Building Invariants from Spinors

Consider a spin half particle in quantum mechanics. We can write the general state of such a particle (considering only the spin degree of freedom) as

$$|\psi\rangle = \psi_{\frac{1}{2}}|\uparrow\rangle + \psi_{-\frac{1}{2}}|\downarrow\rangle$$

or, in vector notation for the J_z basis, $\psi = \begin{pmatrix} \psi_{\frac{1}{2}} \\ \psi_{-\frac{1}{2}} \end{pmatrix}$.

Q: Under an small rotation, what is the infinitesimal change in the quantity ψ_a $(a = \pm\frac{1}{2})$ *(or in the column vector ψ with these two components)?*

Q: Can you think of a quantity built from $\psi_{\frac{1}{2}}$ and $\psi_{-\frac{1}{2}}$ that is unchanged when we make a rotation? *Hint: In terms of the state $|\psi\rangle$, is there some quantity that doesn't change under symmetry tranformations?*

Q: Can you think of a quantity built from $\psi_{\frac{1}{2}}$ and $\psi_{-\frac{1}{2}}$ that transforms like a vector under rotations? *Hint: Can you think of some expectation value involving $|\psi\rangle$ that is a vector quantity?*

Q: A spinor field ψ_α has components (η_a, χ_a) where η_a and χ_a each transform like the ψ in the first question under rotations, but transform under infinitesimal boosts as

$$\delta\eta = \epsilon\frac{1}{2}(\sigma^i)\eta \qquad \delta\chi = -\epsilon\frac{1}{2}(\sigma^i)\chi$$

Which of the combinations $\eta^\dagger\eta$, $\eta^\dagger\chi_a$, $\chi^\dagger\eta$, $\chi^\dagger\chi$ are invariant under boosts? Hint: recall that $\sigma_i^\dagger = \sigma_i$

Q: Acting on a spinor field, parity switches $\eta \leftrightarrow \chi$. What linear combinations of the terms in the previous question are invariant under rotations, boosts, and parity transformations?

Q: Now consider a state of two spin half particles. We can write the general state as

$$|\psi\rangle = A|\uparrow\rangle \otimes |\uparrow\rangle + B|\downarrow\rangle \otimes |\uparrow\rangle + C|\uparrow\rangle \otimes |\downarrow\rangle + D|\downarrow\rangle \otimes |\downarrow\rangle \ .$$

Which linear combination of A, B, C, and D is unchanged when we do a rotation?

Q: An alternative way to write the same state is

$$|\psi\rangle = (\psi_{\frac{1}{2}}|\uparrow\rangle + \psi_{-\frac{1}{2}}|\downarrow\rangle) \otimes (\chi_{\frac{1}{2}}|\uparrow\rangle + \chi_{-\frac{1}{2}}|\downarrow\rangle)$$

Using your previous answer, write down a quantity built from $\psi_{\frac{1}{2}}$, $\psi_{\frac{-1}{2}}$, $\chi_{\frac{1}{2}}$, and $\chi_{-\frac{1}{2}}$ that is unchanged when we do a rotation.

Q: Based on your result from the previous page, can you write down some quantities involving the components (ψ_a, χ_a) of a Dirac spinor field (but NOT involving complex conjugates) that is invariant under rotations and boosts?