Building Lorentz-Invariant Actions, part I: Scalar, Vector and Tensor Fields

So far, we've seen that quantizing fields allows us to describe systems with arbitrary numbers of particles, and that symmetries in local field theories lead to local conservation laws. Last time, we understood that different kinds of particles are distinguished by how the quantum symmetry transformations act on the particle states, while different kinds of fields are also classified by their transformation rules under symmetries. As we will see, field theories based on fields with different transformation properties will describe different kinds of particles. But the first step will be to understand what the allowed transformation properties are (fora given set of symmetries) and how to write down actions that are invariant under these symmetry transformations.

In particle physics, the most fundamental symmetries are those implied by Einstein's principle of relativity: "The laws of physics are the same in all inertial reference frames." For field theory, this means that given some physical field trajectory (i.e. the field as a function of space and time) a new field trajectory equivalent to how the original one would be perceived in another frame of reference is also physical. This will be guaranteed by demanding that the action should be invariant under translations, rotations, and boosts, collectively known as the Poincaré transformations (see the notes on Special Relativity for a review).

We recall that the Poincaré transformations may be defined by their action on the coordinates of an event,

$$x^\mu \to \Lambda^\mu{}_\nu x^\nu + a^\mu \tag{1}$$

where a^μ is a constant vector parameterizing translations and Λ is a matrix satisfying

$$\Lambda^T \eta \Lambda = \eta \; . \tag{2}$$

In a field theory, symmetries should always be considered to act on the fields (i.e. they transform the physical variables to new ones), so for example a translation is a transformation

$$\tilde{\phi}(x^\mu) = \phi(x^\mu + a^\mu)$$

that gives us a new field in terms of an old one. Note in particular that in the expression for an action , the symmetry transformation *does not act on the coordinates used as integration variables, only on the fields.*

Translation Invariance

Let's begin by thinking about translations:

What property of an action will ensure that the physics is translation-invariant?

Answer: The action should treat all points in space and time equivalently. For a field theory, this will be ensured by writing the action as an integral over all space and time *of some quantity that doesn't explicitly depend on the coordinates (e.g. $\phi^2(x,t)$ but not $x^2\phi(x,t)$).*

Invariants under Lorentz tranformations

We'd now like to understand how to write actions that are Lorentz-invariant. It is useful to begin by recalling that the basic invariant quantity in special relativity is the combination

$$(\Delta s)^2 = c^2(\Delta t)^2 - (\Delta\vec{x})^2 = \Delta x^\mu \eta_{\mu\nu} \Delta x^\nu$$

where we have defined

$$\eta_{\mu\nu} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} .$$

Since $\Delta x^\mu \eta_{\mu\nu} \Delta x^\nu$ is invariant, it follows that $a^\mu \eta_{\mu\nu} b^\nu$ will be invariant for any quantities a and b that transform like the coordinates. Recalling that Alternatively, we can say that $b^\mu a_\mu$ will be invariant, where b^μ transforms like x^μ and a_μ transforms like $\eta_{\mu\nu}x^\nu$.

What is the transformation rule for $x_\mu = \eta_{\mu\nu}x^\nu$ under a boost with

$$\Lambda^\mu{}_\nu = \begin{pmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} ?$$

Answer:

$$x_\mu \to \begin{pmatrix} \gamma & \gamma\beta & 0 & 0 \\ \gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} x_\mu .$$

In matrix language, what is the transformation rule for ηx?

Answer: We have

$$\eta x \to \eta \Lambda x = (\eta \Lambda \eta)(\eta x) ,$$

where we have used that $\eta^2 = 1$. From (2), we have that $\eta\Lambda\eta = (\Lambda^{-1})^T$, so we can say $x_\mu \to (\Lambda^{-1})^T{}_\mu{}^\nu x_\nu$ or

$$x_\mu \to (\Lambda^{-1})^\nu{}_\mu x_\nu . \qquad (3)$$

Lorentz Tranformations of Fields

In order to understand how to write down actions that are invariant under Lorentz transformations, we first need to understand how the fields transform. Under a Lorentz transformation, an event that happened at x before the transformation will happen at Λx in the transformed configuration. Thus, the fields in the new configuration at the coordinates Λx should be determined in terms of the old fields at the coordinates x:

$$\{\tilde{\phi}_i(\Lambda x)\} \qquad \text{determined by} \qquad \{\phi_i(x)\}$$

As for rotations, the simplest possibility is a SCALAR FIELD,

$$\tilde{\phi}(\Lambda x) = \phi(x)$$

or

$$\tilde{\phi}(x) = \phi(\Lambda^{-1} x) \ . \tag{4}$$

However, as for rotations, it's also possible that the symmetry may mix up the components of the field. By analogy with rotations, we can also have VECTOR FIELDS

$$\tilde{\phi}^\mu(x) = \Lambda^\mu{}_\nu \phi^\nu(\Lambda^{-1} x) \ .$$

and TENSOR FIELDS, for example

$$\tilde{\phi}^{\mu_1 \mu_2}(x) = \Lambda^{\mu_1}{}_{\nu_1} \Lambda^{\mu_2}{}_{\nu_2} \phi^{\nu_1 \nu_2}(\Lambda^{-1} x) \ .$$

A tensor field with two indices generally has 16 independent components, but we can specialize to fields that are ANTISYMMETRIC $\phi^{\mu\nu} = -\phi^{\nu\mu}$ (with 6 independent components), or SYMMETRIC $\phi^{\mu\nu} = \phi^{\nu\mu}$ (with 10 independent components) or symmetric and TRACELESS $\eta_{\mu\nu}\phi^{\mu\nu} = 0$ (with 9 independent components), since the symmetry transformations respect these extra constraints.

We want also want the actions to depend on derivatives of the fields, so we should understand how these transform under Lorentz transformations.

Does $\frac{\partial \phi}{\partial x^\mu}$ transform in the same way as a vector field ϕ^μ?

Answer: No. From (4) we find that

$$\frac{\partial}{\partial x^\mu} \tilde{\phi}(x) = \frac{\partial}{\partial x^\mu} (\phi(\Lambda^{-1} x)) = (\Lambda^{-1})^\nu{}_\mu \frac{\partial \phi}{\partial x^\mu} (\Lambda^{-1} x) \ .$$

The components here transform in the same way as x with a lower index (3), so $\frac{\partial \phi}{\partial x^\mu}$ transforms like a field $\phi_\mu = \eta_{\mu\nu}\phi^\nu$.

Building Lorentz-Invariant actions

We'd now like to figure out how to build Lorentz-invariant actions. We have seen that the Lorentz transformations result in shifts of the field in spacetime but also a mixing of components. To understand how to build quantities that are invariant, consider first this question:

How does $\int d^d x \phi(x)$ transform under a Lorentz transformation if $\phi(x)$ is a scalar field?

Answer: It is invariant. The transformed action is

$$\int d^d x \tilde{\phi}(x) = \int d^d x \phi(\Lambda^{-1} x)$$

Making the change of variables $x = \Lambda \tilde{x}$ (note that this is not part of the symmetry transformation), we find $d^d x = |\det(\Lambda)| d^d \tilde{x}$, but (2) implies

$$\det(\Lambda^T \eta \Lambda) = \det(\eta) \Rightarrow \det(\Lambda) = \pm 1 \ . \tag{5}$$

We conclude that

$$\int d^d x \tilde{\phi}(x) = \int d^d \tilde{x} \phi(\tilde{x})$$

Our demonstration that this expression is invariant relied only on the fact that $\phi(x)$ transforms as a scalar field. Thus, any action of the form

$$S = \int d^d x \mathcal{L}(x)$$

will be invariant so long as $\mathcal{L}(x)$ is some expression built from the various fields that transforms in the same way as a scalar field,

$$\mathcal{L}(x) \to \mathcal{L}(\Lambda^{-1} x) \ .$$

We have seen that $\mathcal{L} = \phi$ works, but it is easy to check that $\mathcal{L} = f(\phi)$ also has the same transformation law as a single scalar. To construct scalar $\mathcal{L}$s using vector and tensor fields or using derivatives, we can use the fact that the components of the fields we have defined are mixed up in precisely the same way as the coordinates x^μ (for the case of a vector field), as products of coordinates (e.g. $x^\mu y^\nu$ for a 2-index tensor), or as coordinates x_μ with lower a lower index (as for derivatives of a scalar field). To come up with combinations that are scalar fields, we can therefore just take products of vectors using the η tensor, or contract all the upper and lower indices.[1]

[1] Here, to "contract the indices" means to label an upper index of one expression with the same label as a lower index on another (or the same) expression and to sum this label from 0 to 3. For example, in the expression $\phi^\mu \partial_\mu \phi = \phi^t \partial_t \phi + \phi^x \partial_x \phi + \phi^y \partial_y \phi + \phi^z \partial_z \phi$ the upper and lower indices are contracted.