Problem Set 5

Problem 1

Starting with the classical expression

$$P_x = -\int d^3x \dot{\phi} \partial_x \phi$$

for the conserved x-momentum in a 3+1 dimensional scalar field theory, show that the momentum operator is:

$$P_x = \int \frac{d^3p}{(2\pi)^3} p_x a_{\vec{p}}^\dagger a_{\vec{p}} \, .$$

From this (and using the commutation relations for a and $a^\dagger$, show that the state

$$a_{\vec{p}}^\dagger |0\rangle$$

is an eigenstate of momentum with eigenvalue p_x.

Problem 2

Starting from $[a_p, a_q^\dagger] = 2\pi\delta(p-q)$ for a 1+1 dimensional field theory, calculate $[\phi(x_1, t_1), \phi(x_2, t_2)]$. What does the result give when $t_2 = t_1$?

Problem 3

Starting with the expression for the field in terms of creation and annihilation operators, calculate the correlation function $\langle 0|\phi(\vec{x})\phi(0)|0\rangle$. Express your result explicitly as a function of $|x|^2$ using Bessel functions and determine explicitly the asymptotic form of the function for large $|x|$. What happens to the correlation function as we increase the mass? If you have trouble evaluating the integral explicitly, do it numerically, produce a plot, and determine the large $|x|$ behaviour.

Tip: For integrals like

$$\int d^d k e^{ik\cdot x} f(|k|^2)$$

a useful trick is to recognize that they can only depend on the length of $\vec{x}$, since a rotation on $\vec{x}$ leaves the integral invariant (we can compensate by rotating the integration variable). Thus, it must be that

$$\int d^d k e^{ik\cdot x} f(|k|^2) = F(|x|^2)$$

To find F, we can use the same equation but assume that $\vec{x}$ points only along the x direction:

$$\int d^d k e^{ik_x x} f(|k|^2) = F(x^2) \, .$$

You are welcome to use Maple/Mathematica/Wolfram Alpha for integrals that you find here.

Problem 4

a) Write the most general action for a vector field A_μ in 3+1 dimensions that is local, Poincaré- invariant, gives rise to a linear equation of motion, and is such that the classical evolution is completely determined by A and $\dot{A}$. Treat as equivalent terms that differ by a total derivative, since these give the same result for field configurations that vanish at infinity.

b) Find the most general action if we additionally require that the theory has a symmetry under transformations $A_\mu \to A_\mu + \partial_\mu \lambda$, where λ is an arbitrary function of space and time.

c) In 2+1 dimensions, there is another possible term for part a). What is it?