General Fields

We've now seen that fields with different transformation properties under the Lorentz group describe the physics of particles with different spins. Scalar and vector fields describe particles with spin zero and spin one respectively. It turns out that tensor fields with more indices can be used to describe particles with higher integer spins. But how do we describe particles with half-integer spin?

Representations of the Lorentz Group

To answer this, we need to go back and understand whether there are any more general possibilities for how fields transform under Lorentz transformations. Suppose we have a set of field components ϕ_i. We understood earlier that under the transformation $x \to \Lambda x$, the new fields $\tilde{\phi}_i$ at the transformed point Λx will be determined in terms of the old fields at the point x. But (as we saw for vector and tensor fields), the transformation can also mix up the field components. In general, we might have:

$$\tilde{\phi}_i(\Lambda x) = M_{ij}(\Lambda)\phi_j(x) \ .$$

Here, if we have N field components, then M_{ij} is an $N \times N$ matrix that depends on the Λ. Thus, M defines a map between the set of Lorentz transformations and the the space of $n \times N$ matrices. Not just any map will do. For example, when Λ is the identity (i.e the Lorentz transformation that does nothing), then the fields should remain unchanged, so M must be the $N \times N$ identity matrix. More generally, if we consider a Lorentz transformation that results from two successive Lorentz transformations, $\Lambda = \Lambda_1\Lambda_2$, it must be that the transformation on the fields under the Lorentz transformation Λ must be the same as the transformation on fields that results from performing the Lorentz transformation Λ_2 followed by the Lorentz transformation Λ_1.

Q: What condition on $M(\Lambda)$ does this imply?

Answer: In order for this to be true, we must have

$$M(\Lambda_1\Lambda_2) = M(\Lambda_1)M(\Lambda_2) \tag{1}$$

. Thus, the matrix associated with the product of two Lorentz transformations must equal the product of the matrices associated with the individual transformations. When this holds, we say that the matrices M provide a REPRESENTATION of the Lorentz group.

Q: For rotations around a single axis, the group elements can be labeled by the rotation angle θ, where rotations by angles differing by a multiple of 2π are equivalent. If the rotation by angle θ acting on some field components is described by a matrix $M(\theta)$, what conditions must this $M(\theta)$ satisfy to be a valid representation?

Answer: Two successive rotations by angles θ_1 and θ_2 are equivalent to a single rotation by angle $\theta_1 + \theta_2$. Thus, we must have

$$M(\theta_1 + \theta_2) = M(\theta_1)M(\theta_2) \ .$$

Since rotations through angles differing by 2π are equivalent, we must also have $M(\theta + 2\pi) = M(\theta)$.

Infinitesimal Transformations and Generators

Since there are an infinite number of possible Lorentz transformations, it sounds at first like a difficult problem to find matrices that satisfy the multiplication rule (1) for all possible Λ. It is easy to see that the simple choices $M(\Lambda) = 1$ and $M(\Lambda) = \Lambda$ work, but how can we go about finding all possible solutions?

A crucial step is to realize that general Lorentz transformations can be built up by multiplying together lots of infinitesimal transformations. For example, a rotation around the z axis can be obtained by performing in succession a large number of very small rotations. According to (1), if we know how the very small rotation acts on the fields, then we can figure out how the large rotation acts also. We can see this directly in a couple of ways:

Q: Suppose that an infinitesimal rotation acts on the field components as

$$\delta\phi_i \equiv \tilde{\phi}_i(\Lambda x) - \phi_i(x) = \delta\theta L_{ij}\phi_j \ .$$

How does the field transform under a rotation by angle θ?

Answer: We have (suppressing the vector indices)

$$\frac{\delta\phi}{\delta\theta} = L\phi \ .$$

Considering this as a differential equation, we can write the solution as:

$$\tilde{\phi}(\Lambda x) = e^{\theta L}\phi(x) \ .$$

Thus, if L is the matrix that describes the infinitesimal transformation on the field, the matrix corresponding to any finite rotation is $M(\theta) = e^{\theta L}$.

We can also obtain this result directly from (1), by noting that

$$\begin{aligned} M(\theta) &= [M(\theta/N)]^N \\ &= \lim_{N\to\infty}[M(\theta/N)]^N \\ &= \lim_{N\to\infty}[1 + (\theta/N)M'(0) + \mathcal{O}((\theta/N)^2)]^N \\ &= e^{\theta M'(0)} \end{aligned}$$

Here, the matrix $M'(0)$ is the one we previously called L. We have used $e^x = \lim_{N\to\infty}(1 + x/N)^N$ and the fact that the $\mathcal{O}((\theta/N)^2)$ terms in the third line will not contribute to the result in the limit $N \to \infty$.

In summary, we have learned that for simple rotations about a single axis, the action on fields is completely determined once we know the action of an infinitesimal transformation. The problem of finding the infinite set of matrices $M(\theta)$ is reduced to finding the single matrix L. In this simple case, the only other restriction on L comes from the extra condition $M(\theta + 2\pi) = M(\theta)$, and this leads to the restriction that the eigenvalues of L must all be integer multiples of i.

Generators for the Lorentz Group

For more complicated groups such as the Lorentz group, there is more than one type of infinitesimal transformation, but the set of all such transformations have the structure of a finite-dimensional vector space. To see this, recall that the Lorentz group may be defined as the set of 4×4 matrices satisfying

$$\Lambda^T \eta \Lambda = \eta \ . \tag{2}$$

An infinitesimal Lorentz transformation is one that is very close to the identity matrix, so we can write

$$\Lambda = 1 + \epsilon\omega$$

The condition (2) on Λ implies that if ϵ is infinitesimal, the matrix ω must satisfy[1]

$$\omega^T \eta + \eta\omega = 0 \ . \tag{3}$$

This condition is *linear* in ω, so the the allowed set of matrices ω form a *vector space*. Explicitly, we have

$$\omega = \begin{pmatrix} 0 & b_1 & b_2 & b_3 \\ b_1 & 0 & a_3 & -a_2 \\ b_2 & -a_3 & 0 & a_1 \\ b_3 & a_2 & -a_1 & 0 \end{pmatrix} \equiv i a_i J^i + i b^i K^i$$

As with any vector space, we can write a general element as a linear combination of basis elements. Here, we have defined basis elements J^i and K^i, which correspond to the infinitesimal rotations around the x y and z axes and infinitesimal boosts in the x, y, and z directions respectively.

A general Lorentz transformation[2] can be built up by a combination of these six infinitesimal transformations, so a representation $M(\Lambda)$ of the Lorentz group will be completely specified once we understand how these six infinitesimal transformations act on the fields. Explicitly, if the matrices $\mathcal{J}^i$ and $\mathcal{K}^i$ describe the action of infinitesimal rotations and boosts on the components of some field (e.g. the change in the field

[1] To see this, we simply plug in the previous formula into (2) and look at the terms of order ϵ.

[2] For the moment we are speaking about the "proper orthocronous Lorentz group" i.e. not including the parity or time-reversal transformations.

under a rotation around the x-axis would be $\tilde{\phi}_i(\Lambda x) - \phi_i(x) = i\delta\theta(\mathcal{J}^1)_{ij}\phi_j)$, then the action for some general Lorentz transformation[3]

$$\Lambda = e^{ia_i J^i + ib_i K^i}$$

will be determined by the matrix

$$M(\Lambda) = e^{ia_i \mathcal{J}^i + ib_i \mathcal{K}^i} \ .$$

Note in particular that the general infinitesimal transformation

$$\Lambda = 1 + \epsilon(ia_i J^i + ib_i K^i)$$

will correspond to

$$M(\Lambda) = 1_{N\times N} + \epsilon(ia_i \mathcal{J}^i + ib_i \mathcal{K}^i) \ .$$

so the matrices representing general infinitesimal transformations form a vector space with basis $\mathcal{J}^i$ and $\mathcal{K}^i$ and the infinitesimal transformation corresponding to a linear combination of Js and Ks will be represented by the same linear combination of $\mathcal{J}$s and $\mathcal{K}$s.

Representing the generators

We have seen that the complete representation $M(\Lambda)$ will be determined once we specify the six matrices $\mathcal{J}^i$ and $\mathcal{K}^i$ that represent the infinitesimal transformations or GENERATORS of the Lorentz group. We would now like to understand what restrictions there are for choosing these matrices. Consider the following

Q: Consider two infinitesimal Lorentz transformations, whose actions on some field components are specified by the matrices $M_1 = e^{\epsilon\Omega_1}$ and $M_2 = e^{\epsilon\Omega_2}$. For $M = M_1 M_2 M_1^{-1} M_2^{-1}$, what is the first non-zero term in $M - 1$ in an expansion in powers of ϵ?

Answer: We have

$$\begin{aligned} M = M_1 M_2 M_1^{-1} M_2^{-1} &= e^{\epsilon\Omega_1} e^{\epsilon\Omega_2} e^{-\epsilon\Omega_1} e^{-\epsilon\Omega_2} \\ &= (1 + \epsilon\Omega_1 + \frac{1}{2}\epsilon^2\Omega_1^2 + \cdots)(1 + \epsilon\Omega_2 + \frac{1}{2}\epsilon^2\Omega_2^2 + \cdots) \\ &\qquad (1 - \epsilon\Omega_1 + \frac{1}{2}\epsilon^2\Omega_1^2 + \cdots)(1 - \epsilon\Omega_2 + \frac{1}{2}\epsilon^2\Omega_2^2 + \cdots) \end{aligned}$$

[3]The fact that any Lorentz transformation can be written as such an exponential may seem to be a stronger statement than simply saying that it can be built up from many infinitesimal transformations. To show this, we need to use the fact that $e^{\omega_1} e^{\omega_2}$ can be written as e^{ω_3} if ω_i are all of the form above. This follows from the "Baker-Campbell-Hausdorff relation" $e^A e^B = e^{A+B+\frac{1}{2}[A,B]+\cdots}$ where the dots indicate terms involving nested commutators of As and Bs. As we will see below, the commutator of any two elements in our vector space of infinitesimal transformation matrices must also be in the vector space, so if A and B satisfy the constraint (3), then the expression $A + B + \frac{1}{2}[A, B] + \ldots$ will also satisfy (3).

$$
\begin{aligned}
&= 1+\epsilon^2(\Omega_1\Omega_2-\Omega_2\Omega_1)+\cdots \\
&= 1+\epsilon^2[\Omega_1,\Omega_2]+\cdots\ .
\end{aligned}
$$

In this example, the infinitesimal transformation M takes the form of the identity matrix plus an infinitesimal parameter times the commutator $[\Omega_1, \Omega_2]$. But since the matrices representing infinitesimal transformations form a vector space, it must be that for any choice of Ω_1 and Ω_2 (i.e. any linear combinations of $\mathcal{J}^i$ and $\mathcal{K}^i$), we can rewrite $[\Omega_1, \Omega_2]$ as a linear combination of the same basis elements $\mathcal{J}^i$ and $\mathcal{K}^i$.

For the matrices J^i and K^i that we defined above, we can check explicitly that

$$
\begin{aligned}
[J_i, J_j] &= i\epsilon_{ijk}J_k \\
[J_i, K_j] &= i\epsilon_{ijk}K_k \\
[K_i, K_j] &= -i\epsilon_{ijk}J_k
\end{aligned}
$$

But these same relations must hold for $\mathcal{J}^i$ and $\mathcal{K}^i$ in order for the constraint (1) to be satisfied.[4]

Thus, for the matrices representing the infinitesimal Lorentz transformations, we must have

$$
\begin{aligned}
[\mathcal{J}_i, \mathcal{J}_j] &= i\epsilon_{ijk}\mathcal{J}_k \\
[\mathcal{J}_i, \mathcal{K}_j] &= i\epsilon_{ijk}\mathcal{K}_k \\
[\mathcal{K}_i, \mathcal{K}_j] &= -i\epsilon_{ijk}\mathcal{J}_k\ .
\end{aligned}
$$

Any $N \times N$ matrices satisfying these commutation relations provide a valid representation of the Lorentz group.

[4]For example, we have

$$
\begin{aligned}
1+\epsilon^2[\mathcal{J}^1,\mathcal{K}^2] &= e^{\epsilon\mathcal{J}^1}e^{\epsilon\mathcal{K}^2}e^{-\epsilon\mathcal{J}^1}e^{-\epsilon\mathcal{K}^2} \\
&= M(e^{\epsilon J^1})M(e^{\epsilon K^2})M(e^{-\epsilon J^1})M(e^{-\epsilon K^2}) \\
&= M(e^{\epsilon J^1}e^{\epsilon K^2}e^{-\epsilon J^1}e^{-\epsilon K^2} \\
&= M(1+\epsilon^2[J^1,K^2]\cdots)\ .
\end{aligned}
$$

We have seen above that a given linear combination of Js and Ks maps to the same linear combination of $\mathcal{J}$s and $\mathcal{K}$s, so $[J^1, K^2] = iK_3$ implies $[\mathcal{J}^1, \mathcal{K}^2] = i\mathcal{K}_3$.