi_{n+1-s}>$ in the Fermi sea and a particle in excited single-particle state $\mid \phi_{n+r}>$ above the Fermi sea. The state labels r and s are independent; the particle-hole pair which is created is uncorrelated.

In this section we consider states of an n-particle system which involve correlated particle-hole pairs. These correlated-pair states are defined as a unitary transformation of the independent-particle state $\mid F>$.

The best correlated-pair state is defined as the correlated-pair state which minimizes the average value of an approximate Hamiltonian involving interaction between correlated particle-hole pairs. The average energy obtained is less than the average energy of the system in the independent-particle state $\mid F >$ for any attractive particle-hole interaction, no matter how small, between correlated particle-hole pairs. Furthermore, there is a gap in the energy spectrum of the n-particle system. The expression for the energy gap cannot be obtained from any finite-order perturbation theory involving interaction between correlated particle-hole pairs.

We comment on the similarities and differences between the approach in this section and the classic papers on correlated-particle pairs by Cooper (1956) and on the theory of superconductivity by Bardeen, Cooper, Schrieffer (1957), Bogoliubov (1958) and Valatin (1958).[2]

Correlated particle-hole pairs

For each $r = 1, 2, \cdots, n$ we define a particle-hole annihilator G_r by

$$\boxed{G_r = H_r F_{n+r} \qquad (4.122)}$$

It follows that:

$$\boxed{[N, G_r] = 0 \qquad (4.123)}$$

$$[G_r, G_s] = 0 \qquad (4.124)$$

$$\left[G_r, G_s^\dagger\right] = (1 - N_{pr} - N_{hr})\delta_{rs} \qquad (4.125)$$

[2] Reprints of these papers are in Pines (1961).

$$G_r^2 = 0 \tag{4.126}$$

$$G_r G_r^\dagger G_r = G_r \tag{4.127}$$

$$G_r^\dagger G_r = N_{pr} N_{hr} \tag{4.128}$$

$$G_r G_r^\dagger = 1 - N_{pr} - N_{hr} + N_{pr} N_{hr} \tag{4.129}$$

Comments

1. **Correlated particle-hole creator**

   $G_r^\dagger$ creates a particle in an excited state and a hole in an unexcited state. The particle and hole created are dependent on each other.

   $G_r^\dagger$ creates a correlated particle-hole pair.

2. **Labelling of particle-hole creators**

   $G_r^\dagger$ is constructed such that the energy difference between the created particle and hole increases with increasing r. That is,

   $G_1^\dagger \mid F >$ is an n-particle state with a particle in the first excited state above the Fermi level and a hole in the Fermi level, and

   $G_n^\dagger \mid F >$ is an n-particle state with a particle in the nth excited state above the Fermi level and a hole in the lowest single-particle level.

3. **Cooper pairs**

   Bardeen, Cooper, Schrieffer (1957) have formulated their theory of superconductivity in terms of annihilation operators

$$b_{\mathbf{k}} = c_{-\mathbf{k}\downarrow} c_{\mathbf{k}\uparrow} \tag{4.130}$$

where $c_{\mathbf{k}\uparrow}$ annihilates a spin-up electron with momentum $\mathbf{k}$ and $c_{-\mathbf{k}\downarrow}$ annihilates a spin-down electron with momentum $-\mathbf{k}$.

$b_{\mathbf{k}}$ annihilates correlated pairs of electrons. These correlated pairs are called Cooper pairs (Cooper (1956)).

(4.122) differs from (4.130): (4.122) involves a particle-hole pair, not a two-particle pair, and so does not change the total number of particles; furthermore, the correlated pair is not restricted to having equal and opposite momentum and spin.

Kaempffer transformation

We define

$$U = U_1 U_2 \cdots U_n \tag{4.131}$$

where

$$U_r = 1 - (1 - u_r) S_{0r} - i v_r S_{2r} \tag{4.132}$$

where

$$S_{0r} = G_r G_r^\dagger + G_r^\dagger G_r \tag{4.133}$$

$$S_{2r} = i\Big(G_r^\dagger - G_r\Big) \tag{4.134}$$

and where u_r and v_r are real parameters satisfying

$$u_r^2 + v_r^2 = 1 \tag{4.135}$$

Comments

1. **Kaempffer transformation and Kaempffer parameters**

   Kaempffer (1965) has considered a transformation of the form (4.131).[1]

   Accordingly, we call U a Kaempffer transformation.

   A Kaempffer transformation is labelled by parameters $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$. We call these parameters Kaempffer parameters.

2. **Properties of S_{0r} and S_{2r}**

   It follows from (4.133) and (4.134) that S_{0r} and S_{2r} are Hermitian and satisfy

$$[S_{0r}, S_{0s}] = [S_{2r}, S_{2s}] = [S_{0r}, S_{2s}] = 0 \tag{4.136}$$

---

[1] Kaempffer also defines operators which we would label S_{1r} and S_{3r} and shows that $S_{0r}, S_{1r}, S_{2r}, S_{3r}$ are isomorphic to the unit matrix and the three Pauli matrices.

$$S_{0r}^2 = S_{2r}^2 = S_{0r} \tag{4.137}$$

$$S_{0r}S_{2r} = S_{2r}S_{0r} = S_{2r} \tag{4.138}$$

3. **Family of number-conserving commuting unitary operators**

It follows from (4.132) that $U_1, U_2, \cdots, U_n$ is a family of number-conserving commuting unitary operators. That is,

$$[N, U_r] = 0 \tag{4.139}$$

$$[U_r, U_s] = 0 \tag{4.140}$$

$$U_r U_r^\dagger = U_r^\dagger U_r = 1 \tag{4.141}$$

4. **Number-conserving unitary operator**

It follows from (4.131) that U is a number-conserving unitary operator. That is,

$$[N, U] = 0 \tag{4.142}$$

$$UU^\dagger = U^\dagger U = 1 \tag{4.143}$$

Correlated-pair model

We define the state $\mid G >$ as a Kaempffer transformation (4.131) of the independent-particle state $\mid F >$ (4.79). That is,

$$\mid G >= U \mid F > \tag{4.144}$$

It follows that

$$N \mid G >= n \mid G > \tag{4.145}$$

$$\mid G >= \prod_{\tau=1}^{n} \left(u_\tau + v_\tau G_\tau^\dagger \right) \mid F > \tag{4.146}$$

Comments

1. **Correlated-pair approximation to the ground-state**

   $\mid G >$ is an n-particle state which is linear combination of the independent-particle state $\mid F >$ and of states formed by creating up to n correlated particle-hole pairs from $\mid F >$.

   $\mid G >$ is labelled by a set of Kaempffer parameters.

   In particular, $\mid G >=\mid F >$ when $v_1 = v_2 = \cdots = v_n = 0$.

   $\mid G >$ is correlated-pair approximation to the n-fermion ground-state of the Hamiltonian (4.71).

2. **BCS superconducting state**

In the Bardeen, Cooper, Schrieffer (BCS) (1957) theory of superconductivity, the superconducting state Ψ is the correlated-pair state

$$\boxed{\Psi = \prod_{\mathbf{k}} \left[(1-h_{\mathbf{k}})^{\frac{1}{2}} + h_{\mathbf{k}} b_{\mathbf{k}}^{\dagger}\right] \Phi_0 \qquad (4.147)}$$

where $h_{\mathbf{k}}$ is a real parameter and Φ_0 is the vacuum state. Ψ is not an n-particle state

(4.144) differs from (4.147): (4.144) is an n-particle state; it involves correlated particle-hole creators acting on the independent-particle state, not correlated two-particle creators acting on the vacuum state.

Quasiparticles and quasiholes

We define annihilators Q_{pr} and Q_{hr} as a Kaempffer transformation (4.131) of particle and hole annihilators. That is,

$$Q_{pr} = UF_{n+r}U^{\dagger} \qquad (4.148)$$

$$r = 1, 2, \cdots, \infty \qquad (4.149)$$

$$Q_{hr} = UH_rU^{\dagger} \qquad (4.150)$$

$$r = 1, 2, \cdots, n \qquad (4.151)$$

It follows that

$$Q_{pr} = u_r F_{n+r} - v_r H_r^\dagger \tag{4.152}$$

$$Q_{hr} = u_r H_r + v_r F_{n+r}^\dagger \tag{4.153}$$

$$r = 1, 2, \cdots, n \tag{4.154}$$

$$Q_{pr} = F_{n+r} \tag{4.155}$$

$$r = n+1, n+2, \cdots, \infty \tag{4.156}$$

It follows from (4.152) and (4.153) that

$$F_{n+r} = u_r Q_{pr} + v_r Q_{hr}^\dagger \tag{4.157}$$

$$H_r = u_r Q_{hr} - v_r Q_{pr}^\dagger \tag{4.158}$$

$$r = 1, 2, \cdots, n \tag{4.159}$$

It follows from (4.90) to (4.98) and from (4.148) to 4.151) that

$$\boxed{\begin{aligned} \{Q_{pr}, Q_{ps}\} &= 0 \qquad (4.160) \\ \left\{Q_{pr}, Q_{ps}^{\dagger}\right\} &= \delta_{rs} \qquad (4.161) \end{aligned}}$$

$$\boxed{\begin{aligned} \{Q_{hr}, Q_{hs}\} &= 0 \qquad (4.162) \\ \left\{Q_{hr}, Q_{hs}^{\dagger}\right\} &= \delta_{rs} \qquad (4.163) \end{aligned}}$$

$$\boxed{\{Q_{pr}, Q_{hs}\} = \left\{Q_{pr}, Q_{hs}^{\dagger}\right\} = 0 \qquad (4.164)}$$

Comments

1. **Nomenclature**

Q_{pr} and Q_{hr} are linear combinations of particle and hole operators.

Q_{pr} is a quasiparticle annihilator. Q_{hr} is a quasihole annihilator.

2. **Bogoliubov-Valatin operators**

Bogoliubov (1958) and Valatin (1958) have introduced linear combinations of operators analogous to (4.152) and (4.153) in their theory of superconductivity.

We note that (4.152) and (4.153) arise via the same transformation which led to the correlated-pair state (4.146).

The Kaempffer transformation (4.131) thus provides the link between the correlated-pair state (4.146) and quasiparticle and quasihole annihilators (4.152) and (4.153).

3. **Fundamental dynamical variables**

(4.148) and (4.150) define fundamental dynamical variables Q_{pr} and Q_{hr}.

(4.160) to (4.164) is a fundamental algebra for the system.

4. **Transformation equations**

(4.157) and (4.158) allow transformations from Q_{pr} and Q_{hr} to F_{n+r} and H_r.

5. **Number operators**

We define quasiparticle and quasihole number operators

$$N_{qpr} = UN_{pr}U^\dagger = Q^\dagger_{pr}Q_{pr} \tag{4.165}$$

$$N_{qhr} = UN_{hr}U^\dagger = Q^\dagger_{hr}Q_{hr} \tag{4.166}$$

and

$$N_{qp} = UN_pU^\dagger = \sum_{r=1}^{\infty} N_{qpr} \tag{4.167}$$

$$N_{qh} = UN_hU^\dagger = \sum_{r=1}^{n} N_{qhr} \tag{4.168}$$

N_{qp} is the quasiparticle number operator. N_{qh} is the quasihole number operator.

It follows from (4.105) and (4.142) that the number operator N (4.38) is expressed in terms of quasiparticle and quasihole number operators as

$$N = n + N_{qp} - N_{qh} \tag{4.169}$$

6. Independent-quasiparticle-quasihole Hamiltonian

The independent-quasiparticle-quasihole Hamiltonian $UH_0U^\dagger$ is expressed in terms of quasiparticle and quasihole number operators as follows:

$$UH_0U^\dagger = \epsilon_F + \sum_{r=1}^{\infty} \epsilon_{pr} N_{qpr} - \sum_{r=1}^{n} \epsilon_{hr} N_{qhr} \tag{4.170}$$

The eigenvectors of $UH_0U^\dagger$ form a basis for the Hilbert space of the system.

7. Properties of the correlated-pair state

The correlated-pair state $\mid G >$ satisfies

$$Q_{pr} \mid G >= 0 \tag{4.171}$$

$$r = 1, 2, \cdots, \infty \tag{4.172}$$

$$Q_{hr} \mid G >= 0 \tag{4.173}$$

$$r = 1, 2, \cdots, n \tag{4.174}$$

$$N \mid G >= n \mid G > \tag{4.175}$$

$$N_{qp} \mid G >= N_{qh} \mid G >= 0 \tag{4.176}$$

$$U H_0 U^\dagger \mid G >= \epsilon_F \mid G > \tag{4.177}$$

$\mid G >$ is an $n-$particle state which contains no quasiparticles and no quasiholes.

$\mid G >$ is an eigenvector of the independent-quasiparticle-quasihole Hamiltonian (4.170) belonging to eigenvalue ϵ_F.

8. Quasiparticle-quasihole states

The n-particle states

$$Q^\dagger_{pr} Q^\dagger_{hs} \mid G > \tag{4.178}$$

$$r = 1, 2, \cdots, \infty \tag{4.179}$$

$$s = 1, 2, \cdots, n \tag{4.180}$$

are one-quasiparticle-one-quasihole states.

One can similarly construct two-quasiparticle-two-quasihole states, and so on.

The general n-particle state is a linear combination of $\mid G >$ and of many-quasiparticle-many-quasihole states.

Best correlated-pair state

We assume that the Hamiltonian (4.71) for the system can be approximated by

$$H_{pair} = H_0 + H_{pair} \tag{4.181}$$

where H_0 is given by (4.106) and where

$$H_{pair} = - \sum_{r,s=1}^{n} g_{rs} G^\dagger_r G_s \tag{4.182}$$

$$g^*_{rs} = g_{rs} \qquad g_{sr} = g_{rs} \qquad g_{rr} = 0 \tag{4.183}$$

H_{pair} is an interaction between correlated particle-hole pairs.

The best correlated-pair state is defined as the state $\mid G>$ which minimizes

$$E_G = < G \mid H_{pair} \mid G > \tag{4.184}$$

We show in Section 4.9 that the above condition yields the following expressions for the Kaempffer parameters:

$$u_r^2 = \frac{1}{2}\left[1 + \frac{\varepsilon_r}{\left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}}\right] \tag{4.185}$$

$$v_r^2 = \frac{1}{2}\left[1 - \frac{\varepsilon_r}{\left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}}\right] \tag{4.186}$$

where

$$\varepsilon_r = \epsilon_{pr} - \epsilon_{hr} \tag{4.187}$$

and where $\varepsilon_{01}, \varepsilon_{02}, \cdots, \varepsilon_{0n}$ are real parameters satisfying

$$\varepsilon_{0r} = \frac{1}{2}\sum_{s=1}^{n} \frac{g_{rs}\varepsilon_{0s}}{\left(\varepsilon_s^2 + \varepsilon_{0s}^2\right)^{\frac{1}{2}}} \tag{4.188}$$

We also show that

$$\boxed{E_G - \epsilon_F = \frac{1}{2}\sum_{r=1}^{n}\left[\varepsilon_r - \left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}\right] \qquad (4.189)}$$

Determining $\varepsilon_{01}, \varepsilon_{02}, \cdots, \varepsilon_{0n}$ from (4.188) determines E_G. The values of $\varepsilon_{01}, \varepsilon_{02}, \cdots, \varepsilon_{0n}$ depend upon the values of the parameters g_{rs} which specify the interaction (4.182) between correlated particle-hole pairs.

Special case: constant interaction

We assume that

$$g_{rs} = 0 \quad \text{if} \quad r \text{ or } s > n_c \qquad (4.190)$$

$$g_{rs} = 2g \quad \text{if} \quad r \text{ and } s \leq n_c \qquad (4.191)$$

The integer n_c specifies the number of correlated particle-hole pairs which participate in the pairing interaction H_{pair} (4.182).

It follows from (4.190) that

$$\varepsilon_{0r} = 0 \quad \text{if} \quad r > n_c \qquad (4.192)$$

and from (4.191) that

$$\varepsilon_{0r} = g \sum_{s \neq r=1}^{n_c} \frac{\varepsilon_{0s}}{\left(\varepsilon_s^2 + \varepsilon_{0s}^2\right)^{\frac{1}{2}}} \tag{4.193}$$

if $r \leq n_c$.

(4.193) is solved by

$$\varepsilon_{0r} = \varepsilon_0 \qquad \text{if} \qquad r \leq n_c \tag{4.194}$$

where

$$\frac{1}{g} = \sum_{r=1}^{n_c - 1} \frac{1}{\left(\varepsilon_r^2 + \varepsilon_0^2\right)^{\frac{1}{2}}} \tag{4.195}$$

Special case: high density of single-particle energy levels

We assume that the number of energy levels is sufficiently high near the Fermi level that the summations in (4.189) and (4.195) can be replaced by integrals. That is,

$$E_G - \epsilon_F = \frac{1}{2} \int_0^{\hbar\omega} \left[\varepsilon - \left(\varepsilon^2 + \varepsilon_0^2\right)^{\frac{1}{2}} \right] n(\varepsilon) d\varepsilon \tag{4.196}$$

$$\boxed{\frac{1}{g} = \int_0^{\hbar\omega} \frac{n(\varepsilon)d\varepsilon}{\left(\varepsilon^2 + \varepsilon_0^2\right)^{\frac{1}{2}}}} \tag{4.197}$$

where $\hbar\omega = \varepsilon_{n_c}$ and $n(\varepsilon)d\varepsilon$ is the number of single-particle states with energy ε between ε and $\varepsilon + d\varepsilon$.

We further assume that

$$\boxed{n(\varepsilon) = n(0) \qquad \text{for} \qquad 0 \leq \varepsilon \leq \hbar\omega} \tag{4.198}$$

$n(0)$ is the energy-level density at the Fermi level.

It follows from (4.196) to (4.198) that

$$\boxed{\varepsilon_0 = \hbar\omega / \sinh\left[\frac{1}{n(0)g}\right]} \tag{4.199}$$

and

$$\boxed{E_G - \epsilon_F = \frac{1}{4}n(0)(\hbar\omega)^2\left\{1 - \left[1 + \left(\frac{\varepsilon_0}{\hbar\omega}\right)^2\right]^{\frac{1}{2}}\right\} - \frac{\varepsilon_0^2}{4g}} \tag{4.200}$$

That is,

$$E_G - \epsilon_F = -\frac{1}{2}n(0)(\hbar\omega)^2\left[\frac{1}{e^{2x}-1} + \frac{2x}{e^{2x}-2+e^{-2x}}\right] \qquad (4.201)$$

where

$$x = \frac{1}{n(0)g} \qquad (4.202)$$

Comments

1. **BCS equations**

   (4.199) to (4.201) are similar to equations in Bardeen, Cooper, Schrieffer (BCS) (1957).

2. **Comparison with the independent-particle state**

   It follows from (4.201) that

$$E_G < \epsilon_F \quad \text{if} \quad g > 0 \qquad (4.203)$$

That is, if there is an attractive force between correlated particle-hole pairs, no matter how weak, the average energy of the system in the best correlated-pair state $\mid G >$ is less than the average energy of the system in the independent-particle state $\mid F >$.

3. **Nonperturbative result**

The right side of (4.201) has an essential singularity when $g = 0$.

(4.201) cannot be obtained from any finite-order perturbation theory involving expansion in powers of g.

4. **Energy gap**

We define the n-particle excited state $\mid G^* >$ as the quasiparticle-quasihole state (4.178) when $r = s = a$ where a is an integer and $1 \leq a \leq n$. That is

$$\boxed{\mid G^* >= Q_{pa}^{\dagger} Q_{ha}^{\dagger} \mid G > \qquad (4.204)}$$

It follows on calculation that

$$\boxed{E_{G^*} =< G^* \mid H_{pair} \mid G^* >= E_G + \left(\varepsilon_a^2 + \varepsilon_{0a}^2\right)^{\frac{1}{2}} \qquad (4.205)}$$

In particular, when (4.194) holds,

$$\boxed{E_{G^*} \geq E_G + \varepsilon_0 \qquad (4.206)}$$

for all values of a.

The energy required to form a quasiparticle-quasihole excitation is at least ε_0; there is a gap in the energy spectrum of the n-particle system.

4.9 Some derivations

Derivation of (4.32) and (4.33)

It follows from (4.21) and (4.22) that

$$F_r F_r^\dagger = \sum_{nn_1n_2\cdots}^{f} \mid n\{\cdots 0\cdots\} >< n\{\cdots 0\cdots\}\mid \tag{4.207}$$

and

$$F_r^\dagger F_r = \sum_{nn_1n_2\cdots}^{f} \mid n+1\{\cdots 1\cdots\} >< n+1\{\cdots 1\cdots\}\mid \tag{4.208}$$

and thus

$$F_r F_r^\dagger + F_r^\dagger F_r = 1 \tag{4.209}$$

which is (4.33) when $r = s$. When $r < s$,

$$F_r F_s^\dagger = \sum_{nn_1n_2\cdots}^{f} \mid n+1\{\cdots 0\cdots 1\cdots\} > (-)^{2m_r+1+\lambda_{rs}} < n+1\{\cdots 1\cdots 0\cdots\}\mid \tag{4.210}$$

where we have written

$$m_s = m_r + n_r + \lambda_{rs} \tag{4.211}$$

where

$$\lambda_{rs} = \sum_{t=r+1}^{s} n_t \tag{4.212}$$

is the number of occupied levels between levels r and s. Then

$$\begin{aligned} F_s^\dagger F_r &= \sum_{nn_1n_2\cdots}^{f} \mid n+1\{\cdots 0\cdots 1\cdots\} > (-)^{2m_r+0+\lambda_{rs}} < n+1\{\cdots 1\cdots 0\cdots\}\mid \\ &= -F_r F_s^\dagger \end{aligned} \tag{4.213}$$

(4) also holds when $s < r$. This completes the proof of (4.33). (4.32) follows similarly.

Derivation of (4.35)

The state may be expressed as

$$| n\{\cdots 1 \cdots 1 \cdots\cdots 1 \cdots\} > \tag{4.214}$$

where the 1's appear in the rth, sth and tth, places. Using (4.24) repeatedly yields

$$\begin{aligned} | n\{\cdots 1 \cdots 1 \cdots\cdots 1 \cdots\} > &= F_r^\dagger \mid n-1\{\cdots 0 \cdots 1 \cdots\cdots 1 \cdots\} > \\ &= F_r^\dagger F_s^\dagger \mid n-2\{\cdots 0 \cdots 0 \cdots\cdots 1 \cdots\} > \\ &= \cdots = F_r^\dagger F_s^\dagger \cdots F_t^\dagger \mid 0 > \end{aligned} \tag{4.215}$$

which is (4.35).

Derivation of (4.45)

We show that

$$\underset{n}{<} n\{n_1 n_2 \cdots\} \mid A_n \mid n\{n_1' n_2' \cdots\} \underset{n}{>} = < n\{n_1 n_2 \cdots\} \mid A \mid n\{n_1' n_2' \cdots\} > \tag{4.216}$$

where the left side involves quantities in n-fermion Hilbert space ${}_n^f\mathfrak{H}^s$ and the right side involves quantities in fermion Fock space ${}^f\mathfrak{H}^s$.

That is, $\mid n\{n_1 n_2 \cdots\} \underset{n}{>}$ is a Slater determinant (3.7) and A_n is given by (4.47), and $\mid n\{n_1 n_2 \cdots\} >$ is given by (4.35) and A is given by (4.45).

Now

$$\sum_{r=1}^{\infty} \mid \phi_r \underset{\alpha}{>} \underset{\alpha}{<} \phi_r \mid = 1_\alpha \tag{4.217}$$

so

$$A(\xi_\alpha) = \sum_{r,s=1}^{\infty} \mid \phi_r \underset{\alpha}{>} \underset{\alpha}{<} \phi_r \mid A(\xi_\alpha) \mid \phi_s \underset{\alpha}{>} \underset{\alpha}{<} \phi_s \mid \tag{4.218}$$

But

$$\underset{1}{<} \phi_r \mid A(\xi_1) \mid \phi_s \underset{1}{>} = \underset{2}{<} \phi_r \mid A(\xi_2) \mid \phi_s \underset{2}{>} = \cdots = \underset{n}{<} \phi_r \mid A(\xi_n) \mid \phi_s \underset{n}{>} \tag{4.219}$$

since the n single-particle spaces are identical, so

$$\underset{\alpha}{<} \phi_r \mid A(\xi_\alpha) \mid \phi_s \underset{\alpha}{>} = < r \mid A \mid s > \tag{4.220}$$

and thus

$$A(\xi_\alpha) = \sum_{r,s=1}^{\infty} < r \mid A \mid s > \mid \phi_r \underset{\alpha}{>} \underset{\alpha}{<} \phi_s \mid \tag{4.221}$$

and

$$A_n(\xi_1, \xi_2, \cdots, \xi_n) = \sum_{r,s=1}^{\infty} < r \mid A \mid s > \sum_{\alpha=1}^{n} \mid \phi_r \underset{\alpha}{>} \underset{\alpha}{<} \phi_s \mid \tag{4.222}$$

Now consider $r < s$. Then

$$\sum_{\alpha=1}^{n} \mid \phi_r \underset{\alpha}{>} \underset{\alpha}{<} \phi_s \mid n\{n_1 n_2 \cdots\} \underset{n}{>} = (-)^{\lambda_{rs}} \mid n\{\cdots 1 \cdots 0 \cdots\} \underset{n}{>} \delta_{0 n_r} \delta_{1 n_s} \tag{4.223}$$

where

$$\lambda_{rs} = \sum_{t=r+1}^{s-1} n_t \tag{4.224}$$

The factor $(-)^{\lambda_{rs}}$ in (4.223) accounts for interchanging rows of the Slater determinant (3.7) to get them into the order $r < s < \cdots < t$.

Thus

$$\sum_{\alpha=1}^{n} \mid \phi_r \underset{\alpha}{>} \underset{\alpha}{<} \phi_s \mid = \sum_{n_1 n_2 \cdots}^{f} \mid n\{\cdots 1 \cdots 0 \cdots\} \underset{n}{>} (-)^{\lambda_{rs}} \underset{n}{<} n\{\cdots 0 \cdots 1 \cdots\} \mid \tag{4.225}$$

(4.45) follows since

$$F_r^\dagger F_s = \sum_{nn_1n_2\cdots}^{f} \mid n\{\cdots 1 \cdots 0 \cdots\} > (-)^{\lambda_{rs}} < n\{\cdots 0 \cdots 1 \cdots\} \mid \tag{4.226}$$

Three checks of (4.54)

We check (4.54) by considering three examples which use the one-particle equations (4.45) and (4.46).

Example 1

When

$$A(\xi_\alpha, \xi_\beta) = 1 \tag{4.227}$$

(4.56) becomes

$$A_n(\xi_1, \cdots, \xi_n) = n(n-1) \tag{4.228}$$

The Fock space operator corresponding to (4.228) is

$$A = N(N-1) \tag{4.229}$$

Substitution of (4.227) into (4.55) yields (4.229) as required.

Example 2

When

$$A(\xi_\alpha, \xi_\beta) = \frac{1}{2}[A(\xi_\alpha) + A(\xi_\beta)] \tag{4.230}$$

(4.56) becomes

$$A_n(\xi_1, \cdots, \xi_n) = (n-1)\sum_{\alpha=1}^{n} A(\xi_\alpha) \tag{4.231}$$

The Fock space operator corresponding to (4.231) is

$$A = (N-1) \sum_{r,s=1}^{\infty} < r \mid A \mid s > F_r^\dagger F_s \tag{4.232}$$

Substitution of (4.230) into (4.55) yields (4.232) as required.

Example 3

When

$$A(\xi_\alpha, \xi_\beta) = A(\xi_\alpha) A(\xi_\beta) \tag{4.233}$$

(4.56) becomes

$$A_n(\xi_1, \xi_2, \cdots, \xi_n) = \left[\sum_{\alpha=1}^{n} A(\xi_\alpha) \right]^2 - \sum_{\alpha=1}^{n} [A(\xi_\alpha)]^2 \tag{4.234}$$

The Fock space operator corresponding to (4.234) is

$$A = \left[\sum_{r,s=1}^{\infty} < r \mid A \mid s > F_r^\dagger F_s \right]^2 - \sum_{r,s=1}^{\infty} < r \mid A^2 \mid s > F_r^\dagger F_s \tag{4.235}$$

(4.235) may be written in the form (4.54) where

$$< rs \mid A \mid ut > = < r \mid A \mid u > < s \mid A \mid t > \tag{4.236}$$

as required.

Derivation of (4.82)

Let $F_r^\dagger$ and $\hat{F}_r^\dagger$ be creation operators determined, respectively, from single-particle potentials $\mathcal{V}(\xi_\alpha)$ and $\mathcal{U}(\xi_\alpha)$. Then

$$\hat{F}_r^\dagger = F_r^\dagger + i \sum_{r'=1}^{\infty} a_{rr'} F_{r'}^\dagger + \cdots \tag{4.237}$$

for some real numbers $a_{rr'}$, where $a_{rr} = 0$ for all r.

Let

$$| F >= F_1^\dagger F_2^\dagger \cdots F_n^\dagger \mid 0 > \tag{4.238}$$

and

$$| \hat{F} >= \hat{F}_1^\dagger \hat{F}_2^\dagger \cdots \hat{F}_n^\dagger \mid 0 > \tag{4.239}$$

(4.238) and (4.239) are the independent-particle vectors corresponding to single-particle potentials $\mathcal{V}(\xi_\alpha)$ and $\mathcal{U}(\xi_\alpha)$, respectively.

We assume that (4.238) yields the minimum expectation value of (4.71) for all independent-particle vectors. In view of (4.237), this assumption may be expressed as

$$\frac{\partial}{\partial a_{rr'}} < \hat{F} \mid H \mid \hat{F} > |_{a_{rr'}=0} = 0 \tag{4.240}$$

(4.240) will be used to determine the single-particle potential $\mathcal{V}(\xi_\alpha)$.

It follows from (4.240) that the term in

$$< \hat{F} \mid H \mid \hat{F} > \tag{4.241}$$

which is linear in the $a_{rr'}$ must vanish. The term in (4.239) which is linear in the $a_{rr'}$ is

$$| F_{lin} >= i \sum_{r=1}^{n} \sum_{r'=1}^{\infty} a_{rr'} F_1^\dagger F_2^\dagger \cdots F_{r'}^\dagger \cdots F_n^\dagger \mid 0 > \tag{4.242}$$

where the $F_{r'}^\dagger$ occurs in the r th place. Now since

$$\left(F_r^\dagger\right)^2 = 0 \tag{4.243}$$

it follows that

$$| F_{lin} >= i \sum_{r=1}^{n} \sum_{r'=n+1}^{\infty} a_{rr'} F_1^\dagger F_2^\dagger \cdots F_{r'}^\dagger \cdots F_n^\dagger \mid 0 > \tag{4.244}$$

It follows from (4.32) and (4.33) that (4.244) can be rewritten as

$$| F_{lin} >= i \sum_{r=1}^{n} \sum_{r'=n+1}^{\infty} a_{rr'} F_{r'}^{\dagger} F_r \mid F > \tag{4.245}$$

Requiring the term in (4.241) which is linear in the $a_{rr'}$ to vanish thus yields

$$\mathrm{Re} < F \mid H F_{r'}^{\dagger} F_r \mid F >= 0 \tag{4.246}$$

for all

$$r \leq n < r' \tag{4.247}$$

Now[1]

$$F_r^{\dagger} \mid F >= F_{r'} \mid F >= 0 \tag{4.248}$$

$$< F \mid F_r =< F \mid F_{r'}^{\dagger} = 0 \tag{4.249}$$

It follows that (4.246) can be written as

$$< F \mid \left[F_{r'}^{\dagger}, H \right] F_r \mid F >= 0 \tag{4.250}$$

We evaluate the left side of (4.250) when H is given by (4.71). It follows using (C.2) and (C.10) that

$$\begin{aligned} \left[F_{r'}^{\dagger}, H \right] &= -\epsilon_{r'} F_{r'}^{\dagger} + \sum_{s=1}^{\infty} < s \mid \mathcal{V} \mid r' > F_s^{\dagger} \\ &+ \frac{1}{2} \sum_{s,t,u=1}^{\infty} \{< st \mid V \mid ur' > - < st \mid V \mid r'u >\} F_s^{\dagger} F_t^{\dagger} F_u \end{aligned} \tag{4.251}$$

[1] Unprimed and primed single-particle labels obey (4.247) in the following.

Using

$$< F \mid F_{r'}^{\dagger} F_r \mid F >= 0 \tag{4.252}$$

$$< F \mid F_s^{\dagger} F_r \mid F >= \delta_{rs} \tag{4.253}$$

$$< F \mid F_s^{\dagger} F_t^{\dagger} F_u F_r \mid F >= \delta_{rs}\delta_{tu} - \delta_{rt}\delta_{us} \tag{4.254}$$

it follows that

$$< F \mid \left[F_{r'}^{\dagger}, H \right] F_r \mid F >=$$

$$=< r \mid \mathcal{V} \mid r' > - \sum_{t=1}^{n} \left\{ < rt \mid V \mid r't > - < rt \mid V \mid tr' > \right\} \tag{4.255}$$

It follows that (4.250) is satisfied if

$$< r \mid \mathcal{V} \mid r' >= \sum_{t=1}^{n} \left\{ < rt \mid V \mid r't > - < rt \mid V \mid tr' > \right\} \tag{4.256}$$

Extending (4.256) to define all matrix elements of $\mathcal{V}$ completes the derivation of (4.82).

Derivation of (4.185) and (4.186)

It follows using (4.157) and (4.158) that

$$N_{pr} = v_r^2 + \cdots \tag{4.257}$$

$$N_{hr} = v_r^2 + \cdots \tag{4.258}$$

$$G_r^{\dagger} G_s = u_r v_r u_s v_s + \cdots \tag{4.259}$$

where $\cdots$ are terms for which $< G \mid \cdots \mid G >= 0$ and therefore

$$E_G = \sum_{r=1}^{n} \left(u_r^2 \epsilon_{hr} + v_r^2 \epsilon_{pr} - a_r u_r v_r \right) \tag{4.260}$$

where

$$a_r = \sum_{s=1}^{n} g_{rs} u_s v_s \tag{4.261}$$

Minimizing

$$E_G + \sum_{r=1}^{n} \lambda_r (u_r^2 + v_r^2) \tag{4.262}$$

with repect to $u_1, u_2, \cdots, u_n$ and $v_1, v_2, \cdots, v_n$ where $\lambda_1, \lambda_2, \cdots, \lambda_n$ are Lagrange multipliers yields

$$2(\epsilon_{hr} + \lambda_r)u_r - a_r v_r = 0 \tag{4.263}$$

$$2(\epsilon_{pr} + \lambda_r)v_r - a_r u_r = 0 \tag{4.264}$$

$$r = 1, 2, \cdots, n \tag{4.265}$$

It follows from (4.263) and (4.264) that

$$(\epsilon_{hr} + \lambda_r)u_r^2 = (\epsilon_{pr} + \lambda_r)v_r^2 \tag{4.266}$$

and therefore

$$u_r^2 = \frac{\epsilon_{pr} + \lambda_r}{\epsilon_{pr} + \epsilon_{hr} + 2\lambda_r} \tag{4.267}$$

$$v_r^2 = \frac{\epsilon_{hr} + \lambda_r}{\epsilon_{pr} + \epsilon_{hr} + 2\lambda_r} \tag{4.268}$$

We define ε_{0r} by

$$\varepsilon_{0r} = 2(\epsilon_{pr} + \lambda_r)^{\frac{1}{2}}(\epsilon_{hr} + \lambda_r)^{\frac{1}{2}} \tag{4.269}$$

It follows that

$$\epsilon_{pr} + \epsilon_{hr} + 2\lambda_r = \left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}} \tag{4.270}$$

where

$$\varepsilon_r = \epsilon_{pr} - \epsilon_{hr} \tag{4.271}$$

and therefore

$$u_r^2 = \frac{1}{2}\left[1 + \frac{\varepsilon_r}{\left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}}\right] \tag{4.272}$$

$$v_r^2 = \frac{1}{2}\left[1 - \frac{\varepsilon_r}{\left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}}\right] \tag{4.273}$$

$$2u_r v_r = \frac{\varepsilon_{0r}}{\left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}} \tag{4.274}$$

$$a_r = \varepsilon_{0r} \tag{4.275}$$

$$E_G - \epsilon_F = \frac{1}{2}\sum_{r=1}^{n}\left[\varepsilon_r - \left(\varepsilon_r^2 + \varepsilon_{0r}^2\right)^{\frac{1}{2}}\right] \tag{4.276}$$

It follows from (4.261), (4.274) and (4.275) that $\varepsilon_{01}, \varepsilon_{02}, \cdots, \varepsilon_{0n}$ are real parameters satisfying (4.188).

Chapter 5 FOCK SPACE FOR FERMIONS: PART 2

In Chapter 4 we considered a system of identical fermions with Fock space as the Hilbert space for the system. The fundamental dynamical variables are creators $F_r^\dagger$ (4.21) and annihilators F_r (4.22) which satisfy the fundamental algebra (4.32) and (4.33). The $F_r^\dagger$ and F_r are constructed using a denumerable set of vectors which span the one-fermion Hilbert space.

In this chapter we continue the description of a system of identical fermions with Fock space as the Hilbert space for the system by defining creators and annihilators constructed using nondenumerable sets of eigenkets which span the one-fermion Hilbert space.

In Section 5.1 we construct creators and annihilators using coordinate-space/spin eigenkets. This yields quantum field theory for fermions.[1]

In Sections 5.2 and 5.3 we construct creators and annihilators using momentum-space/spin and momentum-space/helicity eigenkets, respectively.

The Hamiltonian for a nonrelativistic system of fermions interacting via two-body potentials is written in terms of the various creators and annihilators in Section 5.4.

5.1 Quantum field operator

Definitions

The annihilator F_r (4.22) in fermion Fock space ${}^f\mathfrak{F}^s$ is defined in terms of

[1] We do not explicitly discuss relativistic quantum field theory. This topic is disussed in *QLB: Relativistic Quantum Field Theory*.

the denumerable set of vectors (3.1) which form an orthonormal basis for the Hilbert space for a one-fermion system with rest mass m and spin s.

The simultaneous eigenkets $\mid \vec{x} m_s >$ of the Cartesian position X^1, X^2, X^3 and the z-component of spin S^3 of a particle may also be used as an orthonormal basis for the Hilbert space for a one-fermion system with rest mass m and spin s. These eigenkets satisfy

$$1 = \sum_{m_s=-s}^{+s} \int d^3x \mid \vec{x} m_s >< \vec{x} m_s \mid \tag{5.1}$$

$$< \vec{x} m_s \mid \vec{y} m_s' >= \delta(\vec{x} - \vec{y}) \delta_{m_s m_s'} \tag{5.2}$$

The function

$$\phi_{r m_s}(\vec{x}) =< \vec{x} m_s \mid \phi_r > \tag{5.3}$$

is the coordinate-space/spin representative of $\mid \phi_r >$.

These functions satisfy

$$\sum_{m_s=-s}^{+s} \int d^3x \phi^*_{r m_s}(\vec{x}) \phi_{u m_s}(\vec{x}) = \delta_{ru} \tag{5.4}$$

$$\sum_{r=1}^{\infty} \phi^*_{r m_s}(\vec{x}) \phi_{r m_s'}(\vec{y}) = \delta(\vec{x} - \vec{y}) \delta_{m_s m_s'} \tag{5.5}$$

Furthermore,

$$| \vec{x} m_s >= \sum_{r=1}^{\infty} | \phi_r > \phi^*_{rm_s}(\vec{x}) \tag{5.6}$$

Accordingly, in ${}^f\maltese^s$ we define

$$F_{m_s}(\vec{x}) = \sum_{r=1}^{\infty} \phi_{rm_s}(\vec{x}) F_r \tag{5.7}$$

It follows from (4.28) and (4.29) that

$$F^{\dagger}_{m_s}(\vec{x}) \mid 0 >= (0, \mid \vec{x} m_s >, 0, \cdots) \tag{5.8}$$

$$F_{m_s}(\vec{x}) \mid 0 >= 0 \tag{5.9}$$

and from (5.4) and (5.5) that

$$F_r = \sum_{m_s=-s}^{+s} \int d^3x \phi^*_{rm_s}(\vec{x}) F_{m_s}(\vec{x}) \tag{5.10}$$

It follows from (4.32) and (4.33) that[1]

$$\{F_{m_s}(\vec{x}), F_{m'_s}(\vec{y})\} = 0 \tag{5.11}$$

$$\left\{F_{m_s}(\vec{x}), F^{\dagger}_{m'_s}(\vec{y})\right\} = \delta(\vec{x}-\vec{y})\delta_{m_s m'_s} \tag{5.12}$$

Comments

1. **Fundamental dynamical variables**

(5.7) defines fundamental dynamical variables $F_{m_s}(\vec{x})$ and $F^{\dagger}_{m_s}(\vec{x})$ for a system of identical fermions each with rest mass m and spin s.

(5.11) and (5.12) are a fundamental algebra for the system.

2. **Transformation equation**

(5.10) allow transformation from $F_{m_s}(\vec{x})$ to F_r given by (4.22).

3. **Quantum field theory**

$F_{m_s}(\vec{x})$ is labelled by the continuous variable $\vec{x}$.

$F_{m_s}(\vec{x})$ is a quantum field operator in the Schrodinger picture.

The description of a system of identical fermions using a quantum field as a fundamental dynamical variable is called quantum field theory of fermions.

4. **One-fermion state**

When acting on the vacuum state, $F^{\dagger}_{m_s}(\vec{x})$ creates an elementary fermion at

[1] Some commutators of products of fermion creators and annihilators are given in the Appendix.

position $\vec{x}$ with rest mass m, spin s and z-component of spin m_s.

The general one-fermion state at time t is

$$\mid \psi(t) >= \sum_{m_s=-s}^{+s} \int d^3x \psi_{m_s}(\vec{x},t) F^{\dagger}_{m_s}(\vec{x}) \mid 0 > \tag{5.13}$$

and

$$\mid \psi_{m_s}(\vec{x},t) \mid {}^2 d^3x \tag{5.14}$$

is the probability that the fermion is in the volume $dxdydz$ about the point (x, y, z) with z-component of spin m_s at time t.

$\psi_{m_s}(\vec{x}, t)$ is the coordinate-space/spin wave function of the fermion.

5. **Second quantization**

Definition (5.7) has led to the notion that the field operator $F_{m_s}(\vec{x})$ is given by a process of "second quantization" applied to the functions $\phi_{rm_s}(\vec{x})$.

The phrase "second quantization" arises because one describes the quantum mechanics of a single fermion in terms of $\phi_{rm_s}(\vec{x})$ representing the state of the particle then one uses the same function to construct operators appropriate for describing the quantum mechanics of an assembly of identical fermions.

"Second quantization" is misleading. It gives the impression that the description of a physical system using field theory is beyond quantum mechanics or that it requires a modification of quantum mechanics. It doesn't.

People say "second quantization" when they mean "the Hilbert space is Fock space".

One- and two-particle operators

The operators (4.45) and (4.54) on ${}^f\Psi^s$ are equal, respectively, to the one- and two-particle operators (4.47) and (4.56) on ${}^f\Psi^s_n$.

It follows using (5.10) that (4.45) may be written as

$$\boxed{A = \int d^3x d^3y F^\dagger(\vec{x}) A(\vec{x},\vec{y}) F(\vec{y}) \qquad (5.15)}$$

where $F(\vec{x})$ is the matrix

$$\boxed{F(\vec{x}) = \begin{pmatrix} F_s(\vec{x}) \\ F_{s-1}(\vec{x}) \\ \vdots \\ F_{-s}(\vec{x}) \end{pmatrix} \qquad (5.16)}$$

and $F^\dagger(\vec{x})$ is the corresponding $2s+1$ row matrix, and where $A(\vec{x},\vec{y})$ is a $2s+1$ by $2s+1$ square matrix of one-particle matrix elements

$$\boxed{(A(\vec{x},\vec{y}))_{m_s m_s'} =< \vec{x} m_s \mid A(\xi) \mid \vec{y} m_s' > \qquad (5.17)}$$

It follows using (5.10) that (4.54) may be written as

$$A = \int d^3x d^3y d^3x' d^3y' F^\dagger(\vec{x}) F^\dagger(\vec{y}) A\left(\vec{x}, \vec{y}, \vec{x}', \vec{y}'\right) F\left(\vec{y}'\right) F\left(\vec{x}'\right) \qquad (5.18)$$

where $A\left(\vec{x}, \vec{y}, \vec{x}'\vec{y}'\right)$ is a tensor of two-particle matrix elements

$$\left(A\left(\vec{x}, \vec{y}, \vec{x}', \vec{y}'\right)\right)_{m_1 m_2 m_1' m_2'} = \qquad (5.19)$$
$$\underset{1}{<} \vec{x} m_1 \mid \underset{2}{<} \vec{y} m_2 \mid A(\xi_1, \xi_2) \mid \vec{x}' m_1' \underset{1}{>} \mid \vec{y}' m_2' \underset{2}{>}$$

5.2 Momentum/spin operator

Definitions

In Section 5.1 we used the simultaneous eigenkets $\mid \vec{x} m_s >$ of the Cartesian position X^1, X^2, X^3 and the z-component of spin S^3 of a particle to define the annihilation operator $F_{m_s}(\vec{x})$ (5.7).

The simultaneous eigenkets $\mid \vec{p} m_s >$ of the Cartesian momentum P^1, P^2, P^3 and the z-component of spin S^3 of a particle may also be used as an orthonormal basis for the Hilbert space for a one-fermion system with rest mass m and spin

s. These eigenkets satisfy

$$1 = \sum_{m_s=-s}^{+s} \int d^3p \mid \vec{p}m_s >< \vec{p}m_s \mid \tag{5.20}$$

$$< \vec{p}m_s \mid \vec{q}m'_s >= \delta(\vec{p}-\vec{q})\delta_{m_s m'_s} \tag{5.21}$$

Furthermore,

$$\mid \vec{p}m_s >= \sum_{m'_s=-s}^{+s} \int d^3x \mid \vec{x}m'_s >< \vec{x}m'_s \mid \vec{p}m_s > \tag{5.22}$$

where

$$< \vec{x}m_s \mid \vec{p}m'_s >= \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar}\delta_{m_s m'_s} \tag{5.23}$$

Accordingly, in ${}^f\maltese^s$ we define

$$F_{m_s}(\vec{p}) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3x e^{-i\vec{p}\cdot\vec{x}/\hbar} F_{m_s}(\vec{x}) \tag{5.24}$$

It follows using from (5.8) and (5.9) that

$$F^{\dagger}_{m_s}(\vec{p}) \mid 0 >= (0, \mid \vec{p} m_s >, 0, \cdots) \tag{5.25}$$

$$F_{m_s}(\vec{p}) \mid 0 >= 0 \tag{5.26}$$

and from (5.24) that

$$F_{m_s}(\vec{x}) = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} \int d^3 p e^{i\vec{p}\cdot\vec{x}/\hbar} F_{m_s}(\vec{p}) \tag{5.27}$$

It follows from (5.11) and (5.12) that[2]

$$\{F_{m_s}(\vec{p}), F_{m'_s}(\vec{q})\} = 0 \tag{5.28}$$

$$\left\{F_{m_s}(\vec{p}), F^{\dagger}_{m'_s}(\vec{q})\right\} = \delta(\vec{p} - \vec{q})\delta_{m_s m'_s} \tag{5.29}$$

Also,

$$\left\{F_{m_s}(\vec{x}), F^{\dagger}_{m'_s}(\vec{p})\right\} = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar}\delta_{m_s m'_s} \tag{5.30}$$

2 Some commutators of products of fermion creators and annihilators are given in the Appendix.

Comments

1. **Fundamental dynamical variables**

(5.24) defines fundamental dynamical variables $F_{m_s}(\vec{p})$ and $F^{\dagger}_{m_s}(\vec{p})$ for a system of identical fermions each with rest mass m and spin s.

(5.28) and (5.29) are a fundamental algebra for the system.

2. **Transformation equation**

(5.27) allow transformation from $F_{m_s}(\vec{p})$ to $F_{m_s}(\vec{x})$ given by (5.7).

3. **One-fermion state**

When acting on the vacuum state, $F^{\dagger}_{m_s}(\vec{p})$ creates an elementary fermion with momentum $\vec{p}$, rest mass m, spin s and z-component of spin m_s.

The general one-fermion state at time t is

$$\boxed{\mid \psi(t) >= \sum_{m_s=-s}^{+s} \int d^3p \psi_{m_s}(\vec{p},t) F^{\dagger}_{m_s}(\vec{p}) \mid 0 > \qquad (5.31)}$$

and

$$\boxed{\mid \psi_{m_s}(\vec{p},t) \mid {}^2 d^3p \qquad (5.32)}$$

is the probability that the fermion has momentum in the volume $dp^1dp^2dp^3$ about (p^1,p^2,p^3) with z-component of spin m_s at time t.

$\psi_{m_s}(\vec{p},t)$ is the momentum-space/spin wave function for the fermion.

One- and two-particle operators

The operators (5.15) and (5.18) on ${}^f\maltese^s$ are equal, respectively, to the one- and two-particle operators (4.47) and (4.56) on ${}^f\maltese^s_n$.

It follows using (5.27) that (5.15) can be written as

$$\boxed{A = \int d^3p d^3q F^\dagger(\vec{p}) A(\vec{p}, \vec{q}) F(\vec{q}) \qquad (5.33)}$$

where $F(\vec{p})$ is the matrix

$$\boxed{F(\vec{p}) = \begin{pmatrix} F_s(\vec{p}) \\ F_{s-1}(\vec{p}) \\ \vdots \\ F_{-s}(\vec{p}) \end{pmatrix} \qquad (5.34)}$$

and $F^\dagger(\vec{p})$ is the corresponding $2s + 1$ row matrix, and where $A(\vec{p}, \vec{q})$ is a $2s + 1$ by $2s + 1$ square matrix of one-particle matrix elements

$$\boxed{(A(\vec{p}, \vec{q}))_{m_s m_s'} =< \vec{p} m_s \mid A(\xi) \mid \vec{q} m_s' > \qquad (5.35)}$$

It follows using (5.27) that (5.18) can be written as

$$A = \int d^3p d^3q d^3p' d^3q' F^\dagger(\vec{p}) F^\dagger(\vec{q}) A\left(\vec{p}, \vec{q}, \vec{p}', \vec{q}'\right) F\left(\vec{q}'\right) F\left(\vec{p}'\right) \tag{5.36}$$

where $A\left(\vec{p}, \vec{q}, \vec{p}', \vec{q}'\right)$ is a tensor of two-particle matrix elements

$$\begin{gathered}\left(A\left(\vec{p}, \vec{q}, \vec{p}', \vec{q}'\right)\right)_{m_1 m_2 m_1' m_2'} = \\ \underset{1}{<} \vec{p} m_1 \mid \underset{2}{<} \vec{q} m_2 \mid A(\xi_1, \xi_2) \mid \vec{p}' m_1' \underset{1}{>} \mid \vec{q}' m_2' \underset{2}{>}\end{gathered} \tag{5.37}$$

5.3 Momentum/helicity operator

Definitions

In Section 5.2 we used the simultaneous eigenkets $\mid \vec{p} m_s >$ of the Cartesian momentum P^1, P^2, P^3 and of the z-component of spin S^3 of a particle to define the annihilation operator $F_{m_s}(\vec{p})$ (5.24).

The simultaneous eigenkets $\mid h^\lambda(\vec{p}) >$ of the Cartesian momentum P^1, P^2, P^3 and the helicity Λ may be used as an orthonormal basis for the Hilbert space for a one-fermion system with rest mass m and spin s. These eigenkets satisfy

$$1 = \sum_{\lambda=-s}^{+s} \int d^3p \mid h^\lambda(\vec{p}) >< h^\lambda(\vec{p}) \mid \tag{5.38}$$

$$< h^\lambda(\vec{p}) \mid h^{\lambda'}(\vec{q}) >= \delta(\vec{p} - \vec{q})\delta_{\lambda\lambda'} \tag{5.39}$$

Furthermore,

$$\mid h^\lambda(\vec{p}) >= \sum_{m_s=-s}^{+s} \int d^3q \mid \vec{q} m_s >< \vec{q} m_s \mid h^\lambda(\vec{p}) > \tag{5.40}$$

where

$$< \vec{q} m_s \mid h^\lambda(\vec{p}) >= \delta(\vec{p} - \vec{q}) D^s_{m_s\lambda}(\varphi, \theta, 0) \tag{5.41}$$

where (θ, φ) are the spherical polar coordinates of $\vec{p}$ and $D^j_{m'm}(\alpha, \beta, \gamma)$ is the matrix element of a rotation matrix. (We follow the convention of Rose (1957) for rotation matrices.)

Accordingly, in ${}^{f}\maltese^{s}$, we define

$$F^\lambda(\vec{p}) = \sum_{m_s=-s}^{+s} D^{s*}_{m_s\lambda}(\varphi, \theta, 0) F_{m_s}(\vec{p}) \tag{5.42}$$

It follows from (5.25) and (5.26) that

$$F^{\lambda\dagger}(\vec{p}) \mid 0 >= \Big(0, \mid h^{\lambda}(\vec{p}) >, 0, \cdots\Big) \tag{5.43}$$

$$F^{\lambda}(\vec{p}) \mid 0 >= 0 \tag{5.44}$$

and from (5.42) that

$$F_{m_s}(\vec{p}) = \sum_{\lambda=-s}^{+s} D^{s}_{m_s\lambda}(\varphi, \theta, 0) F^{\lambda}(\vec{p}) \tag{5.45}$$

It follows from (5.28) and (5.29) that[1]

$$\Big\{F^{\lambda}(\vec{p}), F^{\lambda'}(\vec{q})\Big\} = 0 \tag{5.46}$$

$$\Big\{F^{\lambda}(\vec{p}), F^{\lambda'\dagger}(\vec{q})\Big\} = \delta(\vec{p} - \vec{q})\delta_{\lambda\lambda'} \tag{5.47}$$

Also,

[1] Some commutators of products of fermion creators and annihilators are given in the Appendix.

$$\left\{F_{m_s}(\vec{x}), F^{\lambda\dagger}(\vec{p})\right\} = \left(\frac{1}{2\pi\hbar}\right)^{\frac{3}{2}} e^{i\vec{p}\cdot\vec{x}/\hbar} D^s_{m_s\lambda}(\varphi,\theta,0) \qquad (5.48)$$

$$\left\{F_{m_s}(\vec{p}), F^{\lambda\dagger}(\vec{q})\right\} = \delta(\vec{p}-\vec{q}) D^s_{m_s\lambda}(\varphi,\theta,0) \qquad (5.49)$$

Comments

1. **Fundamental dynamical variables**

(5.42) defines fundamental dynamical variables $F^{\lambda}(\vec{p})$ and $F^{\lambda\dagger}(\vec{p})$ for a system of identical fermions each with rest mass m and spin s.

(5.46) and (5.47) are a fundamental algebra for the system.

2. **Transformation equation**

(5.45) allow transformation from $F^{\lambda}(\vec{p})$ to $F_{m_s}(\vec{p})$ given by (5.24).

3. **One-fermion state**

When acting on the vacuum state, $F^{\lambda\dagger}(\vec{p})$ creates an elementary fermion with momentum $\vec{p}$, rest mass m, spin s and helicity λ.

The general one-fermion state at time t is

$$\mid \psi(t) >= \sum_{\lambda=-s}^{+s} \int d^3p \psi^{\lambda}(\vec{p},t) F^{\lambda\dagger}(\vec{p}) \mid 0 > \qquad (5.50)$$

and

$$| \psi^\lambda(\vec{p}, t) |^2 d^3p \tag{5.51}$$

is the probability that the fermion has momentum in the volume $dp^1 dp^2 dp^3$ about (p^1, p^2, p^3) with helicity λ at time t.

$\psi^\lambda(\vec{p}, t)$ is the momentum-space/helicity wave function for the fermion.

For a fermion with zero rest mass which is described by a Hamiltonian which is not invariant under space inversion

$$\psi^\lambda(\vec{p}, t) = 0 \quad \text{unless} \quad \text{either} \quad \lambda = +s \quad \text{or} \quad \lambda = -s \tag{5.52}$$

One- and two-particle operators

The operators (5.33) and (5.36) on ${}^f\Psi^s$ are equal, respectively, to the one- and two-particle operators (4.47) and (4.56) on ${}^f\Psi^s_n$.

It follows using (5.45) that (5.33) can be written as

$$A = \int d^3p d^3q \mathcal{F}^\dagger(\vec{p}) \mathcal{A}(\vec{p}, \vec{q}) \mathcal{F}(\vec{q}) \tag{5.53}$$

where $\mathcal{F}(\vec{p})$ is the matrix

$$
\mathcal{F}(\vec{p}) = \begin{pmatrix} F^s(\vec{p}) \\ F^{s-1}(\vec{p}) \\ \vdots \\ F^{-s}(\vec{p}) \end{pmatrix} \tag{5.54}
$$

and $\mathcal{F}^\dagger(\vec{p})$ is the corresponding $2s+1$ row matrix, and where $\mathcal{A}(\vec{p},\vec{q})$ is the $2s+1$ by $2s+1$ square matrix of one-particle matrix elements

$$
(\mathcal{A}(\vec{p},\vec{q}))_{\lambda\lambda'} =< h^\lambda(\vec{p}) \mid A(\xi) \mid h^{\lambda'}(\vec{q}) > \tag{5.55}
$$

It follows using (5.45) that (5.36) can be written as

$$
A = \int d^3p d^3q d^3p' d^3q' \mathcal{F}^\dagger(\vec{p})\mathcal{F}^\dagger(\vec{q})\mathcal{A}\left(\vec{p},\vec{q},\vec{p'},\vec{q'}\right)\mathcal{F}\left(\vec{q'}\right)\mathcal{F}\left(\vec{p'}\right) \tag{5.56}
$$

where $\mathcal{A}\left(\vec{p},\vec{q},\vec{p'},\vec{q'}\right)$ is a tensor of two-particle matrix elements

$$
\begin{aligned}
&\left(\mathcal{A}\left(\vec{p},\vec{q},\vec{p'},\vec{q'}\right)\right)_{\lambda_1\lambda_2\lambda_1'\lambda_2'} \\
&=\underset{1}{<} h^{\lambda_1}(\vec{p}) \mid \underset{2}{<} h^{\lambda_2}(\vec{q}) \mid A(\xi_1,\xi_2) \mid h^{\lambda_1'}\left(\vec{p'}\right) \underset{1}{>} \mid h^{\lambda_2'}\left(\vec{q'}\right) \underset{2}{>}
\end{aligned} \tag{5.57}
$$

5.4 Hamiltonian for interacting fermions

As in Section 4.5 we consider a system of identical nonrelativistic fermions each with rest mass m and spin s interacting with each other via two-body potentials. For simplicity here we assume that there are no external potentials acting on the system and we assume that the two-body potentials are central and spin-independent.

The Hamiltonian for the system on ${}^f\maltese_n^s$ is given by (4.64) with $U(\xi_\alpha) = 0$ and

$$V(\xi_\alpha, \xi_\beta) = V\Big(\mid \vec{X}_\alpha - \vec{X}_\beta \mid\Big) \tag{5.58}$$

It follows from (5.15) and (5.18) that the Hamiltonian for system on ${}^f\maltese^s$ is

$$H = T + V \tag{5.59}$$

where

$$T = -\frac{\hbar^2}{2m} \int d^3x F^\dagger(\vec{x}) \nabla^2 F(\vec{x}) \tag{5.60}$$

and

$$V = \frac{1}{2}\int d^3r d^3R$$
$$V(r)F^\dagger\left(\vec{R}+\frac{1}{2}\vec{r}\right)F^\dagger\left(\vec{R}-\frac{1}{2}\vec{r}\right)F\left(\vec{R}-\frac{1}{2}\vec{r}\right)F\left(\vec{R}+\frac{1}{2}\vec{r}\right) \tag{5.61}$$

The variables $\vec{R}$ and $\vec{r}$ in (5.61) are the centre-of-mass position and relative position, respectively, of the interacting pair.

It follows from (5.33) and (5.36) that (5.60) and (5.61) may be expressed alternatively as

$$T = \int d^3p \frac{p^2}{2m} F^\dagger(\vec{p})F(\vec{p}) \tag{5.62}$$

and

$$V = \frac{1}{2}\int d^3k d^3k' d^3K$$
$$V\left(\mid \vec{k'} - \vec{k} \mid\right)F^\dagger\left(\frac{1}{2}\vec{K}+\vec{k}\right)F^\dagger\left(\frac{1}{2}\vec{K}-\vec{k}\right)F\left(\frac{1}{2}\vec{K}-\vec{k'}\right)F\left(\frac{1}{2}\vec{K}+\vec{k'}\right) \tag{5.63}$$

where

$$V\left(\mid \vec{k'} - \vec{k} \mid\right) = \left(\frac{1}{2\pi\hbar}\right)^3 \int d^3r e^{i(\vec{k'}-\vec{k})\cdot\vec{r}/\hbar} V(r) \tag{5.64}$$

The variable $\vec{K}$ in (5.63) is the total momentum of the interacting pair and $\vec{k'}$ and $\vec{k}$ are, respectively, the relative momenta of the interacting pair which is annihilated and created. $\vec{k'} - \vec{k}$ is the relative momentum transferred in the two-body interaction.

It follows from (5.53) and (5.56) that (5.62) and (5.63) may be expressed alternatively as

$$\boxed{T = \int d^3p \frac{p^2}{2m} \mathcal{F}^\dagger(\vec{p})\mathcal{F}(\vec{p}) \qquad (5.65)}$$

and

$$\boxed{\begin{gathered} V = \frac{1}{2}\int d^3k d^3k' d^3K \\ V\left(\mid \vec{k'} - \vec{k} \mid\right)\mathcal{F}^\dagger\left(\frac{1}{2}\vec{K} + \vec{k}\right)\mathcal{F}^\dagger\left(\frac{1}{2}\vec{K} - \vec{k}\right)\mathcal{F}\left(\frac{1}{2}\vec{K} - \vec{k'}\right)\mathcal{F}\left(\frac{1}{2}\vec{K} + \vec{k'}\right) \\ (5.66) \end{gathered}}$$

Comments

1. **Generalization**

(5.59) describes a nonrelativistic system of fermions interacting via central spin-independent two-body potentials given by (5.61).

(5.59) can be generalized to describe a nonrelativistic system of fermions interacting via noncentral spin-dependent two-body potentials.

Hsieh (1979) contains such a generalization for a system of nucleons.

2. **Creating a physical fermion**

It follows from (5.59) that

$$HF_{m_s}^{\dagger}(p) \mid 0 >= \frac{p^2}{2m} F_{m_s}^{\dagger}(p) \mid 0 > \tag{5.67}$$

$F_{m_s}^{\dagger}(p) \mid 0 >$ is an eigenket of the Hamiltonian for the interacting system.

When acting on the vacuum state, $F_{m_s}^{\dagger}(p)$ creates a physical fermion with mass m, spin s, z-component of spin m_s, momentum p and energy $p^2/2m$.

3. **Physical particles and elementary particles**

The elementary particle of the theory is a physical particle when the Hamiltonian is given by (5.59).

This is not surprising. For example, (5.59) and its generalization to include spin-dependent two-body interactions describe nuclear systems for energies below the threshold for producing pions. The elementary particles of the theory are physical nucleons: one says "Nuclei are composed of physical nucleons."

4. **Form of the interaction**

There is no contribution from (5.63) to the right side of (5.67) because (5.63) contains two annihilators.

It follows that

$$VF_{m_s}^{\dagger}(p) \mid 0 >= 0 \tag{5.68}$$

Elementary particles are physical particles for every V for which (5.68) holds.

In Chapter 7 we consider interaction in a system of fermions and bosons for which (5.68) does not hold.

For such a system, the elementary particles are not physical particles.

Chapter 6 FOCK SPACE FOR BOSONS

In this chapter we give the Fock space description of a system of bosons.

The system of bosons may be any of those listed at the beginning of Chapter 3. That is, it may, for example, be a system of

- photons characterizing an electromagnetic field[1]
- phonons characterizing the lattice vibrations of a crystal
- pions or kaons created in collisions of nuclear projectiles
- gluons in nuclear matter.

Whatever the system, each boson has integral spin and all states of the system are symmetric under permutation of the particles.

The Fock space description of a boson system is analogous to the Fock space description of a fermion system given in Chapters 4 and 5.

Fock space and creators and annihilators for bosons are defined in Sections 6.1 and 6.2. These operators are defined in terms of a denumerable set of vectors which form an orthonormal basis for the Hilbert space for a one-boson system.

Creators and annihilators labelled by a continuous variable are defined in Sections 6.3 and 6.4.[2]

---

1 Photons and Maxwell's equations as relativistic quantum field equations are discussed in detail in *QLB: Relativistic Quantum Field Theory*.

2 We do not explicitly discuss relativistic quantum field theory. This topic is disussed in*QLB: Relativistic Quantum Field Theory*.

6.1 Boson Fock space defined

1. Let

$$\psi = (\psi_0, \psi_1, \cdots, \psi_n, \cdots) \tag{6.1}$$

where ψ_n is a vector in the n-boson Hilbert space ${}^{b}\maltese_n^s$ (2.28).[1]

Each ψ_n is of the form (3.31). ψ_n is the component of ψ in ${}^{b}\maltese_n^s$.

2. Addition of ψ and $\chi = (\chi_0, \chi_1, \cdots, \chi_n, \cdots)$ is defined as

$$\psi + \chi = (\psi_0 + \chi_0, \psi_1 + \chi_1, \cdots, \psi_n + \chi_n, \cdots) \tag{6.2}$$

3. Multiplication of ψ by a scalar a is defined as

$$a\psi = (a\psi_0, a\psi_1, \cdots, a\psi_n, \cdots) \tag{6.3}$$

4. The scalar product of ψ and χ is defined as

$$(\psi, \chi) = \sum_{n=0}^{\infty} (\psi_n, \chi_n) \tag{6.4}$$

It is required that $(\psi, \psi) < \infty$ for all ψ.

5. The set of elements ψ is a separable Hilbert space.

Comments

1. **Boson Fock space ${}^{b}\maltese^s$**

The above Hilbert space is called boson Fock space. It will be denoted by ${}^{b}\maltese^s$.

---

1 ${}^{b}\maltese_0^s$ is defined in item 4 of the Comments list.

$^{b}\maltese^{s}$ is the direct sum of the Hilbert spaces $^{b}\maltese_{n}^{s}$ (2.28) for all n. That is,

$$\boxed{\quad {}^{b}\maltese^{s} = {}^{b}\maltese_{0}^{s} \oplus {}^{b}\maltese_{1}^{s} \oplus \cdots \oplus {}^{b}\maltese_{n}^{s} \oplus \cdots \qquad (6.5)\quad}$$

$\oplus$ denotes direct sum.

2. **<u>States of the system</u>**

The unit norm vectors (6.1) in $^{b}\maltese^{s}$ correspond to states of the system.

The probability P_n that the system has n bosons in it is

$$\boxed{\quad P_n = < \psi_n \mid \psi_n > \qquad (6.6)\quad}$$

and

$$\boxed{\quad \sum_{n=0}^{\infty} P_n = 1 \qquad (6.7)\quad}$$

3. **<u>Components of ψ</u>**

There is no conservation law for the number of bosons in a physical system.

ψ can have any number of nonzero components.

4. **<u>Hilbert space $^{b}\maltese_{0}^{s}$</u>**

${}^b\mathfrak{H}^s_0$ is defined to be a one-dimensional space.

The unit norm vector spanning ${}^b\mathfrak{H}^s_0$ is labelled $\mid 0[00\cdots] \underset{0}{>}$.

5. Basis vectors for ${}^b\mathfrak{H}^s$

A basis for ${}^b\mathfrak{H}^s$ is the set of vectors

$$\boxed{\mid n[n_1 n_2 \cdots] > \qquad (6.8)}$$

defined by

$$\mid 0[00\cdots] >= \left(\mid 0[00\cdots] \underset{0}{>}, 0, \cdots\right) \qquad (6.9)$$

$$\mid 1[n_1 n_2\cdots] >= \left(0, \mid 1[n_1 n_2\cdots] \underset{1}{>}, 0, \cdots\right) \qquad (6.10)$$

$$\mid 2[n_1 n_2\cdots] >= \left(0, 0, \mid 2[n_1 n_2\cdots] \underset{2}{>}, 0, \cdots\right) \qquad (6.11)$$

$$\vdots$$

where

$$\boxed{\mid n[n_1 n_2 \cdots] \underset{n}{>} \qquad (6.13)}$$

for all $n = 1, 2, \cdots$ is the symmetric determinant (3.23).

Then

$$\sum_{nn_1n_2\cdots}^{b} \mid n[n_1n_2\cdots] >< n[n_1n_2\cdots] \mid = 1 \tag{6.14}$$

where

$$\sum_{nn_1n_2\cdots}^{b} = \sum_{n=0}^{\infty} \sum_{n_1n_2\cdots}^{b} \tag{6.15}$$

where the second summation on the right side is defined by (3.28).

Furthermore,

$$< n[n_1n_2\cdots] \mid n'[n_1'n_2'\cdots] >= \delta_{nn'}\delta_{n_1n_1'}\delta_{n_2n_2'}\cdots \tag{6.16}$$

6. <u>Vacuum state</u>

(6.9) is the vacuum state of the system. It will be denoted by $\mid 0 >$. That is,

$$\mid 0 >=\mid 0[00\cdots] > \tag{6.17}$$

7. **General state of the system**

The general state of the system has the form

$$\boxed{\mid \psi(t) >= \sum_{nn_1n_2\cdots}^{b} \mid n[n_1n_2\cdots] >< n[n_1n_2\cdots] \mid \psi(t) > \qquad (6.18)}$$

and

$$\boxed{< n[n_1n_2\cdots] \mid \psi(t) > \qquad (6.19)}$$

is the probability amplitude that at time t there are n bosons in the system with n_1 bosons occupying the single-particle state $\mid \beta_1 >$, and n_2 bosons occupying the single-particle state $\mid \beta_2 >$, and so on.

6.2 Creators and annihilators

We define boson creation and annihilation operators in this section. These operators are fundamental dynamical variables for a system of identical bosons. They obey commutation relations.

As with the fermion case, introduction of creation and annihilation operators yields intuitive and elegant expressions for observables and basis states.

Definitions

For each $r = 1, 2, \cdots$, we define

$$\boxed{B_r^\dagger = \sum_{nn_1n_2\cdots}^{b} \mid n+1[n_1n_2\cdots n_r+1\cdots] > \sqrt{n_r+1} < n[n_1n_2\cdots n_r\cdots] \mid \qquad (6.20)}$$

from which

$$\boxed{B_r = \sum_{nn_1n_2\cdots}^{b} \mid n[n_1n_2\cdots n_r\cdots] > \sqrt{n_r+1} < n+1[n_1n_2\cdots n_r+1\cdots] \mid \qquad (6.21)}$$

where $\mid n[n_1n_2\cdots] >$ is the basis vector (6.8) in ${}^{b}\maltese^{s}$.

It follows from (6.20) and (6.21) that

$$B_r^\dagger \mid n[n_1n_2\cdots n_r\cdots] >= \sqrt{n_r+1} \mid n+1\{n_1n_2\cdots n_r+1\cdots\} > \qquad (6.22)$$

$$B_r \mid n+1[n_1n_2\cdots n_r+1\cdots] >= \sqrt{n_r+1} \mid n[n_1n_2\cdots n_r\cdots] > \qquad (6.23)$$

$$B_r^\dagger \mid 0 >= \left(0, \mid \beta_r \underset{1}{>}, 0, \cdots\right) \tag{6.24}$$

$$B_r \mid 0 >= 0 \tag{6.25}$$

Comments

1. **Boson creator**

   $B_r^\dagger$ is a boson creation operator or boson creator.

   When acting on an n-boson basis vector (6.8) with n_r particles occupying single-particle state $\mid \beta_r >$, $B_r^\dagger$ yields an $n+1$-boson basis vector (6.8) with $n_r + 1$ particles occupying single-particle state $\mid \beta_r >$.

2. **Boson annihilator**

   B_r is a boson annihilation operator or boson annihilator.

   When acting on an $n+1$-boson basis state (6.8) with n_r+1 particles occupying single-particle state $\mid \beta_r >$, B_r yields an n-boson basis state (6.8) with n_r particles occupying single-particle state $\mid \beta_r >$.

3. **Creating an elementary boson**

   When acting on the vacuum state, $B_r^\dagger$ creates an elementary boson with rest mass m and spin s in single-particle state $\mid \beta_r >$.

4. **One-boson state**

   The general one-boson state at time t is

$$| \psi(t) >= \sum_{r=1}^{\infty} \psi_r(t) B_r^\dagger \mid 0 > \tag{6.26}$$

where

$$\psi_r(t) =< 0 \mid B_r \psi(t) > \tag{6.27}$$

is the probability amplitude that the boson is in the state $\mid \beta_r >$ at time t.

Commutation relations

It follows from (6.20) and (6.21) that[1]

$$[B_r, B_s] = 0 \tag{6.28}$$

$$\left[B_r, B_s^\dagger\right] = \delta_{rs} \tag{6.29}$$

where $[A.B]$ is the commutator of A and B.[2]

Basis vectors

It follows from (6.22) that the basis vector (6.8) may be expressed as n creators acting on the vacuum state. That is,

1 The proof of (6.28) and (6.29) is similar to the proof of (4.32) and (4.33).

2 Some commutators of products of boson creators and annihilators are given in the Appendix.

$$| n[n_1 n_2 \cdots] >= \frac{1}{\sqrt{n_1! n_2! \cdots}} \left(B_1^\dagger\right)^{n_1} \left(B_2^\dagger\right)^{n_2} \cdots | 0 > \qquad (6.30)$$

where

$$\sum_{r=1}^{\infty} n_r = n \qquad (6.31)$$

The proof of (6.30) follows on evaluating the right side using (6.22).

Comments

1. **Form of the basis vector**

   (6.30) is a compact, intuitive and elegant expression for the basis vector (6.8).

2. **Manifest symmetry**

   In view of (6.28), (6.30) is unchanged when any two particle labels are interchanged.

   (6.30) is manifestly symmetric under particle interchange.

3. **Fundamental dynamical variables**

   Each basis vector (6.8) can been expressed in terms of boson creators acting on the vacuum state. The set of creators and annihilators defined by (6.20) and (6.21) is a set of fundamental dynamical variables for a system of identical bosons.

Commutation relations (6.28) and (6.29) are a fundamental algebra for the system.

6.3 Quantum field operator

Definitions

In Section 5.1 we defined the fermion field operator $F_{m_s}(x)$ (5.7) in terms of the coordinate-space/spin representatives of a denumerable set of vectors which form an orthonormal basis for the Hilbert space for a one-fermion system.

We follow an identical procedure in this section to define the boson field operator $B_{m_s}(x)$ by

$$\boxed{B_{m_s}(\vec{x}) = \sum_{r=1}^{\infty} \beta_{rm_s}(\vec{x}) B_r \qquad (6.32)}$$

where

$$\boxed{\beta_{rm_s}(\vec{x}) =< \vec{x} m_s \mid \beta_r > \qquad (6.33)}$$

is the coordinate-space/spin representative of $\mid \beta_r >$ (3.17). These functions satisfy

$$\sum_{m_s=-s}^{+s} \int d^3x \beta^*_{rm_s}(\vec{x})\beta_{um_s}(\vec{x}) = \delta_{ru} \tag{6.34}$$

$$\sum_{r=1}^{\infty} \beta^*_{rm_s}(\vec{x})\beta_{rm'_s}(\vec{y}) = \delta(\vec{x}-\vec{y})\delta_{m_s m'_s} \tag{6.35}$$

It follows from (6.24) and (6.25) that

$$B^\dagger_{m_s}(\vec{x}) \mid 0 >= \left(0, \mid \vec{x}m_s \underset{1}{>}, 0, \cdots\right) \tag{6.36}$$

$$B_{m_s}(\vec{x}) \mid 0 >= 0 \tag{6.37}$$

and it follows using (6.34) and (6.35) that

$$B_r = \sum_{m_s=-s}^{+s} \int d^3x \beta^*_{rm_s}(\vec{x}) B_{m_s}(\vec{x}) \tag{6.38}$$

It follows from (6.28) and (6.29) that[1]

$$\left[B_{m_s}(\vec{x}), B_{m'_s}(\vec{y})\right] = 0 \tag{6.39}$$

$$\left[B_{m_s}(\vec{x}), B^{\dagger}_{m'_s}(\vec{y})\right] = \delta(\vec{x} - \vec{y})\delta_{m_s m'_s} \tag{6.40}$$

Comments

1. **Fundamental dynamical variables**

   (6.32) defines fundamental dynamical variables $B_{m_s}(\vec{x})$ and $B^{\dagger}_{m_s}(\vec{x})$ for a system of identical bosons each with rest mass m and spin s.

   (6.39) and (6.40) are a fundamental algebra for the system.

2. **Transformation equation**

   (6.38) allow transformation from $B_{m_s}(\vec{x})$ to B_r given by (6.21).

3. **Quantum field theory**

   $B_{m_s}(\vec{x})$ is labelled by the continuous variable $\vec{x}$.

   $B_{m_s}(\vec{x})$ is a quantum field operator in the Schrodinger picture.

   The description of a system of identical bosons using a quantum field as a fundamental dynamical variable is called quantum field theory of bosons.

4. **One-boson state**

   When acting on the vacuum state, $B^{\dagger}_{m_s}(\vec{x})$ creates an elementary boson at

[1] Some commutators of products of boson creators and annihilators are given in the Appendix.

position $\vec{x}$ with rest mass m, spin s and z–component of spin m_s.

The general one-boson state at time t is

$$\mid \psi(t) >= \sum_{m_s=-s}^{+s} \int d^3x \psi_{m_s}(\vec{x},t) B^{\dagger}_{m_s}(\vec{x}) \mid 0 > \tag{6.41}$$

and

$$\mid \psi_{m_s}(\vec{x},t) \mid {}^2 d^3x \tag{6.42}$$

is the probability that the boson is in the volume $dxdydz$ about the point (x, y, z) with z–component of spin m_s at time t

$\psi_{m_s}(\vec{x},t)$ is the coordinate-space/spin wave function of the boson.

One- and two-particle operators

The operators (4.45) and (4.54) on ${}^f\maltese^s$ are equal, respectively, to the one- and two-particle operators (4.47) and (4.56) on ${}^f\maltese^s_n$. (5.15) and (5.18) give these operators in terms of fermion field operators. Similar equations hold for a system of bosons.

The expression on ${}^b\maltese^s$ corresponding to (5.15) is

$$A = \int d^3x d^3y B^{\dagger}(\vec{x}) A(\vec{x},\vec{y}) B(\vec{y}) \tag{6.43}$$

where

$$B(\vec{x}) = \begin{pmatrix} B_s(\vec{x}) \\ B_{s-1}(\vec{x}) \\ \vdots \\ B_{-s}(\vec{x}) \end{pmatrix} \tag{6.44}$$

and $B^\dagger(\vec{x})$ is the corresponding $2s+1$ row matrix and where $A(\vec{x},\vec{y})$ is a $2s+1$ by $2s+1$ square matrix of one-particle matrix elements (5.17)..

The expression on boson Fock space ${}^b\mathfrak{F}^s$ corresponding to (5.18) is

$$A = \int d^3x d^3y d^3x' d^3y' B^\dagger(\vec{x}) B^\dagger(\vec{y}) A\left(\vec{x},\vec{y},\vec{x}',\vec{y}'\right) B\left(\vec{y}'\right) B\left(\vec{x}'\right) \tag{6.45}$$

where $A\left(\vec{x},\vec{y},\vec{x}'\vec{y}'\right)$ is a tensor of two-particle matrix elements (5.19).

6.4 Momentum/spin and momentum/helicity operators

In Sections 5.2 and 5.3 we defined fermion operators $F_{m_s}(\vec{p})$ (5.24) and $F^\lambda(\vec{p})$ (5.42) in terms of $F_{m_s}(\vec{x})$ (5.7). An identical procedure is used to define boson operators $B_{m_s}(\vec{p})$ and $B^\lambda(\vec{p})$ in terms of $B_{m_s}(\vec{x})$ (6.32).

Indeed, equations for bosons are identical to equations in Sections 5.2 and 5.3 but with F replaced by B and anticommutors replaced by commutators.

Chapter 7 FOCK SPACE FOR FERMIONS AND BOSONS

In this chapter we give a Fock space description of a system of fermions and bosons in interaction.

The description given here is not Lorentz invariant and it does not use relativistic quantum fields.[2] Furthermore, for simplicity we suppose that the fermions and bosons are spinless.

The fermion-boson system considered in this chapter is thus not a physical system; it is instead a prototype for the physically interesting systems listed at the beginning of Chapter 3. That is, it is a prototype for a system of

- electrons and photons[3]
- electrons and phonons
- nucleons and pions
- quarks and gluons

Our purpose in describing the prototype system is to illustrate some points about fermions and bosons in interaction which arise is the more physically interesting systems.

The Hilbert space for the prototype system is a direct product of a fermion Fock space and a boson Fock space. Fundamental dynamical variables for the system are given in Section 7.1.

---

2 Relativistic quantum field theory is discussed in *QLB: Relativistic Quantum Field Theory*.

3 An introduction of quantum electrodynamics, the relativistic quantum field theory of interacting electrons, positrons and photons, is given in *QLB: Relativistic Quantum Field Theory*.

The fermion-boson trilinear interaction is discussed in Section 7.2.

The dressing transformation method for expressing the Hamiltonian in terms of creators and annihilators for physical particles is given in Section 7.3.

The dressing transformation for the trilinear interaction is given in Section 7.4. Included there is a derivation using the dressing transformation method of the Yukawa potential for interacting physical fermions.

7.1 Fundamental dynamical variables

We consider a system of spinless fermions and spinless bosons. The Hilbert space for the system is the direct product of the fermion Fock space discussed in Chapters 4 and 5 and boson Fock space discussed in Chapter 6.

Fundamental dynamical variables for the system are the fermion creator $F^\dagger(\vec{p})$ and annihilator $F(\vec{p})$ and the boson creator $B^\dagger(\vec{p})$ and annihilator $B(\vec{p})$. $F(\vec{p})$ and $B(\vec{p})$ are the spinless versions of the operators defined in Chapters 5 and 6.

Fermion and boson variables satisfy anticommutation and commutation relations, respectively. In addition, fermion variables commute with boson variables.

Comments

1. **Number operators**

The fermion and boson number operators are

$$\boxed{N_f = \int d^3p F^\dagger(\vec{p})F(\vec{p}) \qquad (7.1)}$$

$$N_b = \int d^3p B^\dagger(\vec{p}) B(\vec{p}) \tag{7.2}$$

2. **Momentum operator**

The momentum operator for the system is

$$\vec{P} = \int d^3p\, \vec{p} \Big[F^\dagger(\vec{p}) F(\vec{p}) + B^\dagger(\vec{p}) B(\vec{p}) \Big] \tag{7.3}$$

3. **Free-particle Hamiltonian**

The Hamiltonian for free fermions and bosons is

$$H_0 = \int d^3p \Big[\epsilon_f(p) F^\dagger(\vec{p}) F(\vec{p}) + \epsilon_b(p) B^\dagger(\vec{p}) B(\vec{p}) \Big] \tag{7.4}$$

where

$$\epsilon_f(p) = \sqrt{p^2c^2 + m_f^2c^4} \tag{7.5}$$

$$\epsilon_b(p) = \sqrt{p^2c^2 + m_b^2c^4} \tag{7.6}$$

where $p^2 = \vec{p} \cdot \vec{p}$ and where m_f and m_b are the rest masses of the elementary fermion and boson, respectively.

4. **Vacuum state**

The vacuum state $\mid 0 >$ contains no fermions or bosons.

$$\boxed{F(\vec{p}) \mid 0 >= B(\vec{p}) \mid 0 >= 0 \qquad (7.7)}$$

5. **Creating elementary fermions and bosons**

When acting on the vacuum state, $F^{\dagger}(\vec{p})$ creates a elementary fermion with rest mass m_f, momentum $\vec{p}$ and energy $\epsilon_f(p)$.

When acting on the vacuum state, $B^{\dagger}(\vec{p})$ creates an elementary boson with rest mass m_b, momentum $\vec{p}$ and energy $\epsilon_b(p)$.

7.2 Trilinear interaction

We suppose that the Hamiltonian H for fermions and bosons in interaction is

$$\boxed{H = H_0 + H_1 \qquad (7.8)}$$

where H_0 is given by (7.4) and where

$$\boxed{H_1 = \int d^3p d^3q \; h(\vec{p},\vec{q}) F^{\dagger}(\vec{p}-\vec{q}) F(\vec{p}) B^{\dagger}(\vec{q}) + \text{ adjoint} \qquad (7.9)}$$

where $h(\vec{p},\vec{q})$ is some function.

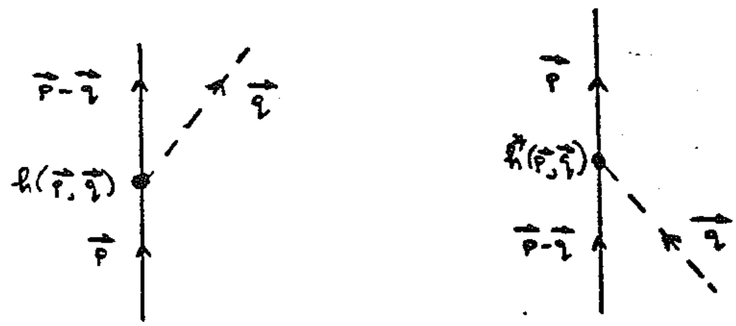

Figure 7.1 Trilinear interaction

The integrand in (7.9) can be represented pictorially as given in Figure 7.1.

Comments

1. Nomenclature

(7.9) is the trilinear interaction. $h(\vec{p},\vec{q})$ is the vertex function for the trilinear interaction.

2. Conservation of fermion number; nonconservation of boson number

(7.9) conserves fermion number

$$\boxed{[H_1, N_f] = 0} \tag{7.10}$$

but not boson number

$$\boxed{[H_1, N_b] \neq 0} \tag{7.11}$$

3. Prototype Hamiltonian

(7.9) is the simplest fermion-boson interaction which conserves fermion number and allows creation and annihilation of bosons.

(7.9) is a prototype for a number of systems of interacting fermions and bosons.

4. Bare and physical fermions

It follows from (7.4) and (7.8) that

$$H_0 F^\dagger(\vec{p}) \mid 0 >= \epsilon_f(p) F^\dagger(\vec{p}) \mid 0 > \tag{7.12}$$

$$H F^\dagger(\vec{p}) \mid 0 > \neq \widetilde{\epsilon}_f(\vec{p}) F^\dagger(\vec{p}) \mid 0 > \tag{7.13}$$

for any function $\widetilde{\epsilon}_f(\vec{p})$.

In view of (7.13), $F^\dagger(\vec{p})$ does not create a physical fermion when acting on the vacuum state.

In view of (7.12), $F^\dagger(\vec{p})$ is said to create a bare fermion when acting on the vacuum state.

m_f in (7.5) is the rest mass of the bare fermion.

5. Creation operator for a physical fermion

The creation operator $\widetilde{F}^\dagger(\vec{p})$ for a physical fermion satisfies

$$N_f \widetilde{F}^\dagger(\vec{p}) \mid 0 >= \widetilde{F}^\dagger(\vec{p}) \mid 0 > \tag{7.14}$$

$$\vec{P}\, \widetilde{F}^\dagger(\vec{p}) \mid 0 >= \vec{p}\, \widetilde{F}^\dagger(\vec{p}) \mid 0 > \tag{7.15}$$

$$H \widetilde{F}^\dagger(\vec{p}) \mid 0 >= \widetilde{\epsilon}_f(\vec{p}) \widetilde{F}^\dagger(\vec{p}) \mid 0 > \tag{7.16}$$

for some function $\widetilde{\epsilon}_f(\vec{p})$. If (7.8) described a Lorentz invariant system it would follow that

$$\boxed{\widetilde{\epsilon}_f(\vec{p}) = \sqrt{p^2c^2 + \widetilde{m}_f^2c^4}} \qquad (7.17)$$

$\widetilde{m}_f$ is the rest mass of the physical fermion.

6. Physical-fermion eigenket

The physical-fermion eigenket $\widetilde{F}^{\dagger}_{m_{s,f}}(p) \mid 0 >$ has the form

$$\boxed{\widetilde{F}^{\dagger}(\vec{p}) \mid 0 >= F^{\dagger}(\vec{p}) \mid 0 > + \sum_{n=1}^{\infty} C_n^{\dagger}(\vec{p}) \mid 0 >} \qquad (7.18)$$

where

$$C_n^{\dagger}(\vec{p}) \mid 0 >= \int_{-\infty}^{+\infty} d^3p_1 \cdots d^3p_{n+1} \delta(\vec{p}_1 + \cdots + \vec{p}_{n+1} - \vec{p}) \cdot$$
$$\cdot c(\vec{p}_1, \cdots, \vec{p}_{n+1}, \vec{p}) B^{\dagger}(\vec{p}_1) \cdots B^{\dagger}(\vec{p}_n) F^{\dagger}(\vec{p}_{n+1}) \mid 0 > \qquad (7.19)$$

where the functions $c(\vec{p}_1, \cdots, \vec{p}_{n+1}, \vec{p})$ are determined by satisfying (7.16).

7. Cloud of bosons

(7.18) is interpreted by the statement:

The physical fermion is a bare fermion surrounded by a cloud of bosons.

7.3 Dressing transformation

In this section we introduce a unitary transformation of the elementary particle operators $F(\vec{p})$ and $B(\vec{p})$ which yields operators $\widetilde{F}(\vec{p})$ and $\widetilde{B}(\vec{p})$ for physical particles. The transformation "dresses" bare variables to yield physical variables.

We write

$$\widetilde{F}(\vec{p}) = UF(\vec{p})U^{\dagger} \tag{7.20}$$

$$\widetilde{B}(\vec{p}) = UB(\vec{p})U^{\dagger} \tag{7.21}$$

where

$$U = e^{D} \tag{7.22}$$

$$D^{\dagger} = -D \tag{7.23}$$

Comments

1. Dressing transformation

The unitary transformation (7.22) is the dressing transformation.

D is the dressing operator.

The strategy for determining D is given in Item 7.

Further details of the dressing transformation method are given in Greenberg (1959), Schweber (1961), Hearn (1981), James (1982), Hearn, McMillan and Raskin (1983) and Cropp (1996).

2. Fundamental dynamical variables

Because U is unitary, $\widetilde{F}(\vec{p})$ and $\widetilde{B}(\vec{p})$ and their adjoints obey the same anticommutation and commutation relations as do $F(\vec{p})$ and $B(\vec{p})$ and their adjoints.

$\widetilde{F}(\vec{p})$ and $\widetilde{B}(\vec{p})$ and their adjoints are fundamental dynamical variables for a system of fermions and bosons.

3. Expressions for operators

Every operator A may be expressed in terms of the variables $F(\vec{p})$ and $B(\vec{p})$ and their adjoints.

We write

$$\boxed{A = A(F, B) \qquad (7.24)}$$

and in view of (7.20) and (7.21),

$$U A(F,B) U^{\dagger} = A\left(\widetilde{F},\widetilde{B}\right) \qquad (7.25)$$

$A\left(\widetilde{F},\widetilde{B}\right)$ is obtained from $A(F,B)$ by replacing F by $\widetilde{F}$ and B by $\widetilde{B}$.

4. **Invariance of the dressing operator**

It follows from (7.22) and (7.25) that

$$D\left(\widetilde{F},\widetilde{B}\right) = D(F,B) \qquad (7.26)$$

5. **Dressed Hamiltonian**

It follows from (7.25) and (7.26) that

$$H(F,B) = \mathcal{H}\left(\widetilde{F},\widetilde{B}\right) \qquad (7.27)$$

where

$$\mathcal{H}\left(\widetilde{F},\widetilde{B}\right) = e^{-D\left(\widetilde{F},\widetilde{B}\right)} H\left(\widetilde{F},\widetilde{B}\right) e^{D\left(\widetilde{F},\widetilde{B}\right)} \qquad (7.28)$$

(7.28) defines $\mathcal{H}$ in terms of H and D.

$\mathcal{H}\left(\widetilde{F},\widetilde{B}\right)$ is the Hamiltonian expressed in terms of dressed creators and annihilators; $\mathcal{H}\left(\widetilde{F},\widetilde{B}\right)$ is the dressed Hamiltonian.

6. **Choice of D; physical-particle eigenkets**

$D\left(\widetilde{F},\widetilde{B}\right)$ is chosen in order that $\widetilde{F}^\dagger(\vec{p}) \mid 0>$ and $\widetilde{B}^\dagger(\vec{p}) \mid 0>$ are eigenkets of the dressed Hamiltonian. That is, $D\left(\widetilde{F},\widetilde{B}\right)$ is chosen in order that

$$\mathcal{H}\left(\widetilde{F},\widetilde{B}\right)\widetilde{F}^\dagger(\vec{p}) \mid 0 >= \widetilde{\epsilon}_f(\vec{p})\widetilde{F}^\dagger(\vec{p}) \mid 0 > \tag{7.29}$$

$$\mathcal{H}\left(\widetilde{F},\widetilde{B}\right)\widetilde{B}^\dagger(\vec{p}) \mid 0 >= \widetilde{\epsilon}_b(\vec{p})\widetilde{B}^\dagger(\vec{p}) \mid 0 > \tag{7.30}$$

for some functions $\widetilde{\epsilon}_f(p)$ and $\widetilde{\epsilon}_b(p)$.

7. **Strategy for determining the dressing operator**

(7.29) and (7.30) hold if, apart from the terms

$$\widetilde{F}^\dagger\widetilde{F} \qquad \text{and} \qquad \widetilde{B}^\dagger\widetilde{B} \tag{7.31}$$

there are no terms in (7.28) which contain only one annihilation operator.

That is, (7.29) and (7.30) hold if (7.28) does not contain terms of the form

$$\widetilde{F}^\dagger\widetilde{F}\widetilde{B}^\dagger, \qquad \widetilde{F}^\dagger\widetilde{F}\widetilde{B}^\dagger\widetilde{B}^\dagger, \qquad \text{etc.} \tag{7.32}$$

D is chosen in order to eliminate terms of the form (7.32) from $\mathcal{H}$.

7.4 Dressing the trilinear interaction

The operator $H\left(\widetilde{F},\widetilde{B}\right)$ for a system of fermions and bosons interacting via the trilinear interaction is

$$H\left(\widetilde{F},\widetilde{B}\right) = H_0\left(\widetilde{F},\widetilde{B}\right) + \lambda H_1\left(\widetilde{F},\widetilde{B}\right) \tag{7.33}$$

where $H_0\left(\widetilde{F},\widetilde{B}\right)$ and $H_1\left(\widetilde{F},\widetilde{B}\right)$ are given by (7.4) and (7.9) with F replaced by $\widetilde{F}$ and B replaced by $\widetilde{B}$.

The real parameter λ in (7.33) is an order-counting parameter which can be set equal to unity.

We write $D\left(\widetilde{F},\widetilde{B}\right)$ in the form

$$D = \sum_{n=1}^{\infty} \lambda^n D_n \tag{7.34}$$

and substituting (7.34) into (7.28) yields

$$
\boxed{
\begin{aligned}
\mathcal{H} = H_0 &+ \lambda\{H_1 + [H_0, D_1]\} \\
&+\lambda^2\left\{[H_1, D_1] + \frac{1}{2}[[H_0, D_1], D_1] + [H_0, D_2]\right\} \\
&+\lambda^3\left\{\frac{1}{2}[[H_1, D_1], D_1] + \frac{1}{6}[[[H_0, D_1], D_1], D_1]\right\} \\
&+\lambda^3\left\{[H_1, D_2] + \frac{1}{2}[[H_0, D_1], D_2] + \frac{1}{2}[[H_0, D_2], D_1] + [H_0, D_3]\right\} + O(\lambda^4)
\end{aligned}
} \tag{7.35}
$$

Comments

1. **Dressed Hamiltonian**

   (7.35) gives the dressed Hamiltonian in terms of the known operators H_0 and H_1 and operators $D_1, D_2, \cdots$ to be determined.

2. **Determining the D_n**

   The D_n are chosen in order to eliminate terms of the form (7.32) from (7.35).

   This can be done order by order in λ.

3. **Expression for D_1**

   $H_1\left(\widetilde{F}, \widetilde{B}\right)$ given by (7.9) contains a term of the form (7.32).

   It follows from (7.35) that D_1 is chosen in order that

$$[H_0, D_1] = -H_1 \tag{7.36}$$

It follows from (7.4) and (7.9) that (7.36) holds if

$$D_1 = \int d^3p d^3q \; d_1(\vec{p}, \vec{q}) \widetilde{F}^\dagger(\vec{p} - \vec{q}) \widetilde{F}(\vec{p}) \widetilde{B}^\dagger(\vec{q}) - \text{ adjoint} \tag{7.37}$$

where

$$d_1(\vec{p}, \vec{q}) = -\frac{h(\vec{p}, \vec{q})}{\Delta(\vec{p}, \vec{q})} \tag{7.38}$$

$$\Delta(\vec{p}, \vec{q}) = \epsilon_f(\mid \vec{p} - \vec{q} \mid) + \epsilon_b(q) - \epsilon_f(p) \tag{7.39}$$

4. **Dressed Hamiltonian**

Substitution of (7.36) into (7.35) yields

$$\begin{aligned} \mathcal{H} = H_0 + \lambda^2 \left\{ \frac{1}{2}[H_1, D_1] + [H_0, D_2] \right\} \\ + \lambda^3 \left\{ \frac{1}{3}[[H_1, D_1], D_1] + \frac{1}{2}[[H_0, D_2], D_1] + \frac{1}{2}[H_1, D_2] + [H_0, D_3] \right\} \\ + O(\lambda^4) \end{aligned} \tag{7.40}$$

(7.40) gives the dressed Hamiltonian with D_1 given by (7.37).

5. **Choice of $D_2, D_3, \cdots$**

   D_2 is chosen in order to eliminate terms of the form (7.32) in the λ^2 term in (7.40).

   Substitution of D_2 in the λ^3 term in (7.40) yields a new equation for the dressed Hamiltonian.

   D_3 is chosen in order to eliminate terms of the form (7.32) in the λ^3 term in the new equation for the dressed Hamiltonian, and so on.

Hamiltonian dressed to second-order

For simplicity in the remainder of this chapter we assume that the vertex function in (7.9) depends only on the boson momentum and is real. That is, we take

$$h(\vec{p}, \vec{q}) = h(q) \tag{7.41}$$

$$h^*(q) = h(q) \tag{7.42}$$

Retaining only the λ^2 term in (7.40) after the above choice of D_2 yields an approximation $\mathcal{H}_2$ to the dressed Hamiltonian $\mathcal{H}$; $\mathcal{H}_2$ is the Hamiltonian dressed to second order.

It follows on substituting (7.37) in the λ^2 term in (7.40) that

$$\mathcal{H}_2 = T + V_{ff} + V_{fb} \tag{7.43}$$

where

$$T = \int dp \left[\widetilde{\epsilon}_f(\vec{p}) \widetilde{F}^\dagger(\vec{p}) \widetilde{F}(\vec{p}) + \widetilde{\epsilon}_b(\vec{p}) \widetilde{B}^\dagger(\vec{p}) \widetilde{B}(\vec{p}) \right] \tag{7.44}$$

$$V_{ff} = \frac{1}{2} \int d^3k d^3k' d^3K$$
$$V_{ff}\left(\vec{k}, \vec{k'}, \vec{K}\right) \widetilde{F}^\dagger\left(\frac{1}{2}\vec{K} + \vec{k}\right) \widetilde{F}^\dagger\left(\frac{1}{2}\vec{K} - \vec{k}\right) \widetilde{F}\left(\frac{1}{2}\vec{K} - \vec{k'}\right) \widetilde{F}\left(\frac{1}{2}\vec{K} + \vec{k'}\right) \tag{7.45}$$

$$V_{fb} = \frac{1}{2} \int d^3k d^3k' d^3K$$
$$V_{fb}\left(\vec{k}, \vec{k'}, \vec{K}\right) \widetilde{F}^\dagger\left(\frac{1}{2}\vec{K} + \vec{k}\right) \widetilde{B}^\dagger\left(\frac{1}{2}\vec{K} - \vec{k}\right) \widetilde{B}\left(\frac{1}{2}\vec{K} - \vec{k'}\right) \widetilde{F}\left(\frac{1}{2}\vec{K} + \vec{k'}\right) \tag{7.46}$$

where

$$\widetilde{\epsilon}_f(\vec{p}) = \epsilon_f(p) - \lambda^2 \int d^3q \frac{h^2(q)}{\Delta(\vec{p},\vec{q})} \tag{7.47}$$

$$\widetilde{\epsilon}_b(\vec{p}) = \epsilon_b(p) \tag{7.48}$$

$$V_{ff}\left(\vec{k},\vec{k'},\vec{K}\right) = \frac{-\lambda^2 h^2\left(\mid \vec{k}-\vec{k'} \mid\right)}{\Delta\left(\frac{1}{2}\vec{K}-\vec{k'},\vec{k}-\vec{k'}\right)} \tag{7.49}$$

$$+k \leftrightarrow k'$$

$$V_{fb}\left(\vec{k},\vec{k'},\vec{K}\right) = \lambda^2 h\left(\mid \frac{1}{2}\vec{K}-\vec{k} \mid\right) h\left(\mid \frac{1}{2}\vec{K}-\vec{k'} \mid\right) \cdot$$

$$\cdot \left[\frac{1}{\Delta\left(\vec{K},\frac{1}{2}\vec{K}-\vec{k}\right)} - \frac{1}{\Delta\left(\frac{1}{2}\vec{K}+\vec{k'},\frac{1}{2}\vec{K}-\vec{k}\right)}\right] \tag{7.50}$$

$$+k \leftrightarrow k'$$

The right sides of (7.49) and (7.50) can be represented pictorially as given in Figures 7.2 and 7.3.

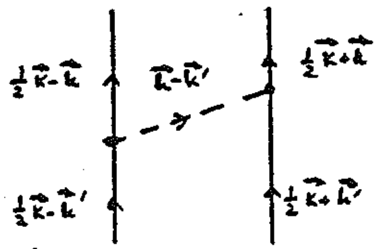

Figure 7.2 Fermion-fermion potential

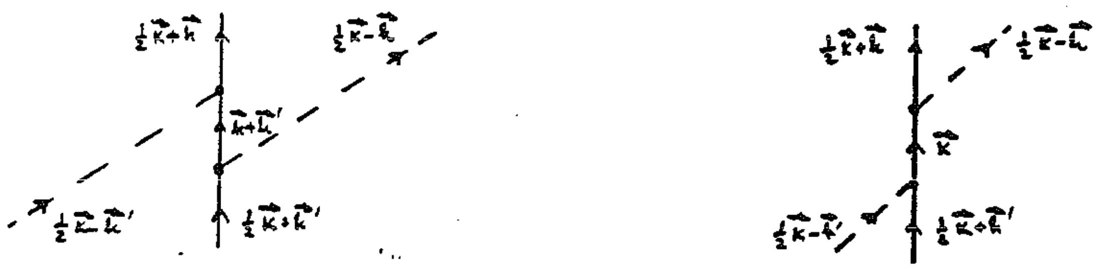

Figure 7.3 Fermion-boson potential

Comments

1. Two-body potentials

(7.43) contains potentials V_{ff} and V_{fb} between two physical fermions and a physical fermion and a physical boson, respectively.

The dressing transformation method thus gives a derivation of these two-body potentials.

2. **Fermion-fermion potential**

(7.49) expresses the momentum-space fermion-fermion potential in terms of the vertex function (7.41).

The right side of (7.49) is negative; the force is attractive.

Regarding Figure 7.2, the fermion-fermion potential arises because of boson exchange between the pair of interacting fermions.

3. **Fermion-boson potential.**

(7.50) expresses the momentum-space fermion-boson potential in terms of the vertex function (7.41).

The two terms in the second line of (7.50) have different signs; one term gives an attractive force, the other gives a repulsive force.

4. **Bare mass in terms of physical mass**

The rest mass $\widetilde{m}_f$ of the physical fermion is defined as

$$\boxed{\widetilde{m}_f = \widetilde{\epsilon}_f(0)/c^2 \qquad (7.51)}$$

The mass m_f of the bare fermion is not observed; it is determined from

$$m_f = \widetilde{m}_f + \frac{\lambda^2}{c^2} \int d^3q \frac{h^2(q)}{\epsilon_f(q) + \epsilon_b(q) - m_f c^2} \tag{7.52}$$

5. **Higher-order terms**

The λ^3 term in (7.40) contains terms of the form

$$\widetilde{F}^\dagger \widetilde{F}^\dagger \widetilde{B}^\dagger \widetilde{F} \widetilde{F} \qquad \text{and} \qquad \widetilde{F}^\dagger \widetilde{F}^\dagger \widetilde{B} \widetilde{F} \widetilde{F} \tag{7.53}$$

These terms correspond to boson production and annhilation in a fermion-fermion interaction.

Terminating the dressing operator series

(7.34) terminates at the first term, that is,

$$D_n = 0 \qquad n = 2, 3, \cdots \tag{7.54}$$

when the approximation

$$\Delta(\vec{p}, \vec{q}) = \epsilon_b(q) \tag{7.55}$$

is made in (7.38).

Comments

1. **Fermion-boson potential**

Approximation (7.55) yields

$$\boxed{V_{fb}(k, k', K) = 0 \qquad (7.56)}$$

2. **Fermion-fermion potential in momentum space**

Approximation (7.55) yields

$$\boxed{V_{ff}\left(\vec{k}, \vec{k'}, \vec{K}\right) = \frac{-2\lambda^2 h^2\left(\mid \vec{k} - \vec{k'} \mid\right)}{\epsilon_b\left(\mid \vec{k} - \vec{k'} \mid\right)} = V_{ff}\left(\mid \vec{k} - \vec{k'} \mid\right) \qquad (7.57)}$$

(7.57) is a momentum-space fermion-fermion potential with depends only on the momentum transferred between the interacting pair.

3. **Fermion-fermion potential in coordinate space**

It follows from (5.64) that the fermion-fermion coordinate-space potential is

$$\boxed{V_{ff}(r) = \int d^3q e^{i\vec{q}\cdot\vec{r}/\hbar} V_{ff}(q) \qquad (7.58)}$$

Yukawa potential

It follows from (7.58) that

$$V_{ff}(r) = -ge^{-\mu r}/r \tag{7.59}$$

when

$$h(q) = \frac{1}{2\pi}\sqrt{\frac{g_0 c^3}{\epsilon_b(q)}} \tag{7.60}$$

where

$$g = g_0 \hbar c \tag{7.61}$$

$$\mu = \frac{m_b c}{\hbar} \tag{7.62}$$

Comments

1. **Yukawa potential**

   (7.59) is the Yukawa potential for interacting fermions.

   (7.59) was derived by H. Yukawa in 1935.

2. **Range of the Yukawa potential**

$1/\mu$ is the range of the Yukawa potential.

The range is equal to the Compton wavelength of the exchanged boson.

Yukawa used the known range of the nucleon-nucleon potential to predict the existence of a boson with rest mass 140 MeV/c^2. The pion was discovered in 1947.

3. **Form of the vertex function**

The factor $1/\sqrt{\epsilon_b(q)}$ in (7.60) appears in the expression for the scalar field $\phi(x)$ defined in *QLB: Relativistic Quantum Field Theory*.

4. **Infinite bare mass**

The integral in (7.52) diverges when approximation (7.55) is used with (7.60).

The bare mass m_f in this approximation is infinite.

Appendix: Two distinguishable particles

In this appendix we consider a system of two distinguishable spinless particles to provide an example of the direct product formalism discussed in Chapter 2.

A.1 Coordinate-space wave function

The Cartesian coordinates $\vec{X}_1$ and $\vec{X}_2$ of the particles are

$$\boxed{\begin{aligned} \vec{X}_1 &= \vec{X} \otimes 1 && \text{(A.1)} \\ \vec{X}_2 &= 1 \otimes \vec{X} && \text{(A.2)} \end{aligned}}$$

with spectral decompositions

$$\boxed{\begin{aligned} \vec{X}_1 &= \int d^3x d^3y \mid \vec{x}\vec{y} > \vec{x} < \vec{x}\vec{y} \mid && \text{(A.3)} \\ \vec{X}_2 &= \int d^3x d^3y \mid \vec{x}\vec{y} > \vec{y} < \vec{x}\vec{y} \mid && \text{(A.4)} \end{aligned}}$$

where

$$\boxed{1 = \int d^3x d^3y \mid \vec{x}\vec{y} >< \vec{x}\vec{y} \mid \qquad \text{(A.5)}}$$

where

$$| \vec{x}\vec{y} >=| \vec{x} \underset{1}{>} | \vec{y} \underset{2}{>} \tag{A.6}$$

and

$$\begin{aligned} < \vec{x}\vec{y} \mid \vec{x'}\vec{y'} > &=< \vec{x} \mid \vec{x'} \underset{1}{>}< \vec{y} \mid \vec{y'} \underset{2}{>} \\ &= \delta\left(\vec{x} - \vec{x'}\right)\delta\left(\vec{y} - \vec{y'}\right) \end{aligned} \tag{A.7}$$

A state $| \psi(t) >$ of the system at time t can be expressed as

$$| \psi(t) >= \int d^3x d^3y \; \psi(\vec{x}, \vec{y}, t) \mid \vec{x}\vec{y} > \tag{A.8}$$

where $\psi(\vec{x}, \vec{y}, t)$ is the coordinate-space wave function of the state, that is,

$$\psi(\vec{x}, \vec{y}, t) =< \vec{x}\vec{y} \mid \psi(t) > \tag{A.9}$$

Then

$$| \psi(\vec{x}, \vec{y}, t) |^2 d^3x d^3y \tag{A.10}$$

is the probability at time t that particle 1 is in the volume d^3x about $\vec{x}$ and particle 2 is in the volume d^3y about $\vec{y}$.

The average positions of particles 1 and 2 in the state $\mid \psi(t) >$ are, respectively,

$$< \psi(t) \mid \vec{X}_1 \mid \psi(t) >= \int d^3x d^3y \, \vec{x} \mid \psi(\vec{x}, \vec{y}, t) \mid^2 \tag{A.11}$$

$$< \psi(t) \mid \vec{X}_2 \mid \psi(t) >= \int d^3x d^3y \, \vec{y} \mid \psi(\vec{x}, \vec{y}, t) \mid^2 \tag{A.12}$$

A.2 Transposition operator

The transposition operator Π for the system corresponds to transposing the two particles and is defined as

$$\Pi = \int d^3x d^3y \mid \vec{y}\vec{x} >< \vec{x}\vec{y} \mid \tag{A.13}$$

It follows that

$$\Pi^\dagger = \Pi \tag{A.14}$$

$$\Pi^2 = 1 \tag{A.15}$$

The two permutation operators Π and 1 form the symmetric group S_2.

It follows from (A.3) and (A.4) that

$$\Pi \vec{X}_1 \Pi^\dagger = \vec{X}_2 \tag{A.16}$$

$$\Pi \vec{X}_1 \Pi^\dagger = \vec{X}_2 \tag{A.17}$$

The state $\Pi \mid \psi(t) >$ is the state of the system when particles 1 and 2 in the state $\mid \psi(t) >$ are interchanged. It follows that

$$\Pi \mid \psi(t) >= \int d^3x d^3y \; \psi_\Pi(\vec{x}, \vec{y}, t) \mid \vec{x}\vec{y} > \tag{A.18}$$

where

$$\psi_\Pi(\vec{x}, \vec{y}, t) =< \vec{x}\vec{y} \mid \Pi \mid \psi(t) >= \psi(\vec{y}, \vec{x}, t) \tag{A.19}$$

A.3 Orthogonal projectors

The symmetrizer Π_s and antisymetrizer Π_a for the system are defined by

$$\boxed{\begin{aligned} \Pi_s &= \frac{1}{2}(1+\Pi) && \text{(A.20)} \\ \Pi_a &= \frac{1}{2}(1-\Pi) && \text{(A.21)} \end{aligned}}$$

It follows that

$$\boxed{\begin{aligned} \Pi_s + \Pi_a &= 1 && \text{(A.22)} \\ \Pi_e \Pi_f &= \Pi_e \delta_{ef} \qquad (e, f = s, a) && \text{(A.23)} \end{aligned}}$$

and

$$\boxed{\begin{aligned} \Pi\Pi_s &= \Pi_s && \text{(A.24)} \\ \Pi\Pi_a &= -\Pi_a && \text{(A.25)} \end{aligned}}$$

Π_s and Π_a are a complete set of orthogonal projection operators for the two-body system.

A.4 General state of the system

A general state $\mid \psi(t) >$ of the system can be written as

$$\mid \psi(t) >=\mid \psi_s(t) > + \mid \psi_a(t) > \tag{A.26}$$

where

$$\mid \psi_e(t) >= \Pi_e \mid \psi(t) > \qquad (e = s, a) \tag{A.27}$$

from which

$$< \psi_s(t) \mid \psi_a(t) >= 0 \tag{A.28}$$

$\mid \psi_s(t) >$ and $\mid \psi_a(t) >$ are the symmetric and antisymmetric components, respectively, of $\mid \psi(t) >$.

$\mid \psi_s(t) >$ and $\mid \psi_a(t) >$ can be written as

$$\mid \psi_s(t) >= \int d^3x d^3y \mid \vec{x}\vec{y} > \psi_s(\vec{x}, \vec{y}, t) \tag{A.29}$$

$$\mid \psi_a(t) >= \int d^3x d^3y \mid \vec{x}\vec{y} > \psi_a(\vec{x}, \vec{y}, t) \tag{A.30}$$

where

$$\psi_s(\vec{x},\vec{y},t) = \frac{1}{2}[\psi(\vec{x},\vec{y},t) + \psi(\vec{y},\vec{x},t)] \tag{A.31}$$

$$\psi_a(\vec{x},\vec{y},t) = \frac{1}{2}[\psi(\vec{x},\vec{y},t) - \psi(\vec{y},\vec{x},t)] \tag{A.32}$$

Then

$$\psi_s(\vec{y},\vec{x},t) = +\psi_s(\vec{x},\vec{y},t) \tag{A.33}$$

$$\psi_a(\vec{y},\vec{x},t) = -\psi_a(\vec{x},\vec{y},t) \tag{A.34}$$

Appendix: Three distinguishable particles

In this appendix we consider a system of three distinguishable particles to provide an example of the comments about the symmetric group S_n given in Chapter 2.

B.1 Permutation operators

We define the three-particle basis state $\mid rst >$ by

$$\boxed{\mid rst >=\mid r \underset{1}{>}\mid s \underset{2}{>}\mid t \underset{3}{>} \qquad \text{(B.1)}}$$

where particle 1 is in single-particle state $\mid r >$, particle 2 is in single-particle state $\mid s >$ and particle 3 is in single-particle state $\mid t >$.

The single-particle states $\mid 1 >, \mid 2 >, \cdots$ are assumed to span the one-particle space, that is,

$$\sum_{r=1}^{\infty} \mid r >< r \mid = 1 \qquad \text{(B.2)}$$

$$< r \mid s >= \delta_{rs} \qquad \text{(B.3)}$$

The six permutation operators $\Pi_1, \Pi_2, \cdots, \Pi_6$ for the system are

$$\Pi_1 = \sum_{r,s,t=1}^{\infty} \mid rts >< rst \mid \tag{B.4}$$

$$\Pi_2 = \sum_{r,s,t=1}^{\infty} \mid tsr >< rst \mid \tag{B.5}$$

$$\Pi_3 = \sum_{r,s,t=1}^{\infty} \mid srt >< rst \mid \tag{B.6}$$

$$\Pi_4 = \sum_{r,s,t=1}^{\infty} \mid trs >< rst \mid \tag{B.7}$$

$$\Pi_5 = \sum_{r,s,t=1}^{\infty} \mid str >< rst \mid \tag{B.8}$$

$$\Pi_6 = \sum_{r,s,t=1}^{\infty} \mid rst >< rst \mid \tag{B.9}$$

It follows that

$$\Pi_\alpha^\dagger = \Pi_\alpha \qquad (\alpha = 1, 2, \cdots, 6) \tag{B.10}$$

and that products of two permutation operators are as given Table B.1.

The six permutation operators $\Pi_1, \Pi_2, \cdots, \Pi_6$ form the symmetric group $\underset{\sim}{S_3}$.

Table B.1 Multiplication table for permutation operators

	Π_1	Π_2	Π_3	Π_4	Π_5	Π_6
Π_1	Π_6	Π_4	Π_5	Π_2	Π_3	Π_1
Π_2	Π_5	Π_6	Π_4	Π_3	Π_1	Π_2
Π_3	Π_4	Π_5	Π_6	Π_1	Π_2	Π_3
Π_4	Π_2	Π_3	Π_1	Π_6	Π_4	Π_4
Π_5	Π_3	Π_1	Π_2	Π_5	Π_6	Π_5
Π_6	Π_1	Π_2	Π_3	Π_4	Π_5	Π_6

Comments

1. Notation

The permutation operators $\Pi_1, \Pi_2, \cdots, \Pi_6$ are alternatively labelled as

$$\Pi_{23},\ \Pi_{31},\ \Pi_{12},\ \Pi_{123},\ \Pi_{321},\ 1 \tag{B.11}$$

respectively. Equivalent labellings result using

$$\Pi_{ab} = \Pi_{ba} \tag{B.12}$$

$$\Pi_{abc} = \Pi_{cab} = \Pi_{bca} \tag{B.13}$$

2. Order of multiplications

Multiplications in the table are done in the following order:

$$\Pi_{first\ column}\Pi_{first\ row} \tag{B.14}$$

3. **Odd and even permutations**

The multiplication table has the general form

$$\Pi_{odd}\Pi_{odd} = \Pi_{even} \tag{B.15}$$

$$\Pi_{even}\Pi_{even} = \Pi_{even} \tag{B.16}$$

$$\Pi_{odd}\Pi_{even} = \Pi_{even}\Pi_{odd} = \Pi_{odd} \tag{B.17}$$

where

$$\Pi_{odd} = \Pi_1 \ \text{ or } \ \Pi_2 \ \text{ or } \ \Pi_3 \tag{B.18}$$

$$\Pi_{even} = \Pi_4 \ \text{ or } \ \Pi_5 \ \text{ or } \ \Pi_6 \tag{B.19}$$

4. **Transposition operators**

The odd permutation operators Π_1, Π_2, Π_3 are transposition operators since they correspond to a transposition of two of the three particles in the system.

It follows from (B.15) that each of the even permutation operators Π_4, Π_5, Π_6 is the product of two transpositions.

B.2 Orthogonal projectors

We define operators Π_s, Π_a, Π_m by

$$\Pi_s = \frac{1}{6}(\Pi_1 + \Pi_2 + \Pi_3 + \Pi_4 + \Pi_5 + \Pi_6) \tag{B.20}$$

$$\Pi_a = \frac{1}{6}(-\Pi_1 - \Pi_2 - \Pi_3 + \Pi_4 + \Pi_5 + \Pi_6) \tag{B.21}$$

$$\Pi_m = \frac{2}{3}\Pi_6 - \frac{1}{3}(\Pi_4 + \Pi_5) \tag{B.22}$$

It follows that

$$\Pi_s + \Pi_a + \Pi_m = 1 \tag{B.23}$$

$$\Pi_e \Pi_f = \Pi_e \delta_{ef} \qquad (e, f = s, a, m) \tag{B.24}$$

Furthermore,

$$\Pi_\alpha \Pi_s = \Pi_s \qquad (\alpha = 1, 2, \cdots, 6) \tag{B.25}$$

$$\Pi_{odd} \Pi_a = -\Pi_a \qquad (\alpha = 1, 2, \cdots, 6) \tag{B.26}$$

$$\Pi_{even} \Pi_a = \Pi_a \qquad (\alpha = 1, 2, \cdots, 6) \tag{B.27}$$

$$\Pi_\alpha \Pi_m = \frac{2}{3}\Pi_\alpha - \frac{1}{3}(\Pi_\beta + \Pi_\gamma) \qquad (\alpha, \beta, \gamma = 1, 2, 3 \quad \text{cyclic}) \tag{B.28}$$

Π_s, Π_a, Π_m are a complete set of orthogonal projection operators for the three-body system.

B.3 Symmetric, antisymmetric, mixed basis states

We define states $\mid rst \underset{s}{>}, \mid rst \underset{a}{>}, \mid rst \underset{m}{>}$ by

$$\mid rst \underset{s}{>} = \sqrt{6}\Pi_s \mid rst > \tag{B.29}$$

$$\mid rst \underset{a}{>} = \sqrt{6}\Pi_a \mid rst > \tag{B.30}$$

$$\mid rst \underset{m}{>} = \frac{\sqrt{6}}{2}\Pi_m \mid rst > \tag{B.31}$$

It follows that

$$| rst >= \frac{1}{\sqrt{6}}\Big(| \underset{s}{rst >} + | \underset{a}{rst >} +2 \,| \underset{m}{rst >}\Big) \qquad \text{(B.32)}$$

and

$$\underset{e}{<} rst \mid \underset{f}{rst >}= \delta_{ef} \qquad (e, f = s, a, m) \qquad \text{(B.33)}$$

Furthermore,

$$\Pi_\alpha \mid \underset{s}{rst >}=\mid \underset{s}{rst >} \qquad (\alpha = 1, 2, \cdots, 6) \qquad \text{(B.34)}$$

$$\Pi_{odd} \mid \underset{a}{rst >}= - \mid \underset{a}{rst >} \qquad \text{(B.35)}$$

$$\Pi_{even} \mid \underset{a}{rst >}=\mid \underset{a}{rst >} \qquad \text{(B.36)}$$

$$\Pi_\alpha \mid \underset{m}{rst >}= \sqrt{\frac{2}{3}}\left[\Pi_\alpha - \frac{1}{2}(\Pi_\beta + \Pi_\gamma)\right] \mid rst > \qquad (\alpha, \beta, \gamma = 1, 2, 3 \ \text{cyclic}) \qquad \text{(B.37)}$$

$| \underset{s}{rst >}, | \underset{a}{rst >}, | \underset{m}{rst >}$ are symmetric, antisymmetric and mixed basis states, respectively.

$\mid rst \underset{s}{>}$ and $\mid rst \underset{a}{>}$ may be written as

$$\mid rst \underset{s}{>} = \frac{1}{\sqrt{6}}\, sym\det \begin{vmatrix} \mid r \underset{1}{>} & \mid r \underset{2}{>} & \mid r \underset{3}{>} \\ \mid s \underset{1}{>} & \mid s \underset{2}{>} & \mid s \underset{3}{>} \\ \mid t \underset{1}{>} & \mid t \underset{2}{>} & \mid t \underset{3}{>} \end{vmatrix} \tag{B.38}$$

$$\mid rst \underset{a}{>} = \frac{1}{\sqrt{6}} \det \begin{vmatrix} \mid r \underset{1}{>} & \mid r \underset{2}{>} & \mid r \underset{3}{>} \\ \mid s \underset{1}{>} & \mid s \underset{2}{>} & \mid s \underset{3}{>} \\ \mid t \underset{1}{>} & \mid t \underset{2}{>} & \mid t \underset{3}{>} \end{vmatrix} \tag{B.39}$$

where det denotes determinant and $sym\det$ denotes a determinant which has plus signs in its definition rather than minus signs.

The right side of (B.38) is manifestly symmetric under particle interchange.

The right side of (B.39) vanishes if any two or r, s, t are equal because the value of a determinant vanishes when any two rows are equal.

The right side of (B.39) is manifestly antisymmetric under particle interchange because the value of a determinant changes sign when any two columns are interchanged.

B.4 General state of the system

A general state $\mid \psi(t) >$ of the system can be written as

$$\mid \psi(t) >=\mid \psi_s(t) > + \mid \psi_a(t) > + \mid \psi_m(t) > \tag{B.40}$$

where

$$| \psi_e(t) >= \Pi_e \mid \psi(t) > \qquad (e = s, a, m) \tag{B.41}$$

It follows that

$$< \psi_e(t) \mid \psi_f(t) >= 0 \qquad (e, f = s, a, m;\ e \neq f) \tag{B.42}$$

$| \psi_s(t) >$, $| \psi_a(t) >$ and $| \psi_m(t) >$ are, respectively, the symmetric, anti-symmetric and mixed components of $| \psi(t) >$.

Appendix: Some commutators

We give some commutators of products of fermion and boson creators and annihilators in this appendix.

The right sides of all equations are written in normal order. That is, all creators are written to the left of all annihilators.

C.1 Commutators for fermions

The following commutators result from the anticommutation relations (4.32) and (4.33) and from the identities given in the appendix of *QLB: Introductory Topics*.

$$\left[F_r, F_s^\dagger F_t\right] = F_t \delta_{rs} \tag{C.1}$$

$$\left[F_r^\dagger, F_s^\dagger F_t\right] = -F_s^\dagger \delta_{rt} \tag{C.2}$$

$$\left[F_r, F_s^\dagger F_t^\dagger\right] = F_t^\dagger \delta_{rs} - F_s^\dagger \delta_{rt} \tag{C.3}$$

$$\left[F_r^\dagger, F_s F_t\right] = F_t \delta_{rs} - F_s \delta_{rt} \tag{C.4}$$

$$\left[F_r^\dagger F_s, F_t^\dagger F_u\right] = F_r^\dagger F_u \delta_{st} - F_t^\dagger F_s \delta_{ru} \tag{C.5}$$

where

$$\boxed{A = \sum_{r=1}^{\infty} a_r F_r^{\dagger} F_r \qquad \text{(C.13)}}$$

Comments

1. Commutators involving field operators

The anticommutation relations (5.11) and (5.12) and the identities given in the appendix of *QLB: Introductory Topics* yield a set of identities similar to the above.

For example, corresponding to (C.1) is

$$\boxed{\left[F_r(\vec{x}_r), F_s^{\dagger}(\vec{x}_s)F_t(\vec{x}_t)\right] = F_t(\vec{x}_t)\delta(\vec{x}_r - \vec{x}_s)\delta_{rs} \qquad \text{(C.14)}}$$

2. Commutators involving momentum and spin

The anticommutation relations (5.28) and (5.29) and the identities given in the appendix of *QLB: Introductory Topics* yield a set of identities similar to the above.

For example, corresponding to (C.1) is

$$\boxed{\left[F_r(\vec{p}_r), F_s^{\dagger}(\vec{p}_s)F_t(\vec{p}_t)\right] = F_t(\vec{p}_t)\delta(\vec{p}_r - \vec{p}_s)\delta_{rs} \qquad \text{(C.15)}}$$

$$\left[F_r^\dagger F_s, F_t^\dagger F_u^\dagger\right] = F_r^\dagger\left(-F_t^\dagger \delta_{su} + F_u^\dagger \delta_{st}\right) \tag{C.6}$$

$$\left[F_r^\dagger F_s, F_t^\dagger F_u^\dagger F_v^\dagger\right] = F_r^\dagger\left(F_t^\dagger\left(F_u^\dagger \delta_{sv} - F_v^\dagger \delta_{su}\right) + F_u^\dagger F_v^\dagger \delta_{st}\right) \tag{C.7}$$

$$\begin{aligned} &\left[F_r^\dagger F_s^\dagger, F_t F_u\right] = \\ &= F_r^\dagger(F_u \delta_{st} - F_t \delta_{su}) + F_s^\dagger(F_t \delta_{ru} - F_u \delta_{rt}) - \delta_{ru}\delta_{st} + \delta_{su}\delta_{rt} \end{aligned} \tag{C.8}$$

$$\left[F_r, F_s^\dagger F_t^\dagger F_u F_v\right] = \left(F_t^\dagger \delta_{rs} - F_s^\dagger \delta_{rt}\right) F_u F_v \tag{C.9}$$

$$\left[F_r^\dagger, F_s^\dagger F_t^\dagger F_u F_v\right] = F_s^\dagger F_t^\dagger (F_v \delta_{ru} - F_u \delta_{rv}) \tag{C.10}$$

$$\begin{aligned} &\left[F_r^\dagger F_s, F_t^\dagger F_u^\dagger F_v F_w\right] = \\ &= F_r^\dagger\left(F_u^\dagger \delta_{st} - F_t^\dagger \delta_{su}\right) F_v F_w + F_t^\dagger F_u^\dagger (F_w \delta_{rv} - F_v \delta_{rw}) F_s \end{aligned} \tag{C.11}$$

$$e^A F_r e^{-A} = e^{-a_r} F_r \tag{C.12}$$

3. **Commutators involving momentum and helicity**

The anticommutation relations (5.46) and (5.47) and the identities given in the appendix of *QLB: Introductory Topics* yield a set of identities similar to the above.

For example, corresponding to (C.1) is

$$\left[F^r(\vec{p}_r), F^{s\dagger}(\vec{p}_s)F^t(\vec{p}_t)\right] = F^t(\vec{p}_t)\delta(\vec{p}_r - \vec{p}_s)\delta_{rs} \tag{C.16}$$

C.2 Commutators for bosons

The following commutators result from the commutation relations (6.28) and (6.29) and from the identities given in the appendix of *QLB: Introductory Topics.*

$$\left[B_r, B_s^\dagger B_t\right] = B_t\delta_{rs} \tag{C.17}$$

$$\left[B_r^\dagger, B_s^\dagger B_t\right] = -B_s^\dagger\delta_{rt} \tag{C.18}$$

$$\left[B_r, B_s^\dagger B_t^\dagger\right] = B_t^\dagger\delta_{rs} + B_s^\dagger\delta_{rt} \tag{C.19}$$

$$\left[B_r^\dagger, B_s B_t\right] = -B_t\delta_{rs} - B_s\delta_{rt} \tag{C.20}$$

$$\left[B_r^\dagger B_s, B_t^\dagger B_u\right] = B_r^\dagger B_u\delta_{st} - B_t^\dagger B_s\delta_{ru} \tag{C.21}$$

$$e^{A}B_{r}e^{-A} = e^{-a_{r}}B_{r} \tag{C.22}$$

where

$$A = \sum_{r=1}^{\infty} a_{r}B_{r}^{\dagger}B_{r} \tag{C.23}$$

Comments

1. **Parastatistics**

That the commutators (C.1) for fermions and (C.17) for bosons have the same form has given rise to the notion of parastatics.

2. **Commutators involving field operators**

The commutation relations (6.39) and (6.40) and the identities given in the appendix of *QLB: Introductory Topics* yield a set of identities similar to the above.

For example, corresponding to (C.17) is

$$\left[B_{r}(\vec{x}_{r}), B_{s}^{\dagger}(\vec{x}_{s})B_{t}(\vec{x}_{t})\right] = B_{t}(\vec{x}_{t})\delta(\vec{x}_{r} - \vec{x}_{s})\delta_{rs} \tag{C.24}$$

3. **Commutators involving momentum and spin**

The commutation relations analogous to analogous to (5.28) and (5.29) and the identities given in the appendix of *QLB: Introductory Topics* yield a set of identities similar to the above.

For example, corresponding to (C.17) is

$$\boxed{\left[B_r(\vec{p}_r), B_s^\dagger(\vec{p}_s)B_t(\vec{p}_t)\right] = B_t(\vec{p}_t)\delta(\vec{p}_r - \vec{p}_s)\delta_{rs} \qquad \text{(C.25)}}$$

4. **Commutators involving momentum and helicity**

The commutation relations analogous to (5.46) and (5.47) and the identities given in the appendix of *QLB: Introductory Topics* yield a set of identities similar to the above.

For example, corresponding to (C.17) is

$$\left[B^r(\vec{p}_r), B^{s\dagger}(\vec{p}_s)B^t(\vec{p}_t)\right] = B^t(\vec{p}_t)\delta(\vec{p}_r - \vec{p}_s)\delta_{rs} \qquad \text{(C.26)}$$

$$\left[B^r(p_r), B^{s\dagger}(p_s)B^t(p_t)\right] = B^t(p_t)\delta(p_r - p_s)\delta_{rs} \qquad \text{(C.27)}$$

C.3 Commutators for fermions and bosons

Commutators for fermions and bosons are derived from (4.32),(4.33), (6.28) and (6.29) and from the results given in the previous sections.

The following commutator arises in the calculations in Chapter 7.

$$\left[F_r^\dagger F_s B_t, F_u^\dagger F_v B_w^\dagger\right] =$$
$$-F_r^\dagger F_u^\dagger F_s F_v \delta_{tw} + F_r^\dagger B_w^\dagger F_v B_t \delta_{su} - F_u^\dagger B_w^\dagger F_s B_t \delta_{rv} + F_r^\dagger F_v \delta_{su} \delta_{tw} \tag{C.28}$$

Commutators involving other variables

Commutators involving fermion and boson field operators, or momentum and spin operators or momentum and helicity operators are similar to the above. The examples in the previous sections illustrate the appropriate correspondence.

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