LAST WEEK: transition amplitudes

$$\langle \psi_f | \mathcal{U}(t, t_0) | \psi_i \rangle \quad \rightarrow \text{Compute using Wick's theorem + diagrams.}$$

$t \to \infty \quad t_0 \to -\infty$: - this is the S-matrix

- always proportional to $\delta^4(p_f - p_i)$ (HW)

Define $\mathcal{M}_{fi}$ via:

$$\langle \psi_f | \mathcal{U}(\infty, -\infty) | \psi_i \rangle = i (2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}$$

with normalization

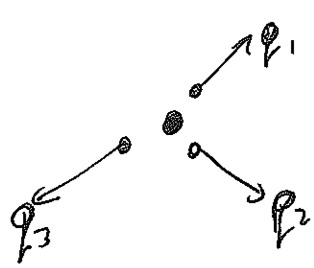
$$|\psi_i\rangle = \sqrt{2\omega_{p_1}}\, a^\dagger_{p_1} \cdots \sqrt{2\omega_{p_n}}\, a^\dagger_{p_n} |0\rangle$$

(preserved by Lorentz transforms)

How does this relate to experiment?

Mostly interested in 1, 2 initial particles

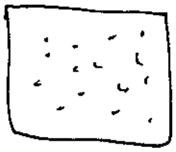
$\boxed{\text{1 initial partice}}$ can decay.



$$\text{decay rate } \Gamma \equiv \frac{\#\text{ decays / unit time}}{\#\text{ particles}}$$

$d\Gamma$ DIFFERENTIAL DECAY RATE — decays to specified set of final particles, e.g. in range $d^3q_1 \cdots d^3q_m$ of momenta.

Q: how does Γ relate to half-life?

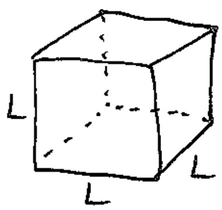
A 

For N particles, have $\frac{1}{N}\frac{dN}{dt} \equiv -\Gamma$

so $N(t) = e^{-\Gamma t} N_0$

By definition $N(t_{\frac{1}{2}}) = \frac{1}{2}N_0$ so $e^{-\Gamma t_{\frac{1}{2}}} = \frac{1}{2} \Rightarrow \Gamma = \frac{\ln(2)}{t_{\frac{1}{2}}}$

How to get $d\Gamma$ from transition amplitude?



simple way: start w. particle in box of volume $V = L^3$ (non-interacting)

- turn on interactions for time T
- find probability for desired final state (e.g. in range $d^3q_1 \ldots d^3q_n$)
  ~proportional to T for T not too large
- divide by T to get rate.

M_{fi}: computed in terms of states

$$|\psi\rangle = \sqrt{2\omega_{p_1}}\, a^\dagger_{p_1} \cdots \sqrt{2\omega_{p_n}}\, a^\dagger_{p_n} |0\rangle \qquad [a_p, a^\dagger_q] = (2\pi)^3 \delta(\vec{p}-\vec{q})$$

How does this relate to properly normalized state of n particles in a box?

continuous		discrete
$\vec{p}$	$\longleftrightarrow$	$\vec{p} = \left(\frac{2\pi}{L}\right)(n_x, n_y, n_z)$
$\int d^3\vec{p}\, f(\vec{p})$	$\longleftrightarrow$	$\sum_p f(\vec{p}) \left(\frac{2\pi}{L}\right)^3$
$\delta^3(\vec{p}-\vec{q})$	$\longleftrightarrow$	$\left(\frac{L}{2\pi}\right)^3 \delta_{\vec{p},\vec{q}}$

$$[a_p, a_q^\dagger] = (2\pi)^3 \delta^3(\vec{p}-\vec{q}) \longleftrightarrow [a_p, a_q^\dagger] = V \delta_{\vec{p},\vec{q}}$$

$$(a_p^\dagger)_{box} = \frac{1}{\sqrt{V}} a_p^\dagger$$

Properly normalized state of n particles is

$$\frac{1}{V^{n/2}} \frac{1}{\sqrt{2\omega_{p_1}}} \cdots \frac{1}{\sqrt{2\omega_{p_n}}} |\psi\rangle \equiv |\psi_{box}\rangle$$

$$(a_{p_1}^\dagger)_{box} \cdots (a_{p_n}^\dagger)_{box} |0\rangle$$

Probability of $1 \to n$ particle transition

$$P = |\langle \psi_f^{box} | \mathcal{U}(\tfrac{T}{2}, -\tfrac{T}{2}) | \psi_i^{box} \rangle|^2$$

$$= \frac{1}{V^{n+1}} \frac{1}{2\omega_{p_1}} \frac{1}{2\omega_{q_1}} \cdots \frac{1}{2\omega_{q_n}} |\langle \psi_f | \mathcal{U}(\tfrac{T}{2}, -\tfrac{T}{2}) | \psi_i \rangle|^2$$

$T, L \to \infty$: $\langle \psi_f | \mathcal{U}(\tfrac{T}{2}, -\tfrac{T}{2}) | \psi_i \rangle \longrightarrow i(2\pi)^4 \delta^4(p_f - p_i) \mathcal{M}_{fi}$

(by definition)

finite T, L :

~~Transition~~

$$[(2\pi)^3 \delta^3(p_f - p_i)]^2 \to V^2 \delta_{p_f, p_i}$$

$$[(2\pi)\delta(E_f - E_i)]^2 \to \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{it(E_f - E_i)} dt \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{it(E_f - E_i)} dt$$

$$\sim T \cdot \int_{-\frac{T}{2}}^{\frac{T}{2}} e^{it(E_f - E_i)} dt \to T^2 \delta_{E_f, E_i}$$

$$\therefore \frac{\text{Prob}}{T} = \frac{1}{V^{1+n}} \frac{1}{2\omega_{p_1}} \frac{1}{2\omega_{q_1}} \cdots \frac{1}{2\omega_{q_n}} |\mathcal{M}_{fi}|^2 \cdot V \cdot (V \delta_{p_f, p_i}) \cdot (T \delta_{E_f, E_i}) \quad \star$$

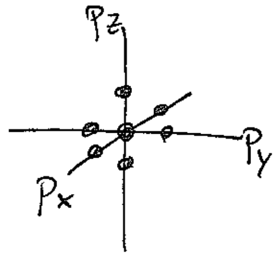
to specific state

$V \to \infty$ prob. to specific momenta $\to 0$

- want probability to go to particles with momenta in range $d^3q_1 \cdots d^3q_n$

Q: # final states in this range?

A: Allowed momenta $\vec{p} = \frac{2\pi}{L}(n_1, n_2, n_3)$



density of points in momentum space:

$$\left(\frac{L}{2\pi}\right)^3 = \frac{V}{(2\pi)^3}$$

In range $d^3q_1 \cdots d^3q_m$ have $\dfrac{d^3q_1 \cdots d^3q_m}{\left(\frac{V}{2\pi}\right)^m}$ possible states

Multiply by this to get prob. into range

$$\boxed{\frac{\text{Prob}}{T} = d\Gamma = |M_{fi}|^2 \frac{1}{\underbrace{2M}_{\substack{\omega_p \\ \text{for particle} \\ \text{at rest}}}} \frac{1}{2\omega_{q_1}} \cdots \frac{1}{2\omega_{q_m}} (2\pi)^4 \delta^4(p_f - p_i) \frac{d^3q_1}{(2\pi)^3} \cdots \frac{d^3q_n}{(2\pi)^3}}$$

total decay rate: integral over q's, sum over possible kinds of final particles