Building Invariants from Spinors

In the previous section, we found that there are two independent 2×2 irreducible representations of the Lorentz group (more specifically the proper orthochronous Lorentz group that excludes parity and time reversal), corresponding to the generators

$$\mathcal{J}^i = \frac{1}{2}\sigma^i \qquad \mathcal{K}^i = -\frac{i}{2}\sigma^i$$

or

$$\mathcal{J}^i = \frac{1}{2}\sigma^i \qquad \mathcal{K}^i = \frac{i}{2}\sigma^i$$

where σ^i are the Pauli matrices (with eigenvalues ± 1). This means that we can have a fields η_a or χ_a with two components such that under infinitesimal rotations

$$\delta\eta = \epsilon\frac{i}{2}(\sigma^i)\eta \qquad \delta\chi = \epsilon\frac{i}{2}(\sigma^i)\chi$$

while under infinitesimal boosts

$$\delta\eta = \epsilon\frac{1}{2}(\sigma^i)\eta \qquad \delta\chi = -\epsilon\frac{1}{2}(\sigma^i)\chi \; .$$

We also saw that there is no way to define a parity transformation in a theory with η or χ alone, but if both types of field are included, a consistent way for the fields to transform under parity is $\eta \leftrightarrow \chi$. In such a theory, we can combine these two fields into a four-component object

$$\psi_\alpha = \begin{pmatrix} \eta_a \\ \chi_a \end{pmatrix} \; , \tag{1}$$

which we call a Dirac spinor. We would now like to understand how to build Lorentz-invariant actions using Dirac spinor fields.

Help from quantum mechanics

Consider a spin half particle in quantum mechanics. We can write the general state of such a particle (considering only the spin degree of freedom) as

$$|\psi\rangle = \psi_{\frac{1}{2}}|\uparrow\rangle + \psi_{-\frac{1}{2}}|\downarrow\rangle$$

or, in vector notation for the J_z basis, $\psi = \begin{pmatrix} \psi_{\frac{1}{2}} \\ \psi_{-\frac{1}{2}} \end{pmatrix}$. Under an small rotation, the infinitesimal change in the two components of ψ is given by

$$\delta\psi = \epsilon\frac{i}{2}(\sigma^i)\psi \tag{2}$$

which follows from the general result that the change in any state under an infinitesimal rotation about the i-axis is

$$\delta|\psi\rangle = \epsilon i J^i|\psi\rangle \; .$$

The rule (2) is exactly the same as the rules for how the two-component spinor fields transform under rotations. Thus, we can understand how to make rotationally invariant quantities with spinor fields if we know how to make rotationally invariant quantities in this quantum mechanics system.

In quantum mechanics, a basic quantity that is invariant under any symmetry transformation is the inner product $\langle\psi_1|\psi_2\rangle$ between state vectors. Translating to the spin-z basis, we can say that if ψ_1 and ψ_2 are the explicit two-component vectors representing any two states, that under a rotation, the quantity

$$\psi_1^\dagger \psi_2 \tag{3}$$

is invariant. If we consider more generally the four quantities $\psi_a^\dagger \psi_b$, what we have seen is that one linear combination of these four quantities is invariant upon performing a rotation. It is easy to show that the remaining three linearly independent quantities are mixed together by rotations in the same way as the three components of a vector. Explicitly, if we write

$$\psi_1^\dagger \sigma^i \psi_2 \ ,$$

these three quantities rotate like an ordinary vector when we perform a rotation. This is simplest to see in the case where $\psi_2 = \psi_1$, in which case the three quantities are simply the expectation values of J^i for $i = x, y, z$.

Back to field theory

Since the quantity (3) is invariant and since fields η and χ transform in exactly the same way as ψ_1 and ψ_2 under rotations, it follows immediately that the quantities $\eta^\dagger\eta$, $\eta^\dagger\chi$, $\chi^\dagger\eta$, $\chi^\dagger\chi$ are all invariant under rotations. We can check this explicitly using the transformation rules given above. We can also use these rules to check which of these four quantities is also invariant under a boost. As an example, under a small boost, we have

$$\begin{aligned}
\delta(\chi^\dagger\chi) &= \delta\chi^\dagger\chi + \chi^\dagger\delta\chi \\
&= -\epsilon\frac{1}{2}\chi^\dagger(\sigma^i)^\dagger\chi - \epsilon\frac{1}{2}\chi^\dagger(\sigma^i)\chi \\
&= -\epsilon\chi^\dagger\sigma^i\chi
\end{aligned} \tag{4}$$

where we have used that $\sigma_i^\dagger = \sigma_i$. Thus, $\chi^\dagger\chi$ is not invariant under boosts. Similarly, we find that $\eta^\dagger\eta$ is not invariant under boosts, but $\eta^\dagger\chi$ and $\chi^\dagger\eta$ are both invariant. Recalling that a parity transformation acts as $\eta \leftrightarrow \chi$ we find that these two quantities are not independently invariant under parity transformations, but the combination

$$\eta^\dagger\chi + \chi^\dagger\eta$$

is invariant under rotations, parity transformations, and boosts. This is our first example of a possible Lorentz-invariant and parity invariant term built from spinor fields.[1]

[1] Here, we really mean that this quantity transforms like a scalar field.

We can write this directly in terms of the Dirac spinor field (1), by noting that it is a linear combination of the quantities $\psi^*_\alpha \psi_\beta$. Specifically, we have

$$\eta^\dagger \chi + \chi^\dagger \eta = M_{\alpha\beta} \psi^*_\alpha \psi_\beta \ \psi^\dagger M \psi$$

where M is the 4×4 matrix

$$M = \begin{pmatrix} 0 & \mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix} \equiv \gamma^0 \ .$$

We have chosen to call this matrix γ^0 for reasons that will become apparent. Using this notation, our invariant quantity is

$$\psi^\dagger \gamma^0 \psi \ .$$

Making the further definition that

$$\bar{\psi} = \psi^\dagger \gamma^0 \ ,$$

we can write the invariant quantity simply as

$$\bar{\psi}\psi$$

Other quantities with two spinors

For constructing actions, it will also be useful to understand how other quantities built from two spinor fields transform under Lorentz transformations. The sixteen quantities $\psi^*_\alpha \psi_\beta$, will mix together when we perform a Lorentz transformation[2] We have already seen that one linear combination, $\bar{\psi}\psi$ is invariant under Lorentz transformations. To understand how the remaining quantities transform, we begin by recalling the transformation (4) for the rotationally invariant quantity $\chi^\dagger \chi$ under a boost in the i-direction:

$$\delta(\chi^\dagger \chi) = -\epsilon \chi^\dagger \sigma^i \chi \ .$$

This transformation rule looks just like the transformation rule

$$\delta t = -\epsilon x^i$$

for the time coordinate under a boost in the i-direction, if we associate $t \to \chi^\dagger \chi$ and $x^i \to \chi^\dagger \sigma^i \chi$. This suggests that the four quantities $(\chi^\dagger \chi, \chi^\dagger \sigma^i \chi)$ transform as a four-vector under under rotations and boosts. We can verify this by checking that the

[2] To see this, note that under a Lorentz transformation,

$$\psi^*_\alpha \psi_\beta \to M(\Lambda)^*_{\alpha\sigma} M(\Lambda)_{\beta\tau} \psi^*_\sigma \psi_\tau \equiv \mathbf{M}_{(\alpha\beta),(\sigma\tau)} \psi^*_\sigma \psi_\tau \ .$$

The matrix $\mathbf{M}$ defines some representation of the Lorentz group known as the TENSOR PRODUCT representation of M and M^*, since $\mathbf{M} = M^\dagger \otimes M$. This representation is reducible, which means that we can split the sixteen quantities $\psi^*_\alpha \psi_\beta$ into subsets such that Lorentz transformations only mix the quantities in a subset.

quantities $\chi^\dagger\sigma^i\chi$ transform under boosts in the same way as the spatial components of a four-vector. In a similar way, we find that the four quantities $(\eta^\dagger\eta, -\eta^\dagger\sigma^i\eta)$ transform as a four-vector under rotations and boosts.

A true four-vector should also transform under parity transformations in the same way as the coordinates, that is $(t, x^i) \to (t, -x^i)$. From the two potential four-vectors we have just found, one linear combination has this property, namely

$$\begin{pmatrix} \eta^\dagger\eta + \chi^\dagger\chi \\ -\eta^\dagger\sigma^i\eta + \chi^\dagger\sigma^i\chi \end{pmatrix}$$

Each of these four quantities is a linear combination of the quantities $\psi^\dagger_\alpha\psi_\beta$. Using the definition of γ^0 above, and defining three new 4×4 matrices γ^i as

$$\gamma^i = \begin{pmatrix} 0 & \sigma^i \\ -\sigma^i & 0 \end{pmatrix}$$

it is easy to check that

$$\eta^\dagger\eta + \chi^\dagger\chi = \psi^\dagger\psi = \bar{\psi}\gamma^0\psi \ ,$$

while

$$-\eta^\dagger\sigma^i\eta + \chi^\dagger\sigma^i\chi = \bar{\psi}\gamma^i\psi \ ,$$

In summary, we have found so far that $\bar{\psi}\psi$ is a scalar quantity while the four quantities $\bar{\psi}\gamma^\mu\psi$ $(\mu = 0, i)$ transform as a four-vector under Lorentz transformations.

Building tensors with gamma matrices

The fact that $\bar{\psi}\psi$ is a scalar quantity under Lorentz transformations implies that if ψ transforms like

$$\psi \to M_D(\Lambda)\psi$$

under a general Lorentz transformation ('D' stands for Dirac spinor), then $\bar{\psi}$ must transform as

$$\bar{\psi} \to \bar{\psi}M_D^{-1}(\Lambda) \ .$$

With these transformation rules, it follows that under a Lorentz transformation

$$\bar{\psi}\gamma^\mu\psi \to \bar{\psi}M_D^{-1}(\Lambda)\gamma^\mu M_D(\Lambda)\psi \ .$$

But we found earlier that $\bar{\psi}\gamma^\mu\psi$ transforms as a vector quantity, which means that

$$\bar{\psi}\gamma^\mu\psi \to \Lambda^\mu{}_\nu\bar{\psi}\gamma^\nu\psi \ .$$

For these two transformation rules to agree with each other for any Λ, it must be that

$$M_D^{-1}(\Lambda)\gamma^\mu M_D(\Lambda) = \Lambda^\mu{}_\nu\gamma^\nu \ .$$

Using this, it immediately follows that quantities with additional γ matrices also transform like tensors. We have

$$\bar{\psi}\gamma^{\mu_1}\cdots\gamma^{\mu_n}\psi \ \ \to \ \ \bar{\psi}M_D^{-1}(\Lambda)\gamma^{\mu_1}\cdots\gamma^{\mu_n}M_D(\Lambda)\psi$$

$$
\begin{aligned}
&= \bar{\psi} M_D^{-1}(\Lambda)\gamma^{\mu_1} M_D(\Lambda) \cdots M_D^{-1}(\Lambda)\gamma^{\mu_n} M_D(\Lambda)\psi \\
&= \Lambda^{\mu_1}{}_{\nu_1} \cdots \Lambda^{\mu_n}{}_{\nu_n} \bar{\psi}\gamma^{\nu_1} \cdots \gamma^{\nu_n}\psi \ .
\end{aligned}
$$

These tensors cannot all be independent of one another, since they are all linear combinations of only sixteen independent quantities $\psi^*_\alpha \psi_\beta$. The reason is that products of γ matrices can be simplified using the relations

$$
\gamma^\mu \gamma^\nu + \gamma^\nu \gamma^\mu = 2\eta^{\mu\nu} \mathbb{1}_{4\times 4} \ , \tag{5}
$$

which can be checked from the definition of γ^μ. So for example, the tensor

$$
\bar{\psi}\gamma^\mu \gamma^\nu \psi \ ,
$$

appears to have 16 independent components, but if we split it into a symmetric part

$$
S^{\mu\nu} = \bar{\psi}\frac{1}{2}(\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu)\psi
$$

and an antisymmetric part

$$
A^{\mu\nu} = \bar{\psi}\frac{1}{2}(\gamma^\mu\gamma^\nu - \gamma^\nu\gamma^\mu)\psi \ ,
$$

then the six independent components of this antisymmetric tensor are independent of the scalar and vector quantities constructed above, but the symmetric part can be rewritten using (5) as

$$
S^{\mu\nu} = 2\eta^{\mu\nu}\bar{\psi}\psi \ .
$$

In a similar way, for tensors built using more γ matrices only the completely antisymmetric combinations give new independent terms. In four dimensions, the quantity

$$
\bar{\psi}\frac{1}{6}(\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\nu\gamma^\mu\gamma^\lambda + \ldots)\psi
$$

has four independent components, while the tensor

$$
\bar{\psi}\frac{1}{24}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\sigma + \ldots)\psi
$$

just has a single independent component (for this to be non-zero, we must have $\{\mu, \nu, \lambda, \sigma\} = \{0, 1, 2, 3\}$ otherwise everything cancels). Explicitly, we have

$$
\bar{\psi}\frac{1}{24}(\gamma^\mu\gamma^\nu\gamma^\lambda\gamma^\sigma - \gamma^\nu\gamma^\mu\gamma^\lambda\gamma^\sigma + \ldots)\psi = \epsilon^{\mu\nu\lambda\sigma}\bar{\psi}\gamma^0\gamma^1\gamma^2\gamma^3\psi \ .
$$

It is standard to define

$$
\gamma^5 = i\gamma^0\gamma^1\gamma^2\gamma^3
$$

so that the single independent quantity that we get by including an antisymmetric product of four gamma matrices between $\bar{\psi}$ and ψ may be written

$$
\bar{\psi}\gamma^5\psi
$$

This quantity transforms like a scalar under rotations and boosts, but changes its sign under parity transformations. This type of representation of the Lorentz group (including parity) is known as a PSEUDOSCALAR. Similarly, the four independent quantities that we get from antisymmetric products of three gamma matrices between $\bar{\psi}$ and ψ together transform like a vector under rotations and boosts, but are invariant under parity transformations (unlike a proper four-vector). Such a representation is known as a PSEUDOVECTOR. Using the identity

$$\frac{1}{6}(\gamma^\mu\gamma^\nu\gamma^\lambda - \gamma^\nu\gamma^\mu\gamma^\lambda + \ldots) = -i\epsilon^{\mu\nu\lambda\sigma}\gamma_\sigma\gamma^5$$

we find that the independent components of this pseudovector can be written as

$$\bar{\psi}\gamma^\mu\gamma^5\psi \ .$$

Summary

In summary, the sixteen independent quantities $\psi^*_\alpha\psi_\beta$ may be split into five different irreducible representations of the Lorentz group,

$$\begin{array}{lll} \bar{\psi}\psi & \textit{scalar} & 1 \textit{ independent component} \\ \bar{\psi}\gamma^\mu\psi & \textit{vector} & 4 \textit{ independent components} \\ \bar{\psi}\sigma^{\mu\nu}\psi & \textit{antisymmetric tensor} & 6 \textit{ independent components} \\ \bar{\psi}\gamma^\mu\gamma^5\psi & \textit{pseudovector} & 4 \textit{ independent components} \\ \bar{\psi}\gamma^5\psi & \textit{pseudoscalar} & 1 \textit{ independent component} \end{array} \tag{6}$$

where we have defined

$$\sigma^{\mu\nu} = \frac{i}{2}[\gamma^\mu, \gamma^\nu] \ .$$

Starting from these tensor quantities, we can construct scalar terms in the same way as for scalar/vector/tensor theories, i.e. by making sure each vector index is contracted with another vector index on a field or derivative.

Invariants without complex conjugation

We have focused on terms with two spinors of the form $\psi^*_\alpha\psi_\beta$, but we could have also considered terms built from the sixteen quantities $\psi_\alpha\psi_\beta$ without complex conjugation. As before, we find that one linear combination of these terms is a Lorentz scalar,

$$\psi^T C\psi$$

where C is a matrix chosen so that

$$M^T(\Lambda)C = CM^{-1}(\Lambda) \ .$$

With our choice of gamma matrices, we can check that an appropriate choice is

$$C = \begin{pmatrix} i\sigma^2 & 0 \\ 0 & i\sigma^2 \end{pmatrix} \ .$$

As above, we can then insert additional gamma matrices to get other types of tensors:

$$
\begin{array}{lll}
\psi^T C\psi & \textit{scalar} & 1 \textit{ independent component} \\
\psi^T C\gamma^\mu\psi & \textit{vector} & 4 \textit{ independent components} \\
\psi^T C\sigma^{\mu\nu}\psi & \textit{antisymmetric tensor} & 6 \textit{ independent components} \\
\psi^T C\gamma^\mu\gamma^5\psi & \textit{pseudovector} & 4 \textit{ independent components} \\
\psi^T C\gamma^5\psi & \textit{pseudoscalar} & 1 \textit{ independent component}
\end{array}
\tag{7}
$$

If we are interested in describing fermions carrying a conserved charge (as is commonly the case), such terms cannot be used in an Lagrangian density, since they are not invariant under the $\psi \to e^{i\alpha}\psi$ symmetry that gives rise to the conserved charge.