Problem Set 6

To start, read through the "notes on quantization" on the course website.

Problem 1

Consider the theory of a vector field A^μ with action

$$S = \int d^4x \{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} m^2 A_\mu A^\mu \}$$

where

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \ .$$

a) Derive the canonically conjugate momenta to A^0 and A^i. Explain why taking

$$[A^0(x), \pi^0(y)] = i\delta(x - y)$$

would not make sense.

b) Derive the equation of motion for A^0 and show that A^0 can be expressed in terms of the momenta π^i conjugate to the spatial components of the field.

c) Determine the Hamiltonian for this theory in terms of the fields $A_i(x)$ and their conjugate momenta $\pi_i(x)$. You should be able to use your result for b) to eliminate A^0. Thus, your result should be a function only of the three spatial components of the field and the conjugate momenta for these.

Problem 2

A key step in understanding the physics of vector field theories is to write the field operator in terms of creation and annihilation operators obeying the standard commutation relations.

In the expression for a *scalar field* in terms of creation and annihilation operators, each independent creation operator is multiplied by an independent solution of the Klein-Gordon equation with positive frequency ($e^{i\omega t}$ time dependence) while each annihilation operator is multiplied by an independent solution of the Klein-Gordon equation with negative frequency ($e^{i\omega t}$ time dependence). If we replace the creation and annihilation operators by ordinary complex functions $z(p)$ and $\bar{z}(p)$, the expression for the field becomes an expression for the most general classical solution of the Klein-Gordon equation.

It turns out that these properties are true for vector fields as well. To come up with an expression for the vector field operators in terms of creation and annihilation operators, we can try to find an expression for the general solution to the vector field equation and then replace the coefficients for each positive/negative frequency solution with a creation/annihilation operator.

a) Derive the equations of motion for A_μ for a vector field in problem 2 (write it in a relativistic notation i.e. don't break it into components).

b) Show that the equations imply the pair of equations

$$\begin{aligned} m^2 \partial_\mu A^\mu &= 0 \\ (\partial_\mu \partial^\mu + m^2) A^\nu &= 0 \end{aligned}$$

Thus, each component of A must satisfy the Klein-Gordon equation, and there is an additional constraint relating the components.

c) Find the most general solution of these equations that takes the form of a plane wave in the x direction. How many linearly independent solutions of this form are there for a given wavelength?

Problem 3

Consider the theory of a massless vector field A_μ with action

$$S = \int d^4x \{ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} \} \tag{1}$$

$$F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu \; . \tag{2}$$

Defining $E_i = F_{0i}$ and $B_i = -\frac{1}{2}\epsilon_{ijk} F_{jk}$ (e.g. $B_z = -F_{xy}$), show that the definition (2) and the equation of motion that follows from the action together imply the source-free Maxwell's equations.

$$\begin{aligned} \vec{\nabla} \cdot \vec{E} &= 0 \\ \vec{\nabla} \times \vec{B} &= \frac{\partial \vec{E}}{\partial t} \\ \vec{\nabla} \cdot \vec{B} &= 0 \\ \vec{\nabla} \times \vec{E} &= -\frac{\partial \vec{B}}{\partial t} \\ & . \end{aligned}$$

In other words, (1) gives the action for an electromagnetic field.