Problem Set 7

Problem 1

Finish the worksheet from Wednesday's class and read over the posted notes on general representations of the Lorentz group. I will summarize the important lessons from the worksheet on Friday. You don't need to hand this in.

Problem 2

If $\mathcal{J}_i$ and $\mathcal{K}_i$ are the matrices corresponding to the rotation and boost generators in some representation of the Lorentz group, they must satisfy the following commutation relations:

$$[\mathcal{J}_i, \mathcal{J}_j] = i\epsilon_{ijk}\mathcal{J}_k \tag{1}$$

$$[\mathcal{J}_i, \mathcal{K}_j] = i\epsilon_{ijk}\mathcal{K}_k \tag{2}$$

$$[\mathcal{K}_i, \mathcal{K}_j] = -i\epsilon_{ijk}\mathcal{J}_k \ . \tag{3}$$

If we define

$$\mathcal{A}_i = \frac{1}{2}(\mathcal{J}^i + i\mathcal{K}^i) \qquad\qquad \mathcal{B}_i = \frac{1}{2}(\mathcal{J}^i - i\mathcal{K}^i) \ , \tag{4}$$

calculate $[\mathcal{A}_i, \mathcal{A}_j]$, $[\mathcal{B}_i, \mathcal{B}_j]$, and $[\mathcal{A}_i, \mathcal{B}_j]$.

Problem 3

You may wish to look over the posted notes on angular momentum in quantum mechanics.

a) Consider a quantum system consisting of two completely independent parts A and B, each with a finite number of quantum states (e.g. two separate particles with some spin). If J_A^i and J_B^i are the rotation operators acting on these two parts, what are $[J_A^i, J_A^j]$, $[J_B^i, J_B^j]$, and $[J_A^i, J_B^j]$? Compare the result to your answer to problem 2.

b) Suppose now that the two systems are each spin half particles, so that each subsystem has a two-dimensional Hilbert space. Write down a basis of states for the full system. In this basis, write down the explicit matrices for J_A^x, J_A^y, J_A^z, J_B^x, J_B^y, J_B^z.

c) Can you use your results and the discussion of part a) to write down a set of matrices $\mathcal{J}_i$ and $\mathcal{K}_i$ that represent the Lorentz group?

Problem 4

Read chapter 4 of Tongs notes:

http://www.damtp.cam.ac.uk/user/tong/qft/four.pdf

and/or sections 3.1-3.4 of Peskin and Schroeder

Simplify the following expressions:

a) $\gamma^3\gamma^3$

b) $\gamma^1\gamma^2 + \gamma^2\gamma^1$

c) $\gamma^0\gamma^2\gamma^0$

d) $\gamma^\mu\gamma^\nu\gamma_\mu$

e) Suppose we have a Dirac spinor field $\psi_\alpha(x) = (At, 0, 0, 0)$. If we perform a boost in the z direction by velocity $v = 3/5c$, write an expression for the field that we obtain after the boost. Your expression should be explicit, but it's okay if it involves the exponential of a matrix. *Hint: For a vector field, the Lorentz transformation matrix corresponding to a boost with velocity v in the x direction is*

$$\begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} = e^{iaK_x}$$

where

$$K_x = -i \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

and $\tanh(a) = \beta$.