Quantizing the Dirac Spinor Field

We have

$$S = \int d^4x \left\{ i\bar{\psi}\gamma^\mu \partial_\mu \psi - m\bar{\psi}\psi \right\}$$

Want to write ψ in terms of cr. and annih. ops so H, P, etc... are simple (this lets us determine particle properties)

Recall: scalar field

real: $$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \left(a_p e^{-ip\cdot x} + a_p^\dagger e^{ip\cdot x} \right)$$

same as expression for general soln to classical field eqn if we interpret a, $a^\dagger$ as complex numbers

complex scalar:
(see HW 2.4)

$$\phi(x,t) = \int \frac{d^3p}{(2\pi)^3} \left(a_{\vec{p}}\, e^{-ip\cdot x} + b_p^\dagger e^{ip\cdot x} \right)$$

2 different creation ops for each p:

$a_p^\dagger$ creates particles of charge +1

$b_p^\dagger$ creates particles of charge −1

for charge associated w. $\phi \to e^{i\alpha}\phi$ symmetry.

We now can guess a similar expression for the Dirac field (as we did for the vector) and see whether it has the right properties.

The general solns to the Dirac equation can be expanded in a basis

$$u_r(\vec{p})\, e^{-ip\cdot x} \quad (r=1,2)$$

$$v_r(\vec{p})\, e^{ip\cdot x} \quad (r=1,2)$$

where $p^0 \equiv \sqrt{m^2+\vec{p}^{\,2}}$

Thus, we try

$$(\star) \quad \boxed{\psi(x,t) = \int \frac{d^3p}{(2\pi)^3}\frac{1}{\sqrt{2E_p}}\left(a^r_{\vec{p}}\, u_r(\vec{p})e^{-ip\cdot x} + b^{\dagger r}_{\vec{p}}\, v_r(\vec{p})e^{ip\cdot x}\right)}$$

NOT THE RIGHT THING TO DO:

Following the usual quantization procedure, we would have:

conjugate momentum: $\dfrac{\delta L}{\delta \dot{\psi}} = i\psi^\dagger$

Hamiltonian $\quad H = \int d^3x \left\{ i\psi^\dagger \dot{\psi} - L \right\}$

$$= \int d^3x \left\{ -i\bar{\psi}\gamma^i\partial_i\psi + m\bar{\psi}\psi \right\}$$

$$= \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a^{\dagger r}_{\vec{p}} a^r_{\vec{p}} + b^{\dagger r}_{\vec{p}} b^r_{\vec{p}} \right)$$

(so far, so good)

But: $\left[\psi_\alpha(\vec{x},0)\; i\,\psi^\dagger_\beta(\vec{x},0)\right] = i\,\delta_{\alpha\beta}\,\delta^3(\vec{x}-\vec{y})$

$$\Downarrow$$

$$[a^{\dagger r}_{\vec{p}}, a^s_{\vec{q}}] = (2\pi)^3\delta_{rs}\,\delta^3(\vec{p}-\vec{q})$$

$$[b^{\dagger r}_{\vec{p}}, b^s_{\vec{q}}] = -(2\pi)^3\delta_{rs}\,\delta^3(\vec{p}-\vec{q})$$

BAD!

Q: Calculate norm of $b_p^{\dagger r}|0\rangle$

$$\text{norm} = \langle 0|\, b_p^r\, b_p^{\dagger r}\, |0\rangle$$
$$= \langle 0|\, [b_p^r, b_p^{\dagger r}]\, |0\rangle$$
$$= -(2\pi)^3\, \delta_{rs}\, \delta^3(0)$$
$$= \text{NEGATIVE!}$$

+ve number in regularized finite volume theory.

+ve norms are a basic requirement in quantum mechanics.

THE RIGHT THING TO DO:

According to the worksheet, physical considerations suggest that creation operators for fermions should have ANTICOMMUTATION RELATIONS. Since we suspect that our spinor fields may describe half-integer spin particles, we might see if we get a consistent quantum theory by assuming:

$$\{a_{\vec p}^r, a_{\vec p}^{\dagger s}\} = (2\pi)^3\, \delta^3(\vec p - \vec q)\, \delta_{rs}$$
$$\{b_{\vec p}^r, b_{\vec q}^{\dagger s}\} = (2\pi)^3\, \delta^3(\vec p - \vec q)\, \delta_{rs} \qquad (+)$$

Using (★), we find that these are equivalent to

$$\{\psi_\alpha(\vec x, 0), \psi_\beta^\dagger(\vec y, 0)\} = \delta^3(\vec x - \vec y)\, \delta_{\alpha\beta}$$

The relations (+) give no problem -ve norm states, so we can now check whether the particle states have physically sensible properties

For energy, we find:

$$H = \int d^3x \left\{ -i\bar{\psi}\gamma^i \partial_i \psi + m\bar{\psi}\psi \right\}$$
$$= \sum_r \int \frac{d^3p}{(2\pi)^3} E_{\vec{p}} \left(a^{\dagger r}_{\vec{p}} a^r_{\vec{p}} + b^{\dagger r}_{\vec{p}} b^r_{\vec{p}} \right) + \text{constant}$$

Momentum: $$\vec{P} = \int \frac{d^3p}{(2\pi)^3} \sum_r \vec{p} \left(a^{\dagger r}_{\vec{p}} a^r_{\vec{p}} + b^{\dagger s}_{\vec{p}} b^s_{\vec{p}} \right)$$

Thus, 4 states w. momentum $\vec{p}$, energy $E_{\vec{p}} = \sqrt{\vec{p}^2 + m^2}$
$a^{\dagger 1}_{\vec{p}}|0\rangle$ $a^{\dagger 2}_{\vec{p}}|0\rangle$ $b^{\dagger 1}_{\vec{p}}|0\rangle$ $b^{\dagger 2}_{\vec{p}}|0\rangle$

Charge associated w. $\psi \to e^{i\theta}\psi$ symmetry

cons. current $J^\mu = \bar{\psi}\gamma^\mu\psi$

charge $Q = \int d^3x\, J^0 = \int d^3x\, \psi^\dagger\psi$

QM $\longrightarrow$ $$Q = \int \frac{d^3p}{(2\pi)^3} \sum_r \left(a^{\dagger r}_{\vec{p}} a^r_{\vec{p}} - b^{\dagger s}_{\vec{p}} b^s_{\vec{p}} \right)$$

Thus $a^\dagger_r$ and $b^\dagger_r$ create particles with charge $+1$, -1 respectively.

Dirac: predicted positron 1931
found experimentally 1932.

QFT predicts every charged particle has corresponding antiparticle with same mass, spin.

Interpret. of 2 +1 charge states at mom. $\vec{p}$: spin states of a spin $\frac{1}{2}$ particle.

check: $$\vec{J} = \int d^3x\, \psi^\dagger (\vec{x} \times (-i\vec{\nabla}\psi)) + \psi^\dagger \vec{S} \psi \qquad S^i = \frac{1}{2}\begin{pmatrix} \sigma^i & 0 \\ 0 & \sigma^i \end{pmatrix}$$

acting on $\vec{p} = 0$ states: 1st term vanishes
2nd term rotates $a^\dagger_1|0\rangle$ $a^\dagger_2|0\rangle$ like basis states for spin $\frac{1}{2}$

Propagator: $\langle 0 | \psi_\alpha(x) \bar{\psi}_\beta(y) | 0 \rangle$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{\sqrt{2E_p}} \int \frac{d^3q}{(2\pi)^3} \frac{1}{\sqrt{2E_q}}$$

$$u^r_\alpha(\vec{p}) e^{-ip\cdot x} u^{\dagger s}_{\beta}(q) e^{iq\cdot y} \langle 0 | a^r_{\vec{p}} a^{\dagger s}_{\vec{q}} | 0 \rangle$$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} e^{-ip\cdot(x-y)} \sum_r u^r_\alpha(p) \bar{u}^r_\beta(\vec{p})$$

$$\parallel$$

HW: $\gamma^\mu p_\mu + m$

$$= \int \frac{d^3p}{(2\pi)^3} \frac{1}{2E_p} (\gamma^\mu p_\mu + m) e^{-ip\cdot(x-y)}$$

$$= (i\gamma^\mu_{\alpha\beta}\partial_\mu + m\delta_{\alpha\beta}) \int \frac{d^3p}{(2\pi)^3} e^{-ip\cdot(x-y)} \frac{1}{2E_p}$$

$$= (i\gamma^\mu\partial_\mu + m)_{\alpha\beta} D(x-y)$$

scalar propagator.

---

similar:

$$\{\psi(x), \psi^\dagger(y)\} = (i\gamma^\mu\partial_\mu + m)(D(x-y) - D(y-x))$$

vanishes for spacelike separation.

$\Rightarrow$ $[O_1(x), O_2(y)]$ vanishes for spacelike separation for any observables with even # of spinors (ops with odd # not physical observables)

ASIDE: if we try to use anticom relations for scalar, vector, tensor fields then $[\phi_i(x), \phi_j(y)] \neq 0$ for spacelike sep.