Appendix: The ϵ tensor.

There is one extra tool that we need if we're interested in building the most general Lorentz invariant actions involving scalars, vectors and tensors. To motivate this, consider the following question:

Given three-dimensional vectors $\vec{A}$, $\vec{B}$, and $\vec{C}$, what rotationally invariant quantities can we build that are linear in A, B, and/or C?

Answer: At first glance, the obvious combinations are $\vec{A}\cdot\vec{B}$, $\vec{A}\cdot\vec{C}$, and $\vec{B}\cdot\vec{C}$. However, we can also have the triple product $\vec{A}\cdot(\vec{B}\times\vec{C})$. This is rotationally invariant, though not invariant under parity transformations.

The triple product of vectors in three dimensions may be written as

$$\vec{A}\cdot(\vec{B}\times\vec{C}) = \epsilon^{ijk} A_i B_j C_k \ .$$

This is rotationally invariant since the ϵ tensor satisfies

$$R_{ii'} R_{jj'} R_{kk'} \epsilon^{i'j'k'} = \epsilon^{ijk} \tag{1}$$

To prove this, note that the determinant of a (proper) rotation matrix is

$$\epsilon^{abc} \epsilon^{i'j'k'} R_{ai'} R_{bj'} R_{ck'} = \det(R) = 1 \ .$$

Multiplying this equation by ϵ^{ijk} and using the relation

$$\epsilon^{abc}\epsilon^{ijk} = \delta^{ai}\delta^{bj}\delta^{ck} + \delta^{aj}\delta^{bk}\delta^{ci} + \delta^{ak}\delta^{bi}\delta^{cj} - \delta^{aj}\delta^{bi}\delta^{ck} - \delta^{ai}\delta^{bk}\delta^{cj} - \delta^{ak}\delta^{bj}\delta^{ci} \ ,$$

we obtain the desired relation.

A completely analogous construction is possible with 4-vectors. If we define $\epsilon^{\mu\nu\lambda\sigma}$ to be ± 1 if $(\mu\nu\lambda\sigma)$ is an even or odd permutation of (0123) respectively, and $\epsilon_{\mu\nu\lambda\sigma} = -\epsilon^{\mu\nu\lambda\sigma}$, then (by a similar argument to (1) case)

$$\epsilon^{\alpha\beta\gamma\delta} \Lambda^{\mu}{}_{\alpha} \Lambda^{\nu}{}_{\beta} \Lambda^{\lambda}{}_{\gamma} \Lambda^{\sigma}{}_{\delta} = \det(\Lambda) \epsilon^{\mu\nu\lambda\sigma}$$

This means that it $T^{\mu\nu\lambda\sigma}$ is any combination of fields with four upper indices (e.g. $\phi^{\mu\nu}\partial^{\lambda}\phi^{\sigma}$, we will have

$$\epsilon_{\mu\nu\lambda\sigma} T^{\mu\nu\lambda\sigma} \to \det(\Lambda) \epsilon_{\mu\nu\lambda\sigma} T^{\mu\nu\lambda\sigma}$$

so the expression on the left is invariant under all Lorentz-transformations with $det(\Lambda) = 1$. These are known as the "proper orthochronous" Lorentz transformations, and include any combination of proper rotations and boosts, but not transformations involving a parity flip or a time reversal.

The most common appearance of the ϵ tensor is in three-dimensional field theories, where it appears in the "Chern-Simons term" for a vector field

$$S_{CS} = \int d^3x \epsilon^{\mu\nu\lambda} A_{\mu} \partial_{\nu} A_{\lambda}$$