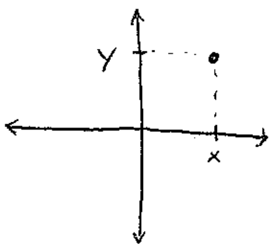
Waves and Complex Numbers



Complex numbers turn out to be extremely useful in every area of physics involving wave phenomena. Here, we'll review a little about complex numbers and then see how they relate to waves.

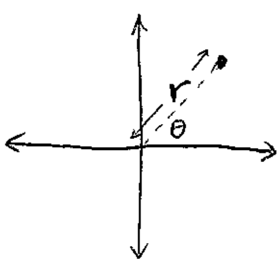
The Basics

A simple way to think about complex numbers is as points in a two dimensional plane where we call the x axis the "real part" and the y-axis the imaginary part. This picture makes it clear that every complex number also has a "magnitude" (the distance r from the origin, also called the "modulus") and a phase (the angle θ measured counterclockwise from the real axis). These are related to the x and y coordinates by

$\Updownarrow$

$z = x + i\,y$

$$\begin{aligned} x &= r\cos(\theta) \\ y &= r\sin(\theta)\ . \end{aligned} \tag{1}$$



It is clear that we can add up two complex numbers (by vector addition, i.e. adding the real parts and adding the imaginary parts), but there is also a natural way to multiply them, by multiplying the magnitudes and adding the phases. So for example

$$(rcos(\theta), rsin(\theta)) \times (r'cos(\theta'), r'sin(\theta')) = (rr'cos(\theta + \phi), rr'sin(\theta + \phi)) \tag{2}$$

$\Updownarrow$

$z = (r\cos\theta) + i(r\sin\theta)$

It turns out that this is the only way to multiply points on the two dimensional plane that satisfies the usual rules for multiplication (commutative, associative, distributive properties).

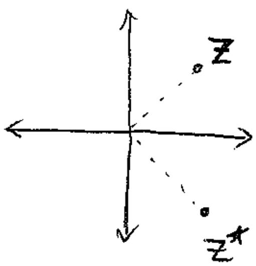
Of course, we usually associate the point $(1, 0)$ on the x axis with the usual real number 1, and call the point $(0, 1)$ on the y-axis i. So a general complex number can be represented by

$$z = (x, y) = x + iy\ .$$

We can then check that the multiplication law above gives $i \times i = -1$ and so it follows (by the distributive property) that

$$z_1 z_2 = (x_1 x_2 - y_1 y_2) + i(x_1 y_2 + x_2 y_1)\ ,$$

as usual.



It is often useful to define the complex conjugate of a complex number as the complex number with the same magnitude but negative the phase. In terms of the real and imaginary parts, the complex conjugate of z just reverses the imaginary part:

$$z^* = x - iy\ .$$

This is useful, since z^*z gives a real number equal to the squared magnitude of z.

$$|z|^2 = zz^* = x^2 + y^2 \tag{3}$$

Also, in terms of z and z^*, we have

$$\begin{aligned} z + z^* &= 2Re(z) \\ -i(z - z^*) &= 2Im(z) \end{aligned}$$

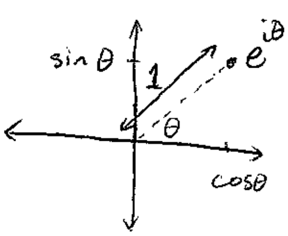
Complex exponentials

An extremely useful fact is that we can relate the real and imaginary part of a complex number to its magnitude and phase by

$$re^{i\theta} = z = x + iy \tag{4}$$

To understand this, we have to remember the definition of the exponential in terms of a power series:

$$e^a = 1 + a + \frac{1}{2}a^2 + \frac{1}{3!}a^3 + \ldots$$



Plugging in $a = i\theta$, and collecting the even and odd terms in the power series, we get

$$\begin{aligned} e^{i\theta} &= (1 - \frac{1}{2}\theta^2 + \frac{1}{4!}\theta^4 + \ldots) + i(\theta - \frac{1}{3!}\theta^3 + \ldots) \\ &= \cos(\theta) + i\sin(\theta) \end{aligned}$$

where we have recognized the two series in brackets as the power series for sin and cos. So geometrically, $e^{i\theta}$ represents a point on the unit circle at an angle θ (counterclockwise) from the x-axis. Since $re^{i\theta}$ is a point in the same direction but with distance r from 0, it has x and y components (real and imaginary parts) $x = r\cos(\theta)$ and $y = r\sin(\theta)$ respectively, showing that (4) agrees with (1).

Representing sinusoidal functions as the components of complex exponentials can make a lot of manipulations a lot simpler, and this is the key to why complex numbers are so useful in describing wave phenomena. As an example, we can derived the law for the cosine of the sum of two angles as

$$\begin{aligned}
\cos(A+B) &= Re(e^{i(A+B)}) \\
&= Re(e^{iA}e^{iB}) \\
&= Re\left[(\cos(A) + i\sin(A))(\cos(B) + i\sin(B))\right] \\
&= \cos(A)\cos(B) - \sin(A)\sin(B)
\end{aligned}$$

This derivation is much easier than the usual one, and the key is that we could use the simple feature of exponentials that $e^{a+b} = e^a e^b$.

Waves using complex numbers

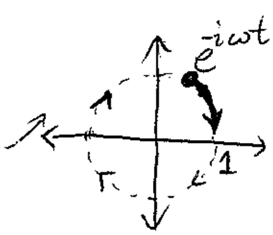
The fact that the components of $e^{i\theta}$ give sine and cosine functions is the key to why complex numbers are so useful when it comes to describing wave phenomena (where sinusoidal functions show up often). A typical sinusoidal wave takes the form

$$A\cos(kx - \omega t + \phi)$$

Assuming a fixed frequency and wavelength (as we have, for example with light of a particular colour), the information that determines the properties of the wave are the amplitude A and the phase ϕ. From what we've just learned, it is clear that this wave can be written as

$$Re(Ae^{i(kx-\omega t+\phi)}) = Re(Ae^{i\phi}e^{i(kx-\omega t)}) = Re(Ze^{i(kx-\omega t)})$$

where we have defined the complex number $Z = Ae^{i\phi}$. So the information about the amplitude and the phase of the wave are nicely combined into a single complex number Z which multiplies $e^{i(kx-\omega t)}$.



We can think about all of this geometrically by noticing that $e^{i(kx-\omega t)}$ is just a complex number on the unit circle (i.e. with fixed magnitude 1) that circles clockwise at a constant rate. The x component of this (the real part) is clearly an sinusoidal function. When we multiple this by the complex number Z, then the new complex number goes around a circle of radius A (so the amplitude of the real part is A) and the angle is shifted by ϕ relative to the original wave, as we can see from our multiplication rule (2).

The real advantage of using the complex numbers comes when we want to add up waves that are not in phase (as we do when considering circular or elliptical polarization of light). Consider the following question: what is the amplitude of a wave that we get by superposing two waves with arbitrary amplitude and phase (but the same wavelength and frequency):

$$A_1 \cos(kx - \omega t + \phi_1) + A_2 \cos(kx - \omega t + \phi_2)$$

To work it out by conventional means is fairly tedious, but we can do it without thinking using complex numbers. If we define $Z_1 = A_1 e^{i\phi_1}$ and $Z_2 = A_2 e^{i\phi_2}$ then the combined wave is just

$$Re\left[(Z_1 + Z_2)e^{i(kx-\omega t)}\right]$$

In other words, if we use complex numbers Z_1 and Z_2 to keep track of the amplitude and phase of the two waves, then the amplitude and phase of the wave that we get by adding the two waves are just the ones corresponding to the complex number $Z_1 + Z_2$. In particular, the amplitude is the magnitude of $Z_1 + Z_2$ which is

$$\begin{aligned}
|Z_1 + Z_2| &= |(A_1 \cos(\phi_1) + A_2 \cos(\phi_2)) + i(A_1 \sin(\phi_1) + A_2 \sin(\phi_2))| \\
&= \sqrt{(A_1 \cos(\phi_1) + A_2 \cos(\phi_2))^2 + (A_1 \sin(\phi_1) + A_2 \sin(\phi_2))^2} \\
&= \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 (\cos(\phi_1)\cos(\phi_2) + \sin(\phi_1)\sin(\phi_2))} \\
&= \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi_1 - \phi_2)}
\end{aligned}$$

so this gives the amplitude of the combined wave. The answer is a bit complicated, but the point is that we didn't have to think very much to get it. We have calculated the magnitude by equating the first and last terms of equation (3), but we could have also used

$$\begin{aligned}
|Z_1 + Z_2| &= \sqrt{(Z_1 + Z_2)^*(Z_1 + Z_2)} \\
&= \sqrt{(A_1 e^{-i\phi_1} + A_2 e^{-i\phi_2})(A_1 e^{i\phi_1} + A_2 e^{i\phi_2})} \\
&= \sqrt{A_1^2 + A_2^2 + A_1 A_2 (e^{i(\phi_1 - \phi_2)} + e^{i(\phi_2 - \phi_1)})} \\
&= \sqrt{A_1^2 + A_2^2 + 2A_1 A_2 \cos(\phi_1 - \phi_2)}
\end{aligned}$$

where in going to the last line, the imaginary parts of the two complex exponentials cancel.