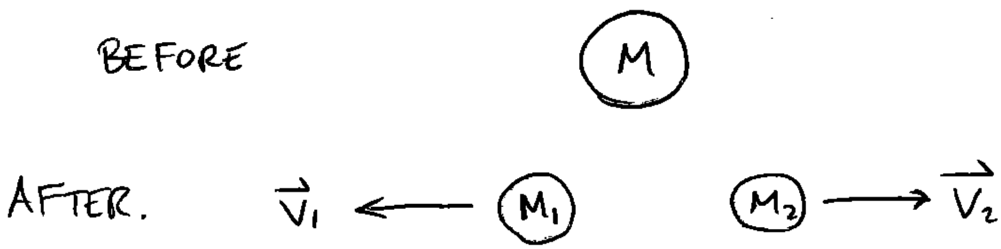
# Energy & momentum conservation example.




Suppose we want to determine the outgoing velocities of two particles that result from the decay of a single unstable particle. Initially, we have:

$$\vec{p} = 0 \qquad E = Mc^2$$

After the decay, we must have:

$$\vec{p}_1 + \vec{p}_2 = 0 \qquad \text{MOMENTUM CONSERVATION}$$

$$E_1 + E_2 = Mc^2 \qquad \text{ENERGY CONSERVATION.}$$

The momentum conservation equation implies that $\vec{p}_1$ and $\vec{p}_2$ have opposite directions and the same magnitude:

$$|p_1| = |p_2| \Longrightarrow \gamma_1 m_1 v_1 = \gamma_2 m_2 v_2 \qquad (1)$$

The energy conservation equation gives (dividing by $c^2$):

$$M = \gamma_1 m_1 + \gamma_2 m_2 \qquad (2)$$

We have two equations for the two unknown velocities, so it only remains to solve them. This example, with two ~~unknown~~ masses not equal to each other, is the most complicated, so the ~~eq~~ examples that you'll have to work out will generally be less messy.

$\boxed{\text{METHOD } ①}$ : Write everything in terms of $\gamma$'s:

We can use the following identity : $\boxed{v\gamma = \sqrt{\gamma^2 - 1}}$ to rewrite equation ① as:

$$m_1 \sqrt{\gamma_1^2 - 1} = m_2 \sqrt{\gamma_2^2 - 1}$$

$$\Rightarrow m_1^2 \gamma_1^2 - m_1^2 = m_2^2 \gamma_2^2 - m_2^2$$

$$\Rightarrow m_2 \gamma_2 = \sqrt{m_1^2 \gamma_1^2 + m_2^2 - m_1^2}$$

We can plug this in to equation ① to get an equation entirely in terms of $\gamma_1$:

$$M = m_1 \gamma_1 + \sqrt{m_1^2 \gamma_1^2 + m_2^2 - m_1^2}$$

To solve an equation like this, we put the square root by itself on one side of the equation and square it:

$$(M - m_1 \gamma_1)^2 = m_1^2 \gamma_1^2 + m_2^2 - M_1^2$$

$$\Rightarrow M^2 \cancel{m} - 2 M m_1 \gamma_1 = M_2^2 - M_1^2$$

$$\Rightarrow \gamma_1 = \frac{M^2 + M_1^2 - M_2^2}{2 M M_1}$$

switching $1 \leftrightarrow 2$ we also have:

$$\gamma_2 = \frac{M^2 + M_2^2 - M_1^2}{2 M M_2}$$

We could now find $v_1$ and $v_2$ from

$$\gamma_1 = \frac{1}{\sqrt{1-\frac{v_1^2}{c^2}}} \Rightarrow v_1 = c\sqrt{1-\frac{1}{\gamma_1^2}}$$

but it's simpler to plug in the numbers to find $\gamma$ first.

METHOD 2: We can also try to solve for the momentum $p = |\vec{p}_1| = |\vec{p}_2|$ and find the velocities from that.

In this case we rewrite equation 2 using

$$\boxed{E^2 = p^2c^2 + m^2c^4}$$

This gives that

$$2 \Rightarrow \quad Mc^2 = E_1 + E_2$$

$$\Rightarrow Mc^2 = \sqrt{p^2c^2 + M_1^2c^4} + \sqrt{p^2c^2 + M_2^2c^4}$$

Now we have two square roots, so it's simplest to move one of them to the other side and square:

$$\left(Mc^2 - \sqrt{p^2c^2+M_1^2c^4}\right)^2 = p^2c^2 + M_2^2c^4$$

$$\Rightarrow M^2c^4 + p^2c^2 + M_1^2c^4 - 2\sqrt{p^2c^2+M_1^2c^4}\cdot Mc^2 = p^2c^2 + M_2^2c^4$$

$$\Rightarrow 2Mc^2\sqrt{p^2c^2+M_1^2c^4} = c^4\left(M^2 + M_1^2 - M_2^2\right)$$

We can now square again to eliminate the square root:

$$4M^2\left(p^2c^2 + M_1^2c^4\right) = c^4\left(M^4 + M_1^4 + M_2^4 - 2M^2M_2^2 - 2M_1^2M_2^2 + 2M^2M_1^2\right)$$

$$\Rightarrow \quad 4M^2p^2 = c^2\left(M^4 + M_1^4 + M_2^4 - 2M^2M_2^2 - 2M_1^2M_2^2 - 2M^2M_1^2\right)$$

this mess factors to this.

$$\Rightarrow \quad p = \frac{c}{2M}\sqrt{(M+M_1+M_2)(M-M_1-M_2)(M_2+M-M_1)(M_1+M-M_2)}$$

We can now find the velocities, for example using

$$v = \frac{pc^2}{E} = \frac{pc^2}{\sqrt{p^2c^2 + m^2c^4}} = \frac{pc}{\sqrt{p^2 + m^2c^2}}$$

For example:

$$v_1 = \frac{c\sqrt{(M+M_1+M_2)(M-M_1-M_2)(M_2+M-M_1)(M_1+M-M_2)}}{\left(M^2 + M_1^2 - M_2^2\right)}$$

In typical applications, the masses are known, so the equations look much simpler, but the steps are the same.

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Summary: useful formulae with energy + momentum:

in terms of $v$: $\quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}} \qquad E = \gamma mc^2 \qquad \vec{p} = \gamma m v$

in terms of $\gamma$: $\quad |\vec{p}| = m\sqrt{\gamma^2 - 1} \qquad v = c\sqrt{1 - \frac{1}{\gamma^2}}$

in terms of $p$: $\quad E^2 = p^2c^2 + m^2c^4$

$$\vec{v} = \frac{\vec{p}c^2}{E}$$