PROBLEM SET 6 SOLUTIONS

① a) We have:

$$S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right\}$$

The terms involving time derivatives in $-\frac{1}{4}F_{\mu\nu}F^{\mu\nu}$ are

$$-\frac{1}{4} F_{0i} F^{0i} - \frac{1}{4} F_{i0} F^{i0}$$

$$= \frac{1}{2}\left(F_{0i}\right)^2$$

$$= \frac{1}{2}\left(\partial_0 A_i - \partial_i A_0\right)^2 = \frac{1}{2}\left(\partial_0 A^i + \partial_i A^0\right)^2$$

remember $A_0 = A^0$, $A_i = -A^i$

So $\frac{\delta L}{\delta \dot{A}_0} = 0$ and $\frac{\delta L}{\delta \dot{A}^i} = \partial_0 A^i + \partial_i A^0$

Thus: $\boxed{\pi^0 = 0}$ and $\boxed{\pi^i = \partial_0 A^i + \partial_i A^0}$ (or $-\partial_0 A_i + \partial_i A_0$)

If $\pi^0 = 0$ identically, then $[A^0, \pi^0]$ must also be zero.

b) For A^0, the equation of motion can be derived using:

$$S = \int d^4x \left\{ \frac{1}{2}\left(\partial_0 A^i + \partial_i A^0\right)^2 + \frac{1}{2} m^2 (A^0)^2 + \text{terms with no } A^0 \right\}$$

$\therefore$ If we take $A^0 + \delta A^0$

$$\delta S = \int d^4x \left\{ \partial_i \delta A^0 \left(\partial_0 A^i + \partial_i A^0\right) + m^2 A^0 \delta A^0 \right\}$$

$$= \int d^4x \, \delta A^0 \left\{ -\partial_i\left(\partial_0 A^i + \partial_i A^0\right) + m^2 A^0 \right\}$$

The equation of motion is thus

$$\boxed{m^2 A^0 - \partial_i\left(\partial_0 A^i + \partial_i A^0\right) = 0}$$

This is equivalent to:

$$A^0 = \frac{1}{m^2} \partial_i \pi^i$$

c) To derive the Hamiltonian, we write:

$$H = \int d^3x\, \dot{A}^i \Pi^i - L$$

now, eliminate $\dot{A}^i$ via

$$\dot{A}^i = \Pi^i - \partial_i A^0$$
$$= \Pi^i - \frac{1}{m^2}\partial_i \partial_j A^j$$

$$= \int d^3x \left(\Pi^i - \frac{1}{m^2}\partial_i(\nabla\cdot\Pi)\right)\Pi^i$$

integrate this by parts to get a term like this

$$- \int d^3x \left(\frac{1}{2}\Pi^i\Pi^i - \frac{1}{4}F_{ij}F_{ij} - \frac{1}{2}m^2 A^i A^i + \frac{1}{2}m^2 \cdot \left(\frac{1}{m^2}\vec{\nabla}\cdot\vec{\Pi}\right)^2\right)$$

$$= \int d^3x \left(\frac{1}{2}\Pi^i\Pi^i + \frac{1}{2m^2}(\vec{\nabla}\cdot\vec{\Pi})^2 + \frac{1}{4}(\partial_i A^j - \partial_j A^i)^2 + \frac{1}{2}m^2 A^i A^i\right)$$

We can see that the final result is always positive

② From

$$S = \int d^{d+1}x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \right\}$$

we find that under a variation of A_μ,

$$\delta S = \int d^{d+1}x \left\{ -\frac{1}{2} F^{\mu\nu} (\partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu) + m^2 A^\mu \delta A_\mu \right\}$$

$$= \int d^{d+1}x \, \delta A_\nu (\partial_\mu F^{\mu\nu} + m^2 A^\nu)$$

so the equations of motion are:

$$\boxed{\partial_\mu (\partial^\mu A^\nu - \partial^\nu A^\mu) + m^2 A^\nu = 0} \quad (*)$$

Acting with ∂_ν, we have

$$\underbrace{\partial_\mu \partial_\nu}_{\text{symmetric}} \underbrace{(\partial^\mu A^\nu - \partial^\nu A^\mu)}_{\text{antisymmetric}} + m^2 \partial_\nu A^\nu = 0$$

$$\Rightarrow \quad \boxed{\partial_\nu A^\nu = 0}$$

Using this in (*), we have the second term vanishing, so (*) becomes

$$\boxed{(\partial_\mu \partial^\mu + m^2) A^\nu = 0}$$

A plane wave in the x-direction (with definite wavelength) takes the form:

$$A^\mu = \varepsilon^\mu e^{i\vec{p}\cdot\vec{x} - \omega t} \qquad \vec{p} = (p_x, 0, 0)$$

$$\equiv \varepsilon^\mu e^{-i p \cdot x} \qquad \text{where } p^\mu = (\omega, \vec{p})$$

Plugging in to $(\partial_\mu \partial^\mu + m^2) A^\nu = 0$, we get

$$-p^2 + m^2 = 0$$

So: $\quad (p^0)^2 = \vec{p}^{\,2} + m^2 \implies p^0 = \pm\sqrt{\vec{p}^{\,2} + m^2}$

Plugging in to $\partial_\mu A^\mu = 0$, we get

$$p_\mu \cdot \varepsilon^\mu = 0$$

Thus, we need ε^μ "perpendicular" to p_μ. There are 3 independent choices for ε^μ for each p^μ. In our case, $p^\mu = (\pm\sqrt{p_x^2 + m^2}, p_x, 0, 0)$ so we could take

$$\varepsilon_1^\mu = (0, 0, 1, 0), \quad \varepsilon_2^\mu = (0, 0, 0, 1),$$

$$\varepsilon_3^\mu = \left(\mp \frac{p_x}{m}, \sqrt{1 + \frac{p_x^2}{m^2}}, 0, 0\right)$$

Then the general solution would be

$$A^\mu = \sum_r c_r \varepsilon_r^\mu \, e^{i p_x \cdot x \mp \sqrt{p_x^2 + m^2}\, t}$$

③ P&S 2.1: a) We have $S = \int d^4x \left\{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right\}$ with $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$

Varying the action with respect to A_μ, we find

$$\delta S = \int d^4x \left\{ -\frac{1}{2} F^{\mu\nu} \{ \partial_\mu \delta A_\nu - \partial_\nu \delta A_\mu \} \right\}$$

$$= \int d^4x \left\{ \partial_\mu F^{\mu\nu} \delta A_\nu \right\}$$

The e.o.m. is then:

$$\partial_\mu F^{\mu\nu} = 0$$

For $\nu = 0$, this gives $\partial_i F^{i0} = 0 \Rightarrow \boxed{\nabla \cdot \vec{E} = 0}$

For $\nu = j$, this gives $\partial_0 F^{0j} + \partial_i F^{ij} = 0$

$$\Rightarrow -\frac{\partial E^j}{\partial t} - \partial_i \varepsilon^{ijk} B^k = 0$$

$$\Rightarrow \boxed{\vec{\nabla} \times \vec{B} = +\frac{\partial \vec{E}}{\partial t}}$$

Note that the remaining equations are automatic from the definition of F in terms of A:

$$F_{0i} = \partial_0 A_i - \partial_i A_0 \Rightarrow E^i = -\frac{\partial A^i}{\partial t} - \nabla_i V$$

$$F_{ij} = \partial_i A_j - \partial_j A_i \Rightarrow B^i = (\nabla \times A)^i$$

These imply $\boxed{\nabla \cdot B = 0}$ and $\boxed{\vec{\nabla} \times \vec{E} = -\frac{\partial B}{\partial t}}$