Tutorial #13: Tunnelling

One of the most dramatic predictions of quantum mechanics is the ability of particles to pass through potential energy barriers which they do not have enough energy to overcome classically. This is the phenomenon of TUNNELING, and in today's tutorial we will understand it quantitatively using the Schrodinger equation.

Question 1

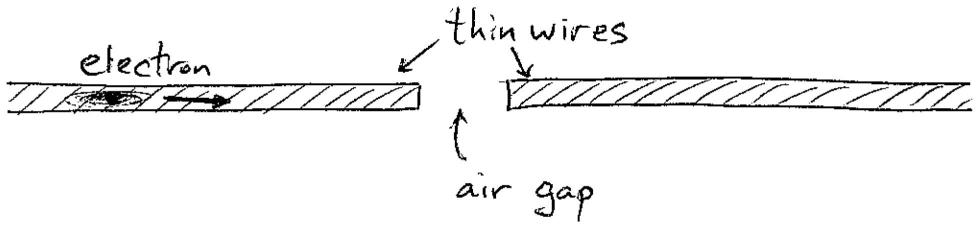
a) As a warm-up, consider the situation where we have a ball rolling towards a hill of height h. What velocity would be required to ensure that the ball will get over the hill?

b) Suppose we sent the ball toward the hill with a smaller velocity than the one in part a. What classical law of physics tells us that the ball can't get over the hill?

Preview: The basic idea of tunnelling is that according to quantum mechanics, the ball has some probability of getting to the other side of the hill *no matter how small* its velocity is. We will now try to understand this better, and also understand why no laws of physics are violated. I should stress that the probability of tunnelling in some macroscopic example like the ball and hill is unimaginably small. On the other hand, microscopic examples of tunnelling (e.g. where an electron gets from one place to another through a region where it does not have enough energy to be classically) happens all the time and can even be put to good use in technologies such as the STM (scanning-tunnelling electron microscope).

Question 2

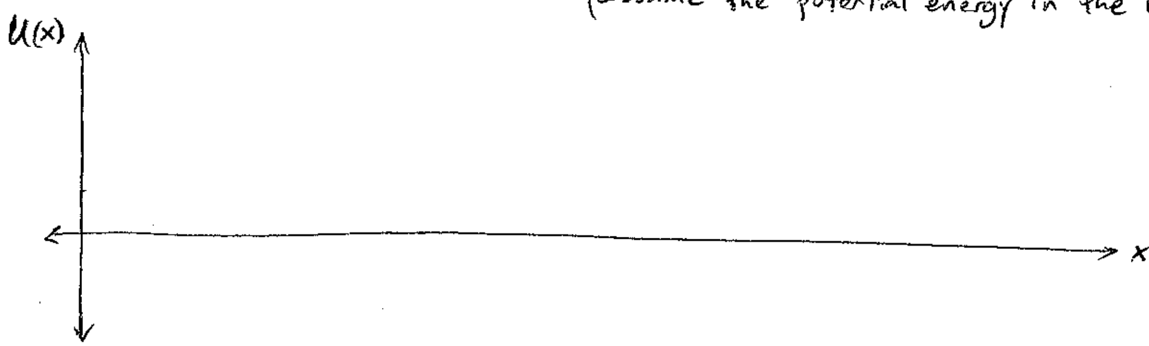
Let's now consider a setup where tunnelling can actually be observed. Consider a situation where we have two thin wires separated by an air gap.



Suppose the work function for the metal is $W=3eV$, that the distance between the wires is d, and that the potential energy for an electron inside the wire is zero.

a) Plot the potential energy of the electron as a function of its position

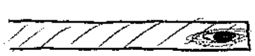
(assume the potential energy in the wires is 0)



b) According to classical physics, what would happen if we sent an electron through the left wire towards the air gap with kinetic energy E < W?

Question 3

In quantum mechanics, the tunnelling phenomenon arises because the wavefunction for an energy eigenstate can be non-zero even in regions where the particle does not have enough energy to be classically. To understand this quantitatively, let's first consider the case where d is very large and investigate the wavefunction of an electron near the end of the first wire. We know from last week that inside the wire, assuming the



electron's energy is E and the potential is 0, the wavefunction is a sinusoidal function $\psi(x) = A\cos(kx) + B\sin(kx)$ with wavelength given by

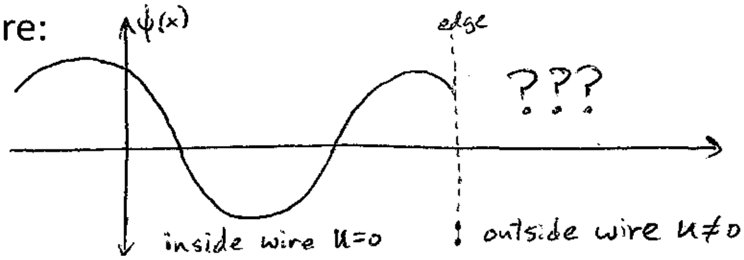
$$h/\lambda = \sqrt{2\,m\,E}$$

This was the general solution to the time-independent Schrodinger equation

$$-\frac{\hbar^2}{2m}\frac{d^2\psi}{dx^2} = (E - U)\,\psi \qquad (*)$$

for U=0. We will get a similar oscillating wavefunction in any region where E > U (i.e. where the electron is allowed to be classically).

Now let's find out what the wavefunction looks like for the region outside the wire:

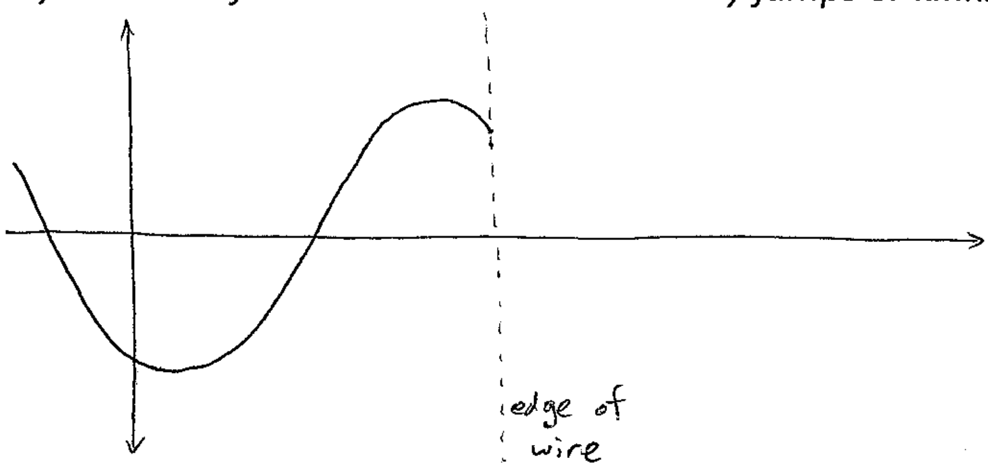


If we are in a region where U = W > E (classically forbidden since it implies a negative kinetic energy), the equation (*) tells us that we need a function whose second derivative is proportional to the function (with a +ve proportionality). Such a function is an exponential $e^{\alpha x}$.

a) For which values of α (you should find 2) does $\psi(x) = e^{\alpha x}$ satisfy the time-independent Schrodinger equation above (with U = W)? *Answer in terms of m, ħ, E, and W.*

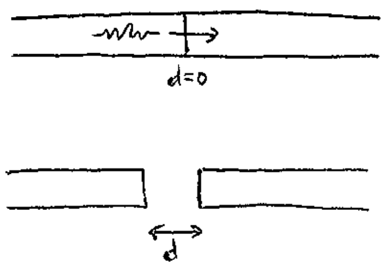
b) Which of the two values of α that you found in part c make sense physically?

c) Complete the plot of the wavefunction near the edge of the wire. *Note that physical wavefunctions should not have any jumps or kinks.*



d) We can see that there is some probability of finding the electron outside the wire, even if its energy is less than the work function. What is the probability density for finding the electron at a distance d into the air gap, relative to (i.e. divided by) the probability density for finding it at the edge of the wire? *Answer in terms of m, ħ, W, E, and d.*

e) Going back to the situation with two wires (i.e. small d), we can now



understand the tunnelling quantitatively. If d=0, the electron will definitely get to the second wire (probability 1). For a finite separation, we can estimate the tunnelling probability by how much the probability density for the electron's wavefunction dies off between the two wires. Thus:

Tunnelling Probability =

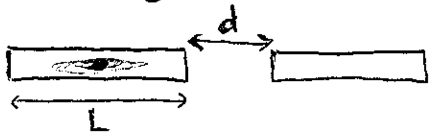
* check your answer at this point *

f) If W = 3eV and the electron has energy 1 eV, how far apart can the wires be if we want at least a 10^{-6} probability of tunnelling?

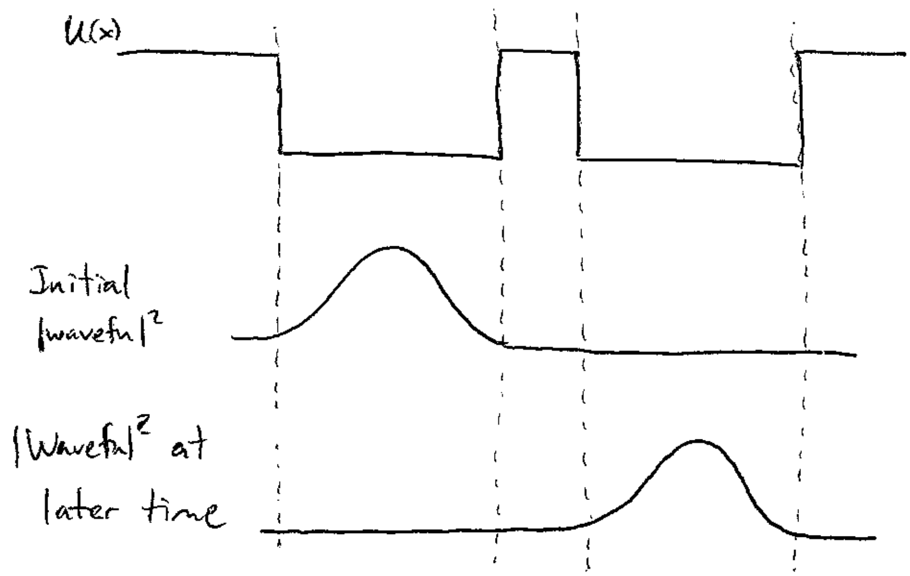
g) If we double this distance, what is the tunnelling probability?

Question 4

A related example is where we have an electron trapped in a thin short wire and we bring another identical wire nearby:

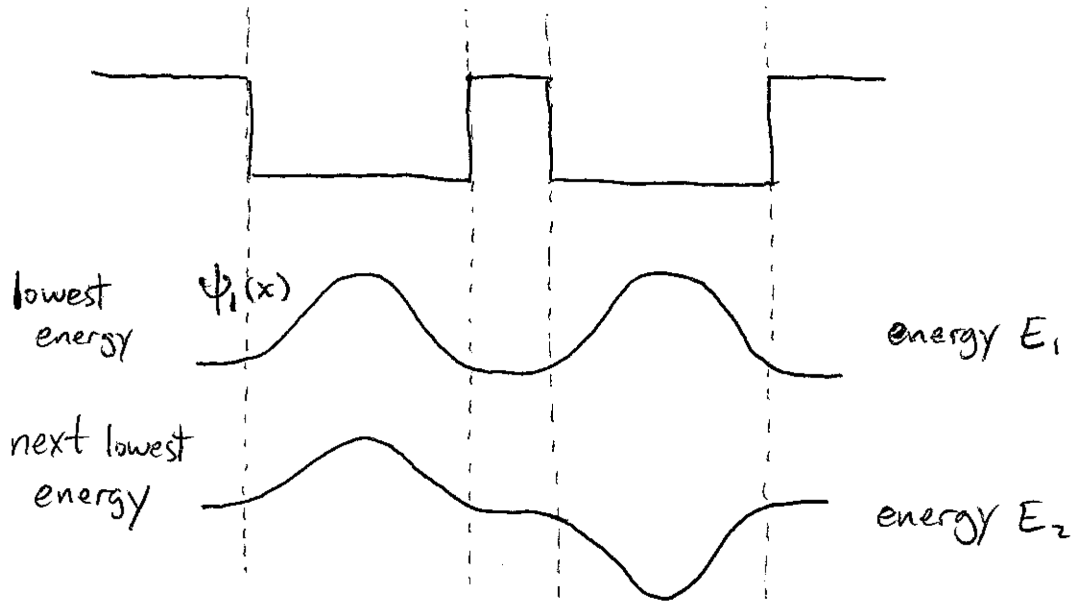


The simulation we saw in class (also set up at the front of the room) showed that as time passes, the probability density for the electron "bounces back and forth" between the two wires:



a) Based on the this time dependence of the probability density, what can we say about whether or not the initial wavefunction is an energy eigenstate for the system with the two wires?

Hopefully, you remembered that energy eigenstates always have constant probability density (that is why we call them STATIONARY STATES), so the wavefunction we started with cannot be an energy eigenstate for the system with two wires. On the other hand, any state can always be written as a sum of energy eigenstates. In this case the true energy eiegnestates of lowest energy look like this:



b) Based on the pictures, how can we write our initial wavefunction in terms of the energy eigenstate wavefunctions $\psi_1(x)$ and $\psi_2(x)$? see picture on prev. page.

c) Based on your answer to part b, what will the wavefunction be at some later time t? *Answer in terms of $\psi_1(x)$, $\psi_2(x)$, E_1, E_2, t, and $\hbar$.*

Question 5 (if you just can't get enough tunnelling):

In this question, we'll estimate how long we have to wait before the electron is likely to be found in the second wire.

a) The energy of the electron in the example above is about

$$E_1 = h^2/(8\ m\ L^2))$$

(this is the ground state energy for an electron in a single wire). If we measure the velocity of the electron, estimate the typical value we would find for $|v|$. *Answer in terms of h,m, and L.*

b) If we imagine a classical particle bouncing back and forth in the wire with this velocity, how often would it collide with the end of the wire near the air gap? *Answer in terms of h, m, and L.*

c) If the electron has a tunnelling probability for each of these collisions given by your answer to part 3g, how long on average will it be before the electron tunnels to the second wire? *Answer in terms of L,m,h,W,and d.*

d) If L = 10nm how large does *d* have to be before the time for tunnelling is greater than the age of the universe ($t \approx 10^{10}$ years)

e) We can actually use our result from 5c to estimate the energy difference between the two true energy eigenstates in question 4. Using your answer to 4c, show that the time before the electron is likely to be found in the second wire is given by $t = \hbar \pi / (E_2 - E_1)$ where E_2 and E_1 are the energies of the eigenstates shown in question 4. Using this result and your answer (5c) for the time t, estimate the energy difference $E_2 - E_1$.