GROUPS, ALGEBRAS, and REPRESENTATIONS

A GROUP is a set of objects together with a rule for multiplying them such that:

i) $(a * b) * c = a * (b * c)$

ii) There is some "identity element" $\mathbb{1}$ such that $\mathbb{1} * a = a * \mathbb{1} = a$ for any a

iii) For every a, there is an inverse a^{-1} such that $a * a^{-1} = a^{-1} * a = \mathbb{1}$

The set of symmetries of a physical system always forms a group, where the product corresponds to performing one symmetry transformation after another, the identity corresponds to doing nothing, and the inverse of any symmetry transformation is simply the map that brings us back to the original configuratio

Groups may be DISCRETE, with a finite number of elements (e.g. the set of rotations that map a cube into itself) or CONTINUOUS, with elements dependi on a set of continuous parameters (e.g. the full set of rotations).

For a given symmetry, there are a set of rules which describe how the symmetry transformations act on the variables describing the configurations of a physical system. In the simplest cases, these rules take the form

$$\phi_i \to M_{ij}(g)\phi_j \qquad (+)$$

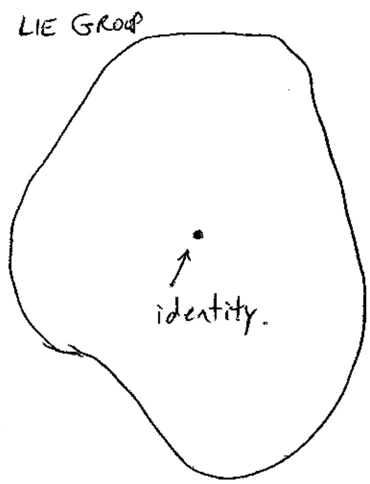
where the ϕ_i's are a set of physical quantities that form a vector space (e.g. a set of components of a wavefunction for a spin j particle, or the components of the momentum vector). For consistency, we must demand that

$$M(g_1)\,M(g_2) = M(g_1 g_2) \qquad (\star)$$

i.e. the action of two successive symmetry transforms must be the same as the action of the product of the two.

Such a map, which associates a linear transformation on a vector space (+) to every element of a group G and satisfies a consistency condition ($\star$) is known as a LINEAR REPRESENTATION.

In physics, we are often interested in the action of a continuous symmetry group (e.g. rotations or Lorentz transformations). In this case, we can visualize the symmetry group as a continuous space on which we can locally put a set of coordinates. Mathematically, we say the group has the structure of a differentiable manifold, and the group is known as a LIE GROUP.



It turns out to be extremely important to understand the infinitesimal transformations. In our picture, these are the ones that lie very close to the identity element (the symmetry transformation that does nothing). If we parameterize the set of group elements near the identity by some coordinates θ_i such that $g(\theta=0) = 1$, then we can define the <u>generators</u> of the group to be a set of independent tangent vectors to the space at the point $\theta=0$. In other words if we write a Taylor expansion of g around $\theta = 0$, we have

$$g(\theta) = 1 + \theta_j h_j + \mathcal{O}(\theta^2)$$

and we can define h_j to be the generators. The set of all tangent vectors $\theta_j h_j$ forms a vector space, corresponding to the set of all possible infinitesimal symmetry transformations.

For any infinitesimal transformation $g = \underline{1} + \theta_i h_i$, we can produce a finite transformation by acting a large number of times. More precisely, we can take

$$\lim_{N \to \infty} \left(1 + \frac{1}{N}\theta_i h_i\right)^N = e^{\theta_i h_i}$$

to see that the exponential of any element $\theta_i h_i$ in the tangent space is some finite element of the group.

Another crucial property of the set of generators h_i is that the commutator of any two of them must give back a linear combination of the generators. To see this, consider two group elements

$$g_1 = e^{\varepsilon h_i} \qquad g_2 = e^{\varepsilon h_j}$$

Then generally $g_1 g_2 \neq g_2 g_1$, but it must be true that $g_1 g_2 g_1^{-1} g_2^{-1}$ is some other group element. Expanding this out in powers of ε, we find:

$$g_1 g_2 g_1^{-1} g_2^{-1} = 1 + \varepsilon^2 [h_i, h_j] + \mathcal{O}(\varepsilon^3)$$

For $\varepsilon \to 0$, this must be some infinitesimal transformatio so we conclude that $[h_i, h_j]$ must be a linear combination of the h's:

$$[h_i, h_j] = C_{ij}^{k} \, h_k \qquad (**)$$

The coefficients C_{ij}^{k} (known as the STRUCTURE CONSTANTS of the group) determine almost everything we need to know about the group. For, if g_1 and g_2 are any two group elements of the form

$$g_1 = e^{\theta_i h_i} \qquad g_2 = e^{\tilde{\theta}_i h_i}$$

Then:

$$g_1 g_2 = e^{\theta_i h_i} e^{\tilde{\theta}_i h_i}$$

$$= e^{(\theta_i + \tilde{\theta}_i) h_i + \frac{1}{2} \theta_i \tilde{\theta}_j [h_i, h_j] + \cdots} \qquad (++)$$

In this formula, known as the Baker-Campbell-Hausdorf formula, all the higher order terms are of the form of nested commutators (e.g. $[h_{i_1}, [h_{i_2}, \cdots, h_{i_n}] \cdots]$) so using the commutation relations (**) we can eventually write the whole expression on the right hand side of (++) in the form:

$$g_1 g_2 = e^{f_i(\theta, \tilde{\theta}) h_i}$$

Thus, the commutation relations (**) guarantee that the product of two group elements is another group element, and tells us which group element we get.

Note that the vector space of all tangent vectors $\theta_i h_i$ together with the commutator operation that maps any two vectors to a third has the mathematical structure of an ALGEBRA. This is called the LIE ALGEBRA of the group.

One of the nice consequences of all this structure is that we can understand the possible representations of a group to a large extent by understanding the representations of the algebra. By this, we mean any set of matrices $\mathcal{H}_i$ which satisfy the same commutation relations as the h_i,

$$[\mathcal{H}_i, \mathcal{H}_j] = C_{ij}^k \mathcal{H}_k$$

If we find such a set of matrices, then we can immediately deduce a representation of the group (or at least the part that can be continuously connected to the identity). For any group element $g = e^{\theta_i h_i}$, we simply define

$$M(g) = e^{\theta_i \mathcal{H}_i}$$

Since the commutation relations for h and $\mathcal{H}$ are the same, and these tell us how group elements are multiplied, it immediately follows that our consistency requirement (*) will be satisfied. The only thing left to worry about are group elements that cannot be written as $e^{\theta_i h_i}$, but these usually can be formed from the product of those that can with a finite number of additional group elements (e.g. the parity transformation in the orthogonal group). The remaining problem is then reduced to finding the matrices representing these additional special transformations.