PROBLEM SET 9 SOLUTIONS

(1) ψ is a linear combination of $a_{\vec p}^{a}$ a $b_{\vec p}^{\dagger a}$. Thus in $\langle 0|\psi_\alpha(x)\psi_\beta(y)|0\rangle$, we only have terms:

$$\langle 0|a_{\vec p}^{a}\, a_{\vec q}^{b}|0\rangle$$

$$\langle 0|a_{\vec p}^{a}\, b_{\vec q}^{\dagger b}|0\rangle$$

$$\langle 0|b_{\vec p}^{\dagger a}\, a_{\vec q}^{b}|0\rangle$$

$$\langle 0|b_{\vec p}^{\dagger a}\, b_{\vec q}^{\dagger b}|0\rangle$$

All of these vanish, since $a|0\rangle = \langle 0|b^\dagger = 0$ and $ab^\dagger = -b^\dagger a$.

2 a) Since $S = \int d^D x \left\{ (\partial\phi)^2 + \bar\psi \gamma \partial \psi + (\partial A)^2 \right\}$ is dimensionless, the dimensions of ϕ, ψ, and A must be such that:

$$-D + 2 + 2\dim(\phi) = 0$$

here, we define $\dim(\phi) = n$ if $\phi \sim E^n$

$$-D + 1 + 2\dim(\psi) = 0$$

$$-D + 2 + 2\dim(A) = 0$$

$\therefore \dim\phi = \dim A = \frac{D}{2} - 1$ i.e. dimensions $E^{\frac{D}{2}-1}$

$\dim\psi = \frac{D}{2} - \frac{1}{2}$ dimensions $E^{\frac{D}{2}-\frac{1}{2}}$

b) For $S_{int} = \sum_{n\geq 2} \int d^D x\, \lambda_n \phi^n$, we must have

$$\dim(\lambda_n) + n\dim(\phi) - D = 0$$

$$\Rightarrow \dim(\lambda_n) + n\left(\frac{D}{2} - 1\right) - D = 0$$

$$\Rightarrow \dim(\lambda_n) = n\left(1 - \frac{D}{2}\right) + D$$

Thus, the dimensions of λ_n are $E^{n(1-\frac{D}{2})+D}$.

c) If $\lambda_n \Delta_n$ is dimensionless, then the quantity Δ_n has dimensions $E^{n(\frac{D}{2}-1)-D}$. Δ_n is some function of the experimental inputs (energies/momenta). Thus, if we scale all the energies by ε, Δ_n must scale like

$$\Delta_n \to \varepsilon^{n(\frac{D}{2}-1)-D} \cdot \Delta_n$$

d) For $D=4$, this gives

$$\Delta_n \to \varepsilon^{n-4} \Delta_n$$

so for $n-4 \leq 0$, the size of Δ_n does not get smaller if we decrease all the energies. So the interaction terms that will be most important at low energies are $\lambda_3\phi^3$ and $\lambda_4\phi^4$.

e) Now, suppose we have a term schematically of the form $\int d^4x \, \lambda \partial^k \psi^n \phi^m A^\ell$. The dimension of λ must be such that

$$0 = -4 + \dim(\lambda) + k + n\dim(\psi) + m\dim(\phi) + \ell \dim A$$

$$\Rightarrow 0 = -4 + \dim(\lambda) + k + \tfrac{3}{2}n + m + \ell$$

$$\Rightarrow \dim(\lambda) = 4 - k - \tfrac{3}{2}n - m - \ell$$

Similar to part c), the effects of this term will become smaller and smaller as we decrease the energy unless $\dim(\lambda) \geq 0$

$$\Rightarrow k + \tfrac{3}{2}n + m + \ell \leq 4$$

Since we need at least 3 fields to give interactions, $n+m+\ell \geq 3$. This means k can be 0 or 1. We always need an even number of spinors, so n must be 0 or 2. For $n=2$, we have: $k+m+\ell \leq 1$ but we need another field so $m=1$ or $\ell=1$:

$$\phi\bar{\psi}\psi \qquad A_\mu \bar{\psi}\gamma^\mu\psi$$

← or similar with γ^5 and or ψ^C

For $n=0$, we can have $(m=\ell=2)$, $(m=1, \ell=2)$, $(m=0, \ell=4)$, $(m=3, \ell=0)$, $(m=4, \ell=0)$, all with $k=0$:

$$\phi^3, \ \phi A_\mu A^\mu, \ \phi^4 \qquad \phi^2 A_\mu A^\mu \qquad A_\mu A^\mu A_\nu A^\nu$$

$m=1, \ell=3$
$m=3, \ell=1$
$\ell=1, m=2$
→ no Lorentz-invar. possibilities

Or, with $k=1$, we can have:

$$\phi^2 \partial_\mu A^\mu \quad , \quad A_\mu A^\mu \partial_\nu A^\nu$$

③ a) We have: $\gamma^5 = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}$ for our usual choice of gamma matrices. Thus, if $\psi = \begin{pmatrix} \eta \\ \chi \end{pmatrix}$ then:

$$\psi_L = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \psi = \begin{pmatrix} \eta \\ 0 \end{pmatrix}$$

$$\psi_R = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \psi = \begin{pmatrix} 0 \\ \chi \end{pmatrix}$$

The proper Lorentz-transformations are generated by $\mathcal{J}^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$ $\quad \mathcal{K}^i = \frac{i}{2}\begin{pmatrix} -\sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$

and these clearly don't mix η and χ.

b) We have: $\psi = \psi_L + \psi_R$ so

$$\bar\psi \gamma^\mu \partial_\mu \psi = \bar\psi_L \gamma^\mu \partial_\mu \psi_L + \bar\psi_R \gamma^\mu \partial_\mu \psi_R$$
$$+ \bar\psi_L \gamma^\mu \partial_\mu \psi_R + \bar\psi_R \gamma^\mu \partial_\mu \psi_L$$

but $\bar\psi_L \gamma^\mu \partial_\mu \psi_R = \psi^\dagger \frac{1}{2}(1-\gamma^5)^\dagger \gamma^0 \gamma^\mu \frac{1}{2}(1+\gamma^5)\partial_\mu \psi$

$$= \frac{1}{4}\psi^\dagger \gamma^0 (1+\gamma^5)\gamma^\mu (1+\gamma^5)\partial_\mu \psi$$

$$= \frac{1}{4}\psi^\dagger \gamma^0 \gamma^\mu (1-\gamma^5)(1+\gamma^5)\partial_\mu \psi$$

$$= 0$$

and similarly $\bar\psi_R \gamma^\mu \partial_\mu \psi_R = 0$ (we can also write everything in terms of χ, η to see this)

Thus: $\bar\psi \gamma^\mu \partial_\mu \psi = \bar\psi_L \gamma^\mu \partial_\mu \psi_L + \bar\psi_R \gamma^\mu \partial_\mu \psi_R$

c) We have:

$$\bar{\psi}\psi = \bar{\psi}_L\psi_L + \bar{\psi}_R\psi_R + \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$$

BUT: $\bar{\psi}_L\psi_L = \psi_L^\dagger \frac{1}{2}(1-\gamma^5)^\dagger\gamma^0 \cdot \frac{1}{2}(1-\gamma^5)\psi_L$

$$= \frac{1}{4}\psi_L^\dagger \gamma^0 (1+\gamma^5)(1-\gamma^5)\psi_L$$

$$= 0$$

similarly $\bar{\psi}_R\psi_R = 0$ but $\bar{\psi}_L\psi_R$ and $\bar{\psi}_R\psi_L$ are non-zero.

Thus: $\bar{\psi}\psi = \bar{\psi}_L\psi_R + \bar{\psi}_R\psi_L$

d) We have:

$$S = \int d^4x \left\{ \frac{im}{2}\left(\psi_L^T C\psi_L - \psi_L^\dagger C\psi_L^*\right)\right\}$$

$$\mathcal{L} = \frac{im}{2}\left(\begin{pmatrix}\eta^T & 0\end{pmatrix}\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2\end{pmatrix}\begin{pmatrix}\eta \\ 0\end{pmatrix} - \begin{pmatrix}\eta^\dagger & 0\end{pmatrix}\begin{pmatrix} i\sigma_2 & 0 \\ 0 & i\sigma_2\end{pmatrix}\begin{pmatrix}\eta \\ 0\end{pmatrix}\right)$$

$$= -\frac{m}{2}\eta^T\sigma_2\eta + \frac{m}{2}\eta^\dagger\sigma_2\eta^*$$

Now $(\sigma_2)_{ab} = -i\,\varepsilon_{ab}$ so $\eta^T\sigma_2\eta = -i\,\eta_a\eta_b\cdot\varepsilon_{ab}$. This would vanish for a classical function η_a, but is non-zero if η_a and η_b anticommute.