Computers are your friends

A Science One Math/Physics Tutorial

In many real-world applications of physics, mathematics, or other sciences, it is straightforward to write down a set of equations that provides an accurate model of some system that we are interested in, but the equations are too difficult or impossible to solve by hand, even if we know what to do in principle. In many such cases, we can use computers to figure out at least an approximate solution by "brute force," that is, by doing large numbers of tedious calculations.

As an important example, Newton's Second Law leads to a set of differential equations

$$\begin{aligned} \frac{d\vec{r}}{dt} &= \vec{v} \\ \frac{d\vec{v}}{dt} &= \frac{1}{m}\vec{F}(\vec{r},\vec{v}) \end{aligned}$$

where the force $\vec{F}$ is the net force on the object (which can depend on its position and velocity). In principle, these equations determine $\vec{r}(t)$ and $\vec{v}(t)$ for all times once we know the initial position $\vec{r}(0)$ and velocity $\vec{v}(0)$. However, except when the function F is really simple (e.g. constant forces), it's simply not possible to write down a formula for $\vec{r}(t)$ and $\vec{v}(t)$ in terms of ordinary functions that we have names for. However, with enough calculational power, we can often get a solution that is as accurate as we wish. The basic idea is to use the definition of the derivative to say that for small enough Δt,

$$\begin{aligned} \vec{r}(t+\Delta t) &\approx \vec{r}(t) + \Delta t \cdot \vec{v}(t) \\ \vec{v}(t+\Delta t) &\approx \vec{v}(t) + \Delta t \cdot a(\vec{r}(t),\vec{v}(t)) \end{aligned}$$

where $\approx$ means approximately equal (more precisely, the ratio of the two sides approaches 1 as $\Delta t \to 0$). Using this formula, if we know the position and velocity now, we can find the position and velocity at a slightly later time. If we want the position and velocity at a substantially later time, we just need to use this formula over and over again, each time plugging in the new position and velocity on the right hand side.[1]

Such calculations are easily performed by computer. We can use a programming language like Python, specialized calculational software like MatLab, Mathematica, or Maple, or (as we will do today), a simple spreadsheet package.

To get started, open up a blank spreadsheet. If you don't have Excel or a similar package, you can sign up for a free Google account, go to Google Drive, select Create, and select Spreadsheet.

[1] Our result will be more accurate the smaller we take ΔT to be. So if we want to find the position at 1 second after the initial time, we could take $\Delta t = 0.01s$ and use the formula 100 times, or we can take $\Delta t = 0.001s$ and use the formula 1000 times. If we want a certain amount of accuracy, we need to keep making Δt smaller until the answer doesn't change any more at that level of accuracy.

Problem 1

For our physics problem, our goal will be to have one column with all the times $0, \Delta t, 2\Delta t$, etc..., another column with the x-coordinate at each of these times, another column with the x-velocity at each of these times, and similar columns for y and z if these are needed. As a warm-up, lets make a column with all the times.

a) Type "TIME" in the A1 box ("cell") so we remember what this column is (don't type the quotes, just the word in quotes).

b) Type "0" into box A2. This will be the initial time.

c) Click on box A3, type "=", then click the box A2, then type "+ 1" (again, just type the things in quotes). Or you can just type "=A2+1" into box A3. This tells the spreadsheet that box A3 should be whatever is in box A2, plus 1. Try changing the number in box A2 to see what happens, and then change it back to 0.

d) Now, we want to calculate box A4 from A3 in the same way that we calculated box A3 from A2. We can click box A3, copy (CTRL C for Windows, Command-C for Mac), click box A4, and paste (CTRL V for Windows, Command-V for Mac). But this would get tedious if we wanted to do it many times. So instead, we can click the lower-right corner of box A3 and drag downwards for as many boxes as we want. Try dragging it down to fill all the A cells up to A30. Cell A30 should now have the number 28 in it.

But what if we want to change the time increment to 0.1 instead of 1? We could just do everything again with 0.1, but wouldn't it be nice if we could just write the time increment we want in some box and have the times in the first column update automatically when we change that? That feature would be worth some serious $$, wouldn't it? Okay, that last sentence sounded dorky, but I put it in to help you remember that we'll need the $ symbol to accomplish our task.

e) Select cells A3 through A30 and delete them for now. In cell B1, write "TIME STEP" and in cell B2, type "1". Now, for cell A3, instead of "=A2+1", we type "=A2+B2". Your cell A3 should now have the number 1 in it. Click the lower right corner and drag down to fill all the cells up to A30. This should look the same as after step d), with the number 28 in the A30 box. But now try changing box B2 to 0.1. If everything went according to plans, you should now have a series of numbers in column A with increments of 0.1.

You might be wondering why we didn't just type "=A2+B2" into box A3. The reason is that without the dollar signs, the computer interprets this to mean "add the number in the box above to the number in the box above and to the right". This gives the desired answer for box A3, but if we try to copy that to A4, it will give us A3 + B3. But what we want to do is just keep adding the number in the specific box B2 to the number above. That's what typing "=A2 + B2" accomplishes.

Problem 2

We're now ready to do some physics. Let's use the spreadsheet to calculate the trajectory of an object initially without air drag. This is a two-dimensional problem, so type "X", "Y", "Vx", and "Vy" in C1,D1,E1,F1. We'll use columns G and H to calculate accelerations (i.e. the quantities F_x/m and F_y/m), so type "Ax" and "Ay" in cells G1 and H1.

a) Let's assume that the initial X and Y positions are 0 and that the initial X and Y velocities are each 10 m/s. The x-acceleration will always be 0 and the y acceleration will always be $-9.8m/s^2$. Also, let's start with a time step of 0.1s. Enter all these initial values in the appropriate places in row 2.

b) Now, using using the formulas on the bottom of the first page, fill in cells C3,D3,E3,and F3 so that they correctly calculate the new positions and velocities based on the values in the first row. Remember, we want to put in something like "=C2 + ..." so that the spreadsheet will calculate the values for us. The accelerations haven't changed, so we can put in the values 0 and 9.8 in G3 and H3. Even better would be to put "=G2" in G3 and "=H2" in H3, so that we can change all the accelerations all at once by changing the initial values.

c) We're now ready to calculate the positions and velocities for any later times. To do this, simply highlight cells C3 to H3, click on the lower-right corner of this block of cells, and drag downward to fill as many cells as you like.

d) Using these results, determine how long the object is in the air, and how far it travels horizontally before it hits the ground ($Y = 0$).

e) Calculate the exact answers directly and compare these to your values in d).

f) The discrepancy you found has to do with the fact that the formulae on page 1 are only exact in the limit of small time step. To get more accurate results, try making the time step 0.05 instead (you might have to drag down as in step c) to fill in more rows). How much more accurate did your answer become when you cut the time step in half?

Problem 3

Now let's try a problem that can't be solved by hand.

On a remote tropical island, the residents have contracted a highly contagious, deadly virus that threatens to wipe out their entire population. Canadian scientists have developed a treatment that will neutralize the virus and prevent transmission, but the only way to get it to the islanders in time is by an air drop (there is no runway on the island). To ensure the best chance of recovery, the package should be dropped into a small clearing in the center of the village. The plane can safely fly as low as 100m, and the medicine can be ejected (in a padded package) from the back of the plane so that its initial forward velocity relative to the ground is 10 m/s, with zero initial vertical velocity. If the drag force on the package is 0.155 kg/m and the package weighs 1kg, at what horizontal distance from the clearing should the package be released in order for it to land in the clearing?

As a reminder, the accelerations in this case should be $A_x = -0.155V_x\sqrt{V_x^2 + V_y^2}$ *and* $A_y = -0.155V_y\sqrt{V_x^2 + V_y^2}$*. You should start by erasing everything in the fourth row and below. You'll need to change the initial values in the second row, and the accelerations in G2, G3, H2, and H3, then drag to recalculate the other boxes. As an example, G2 should be "*$= -0.155 * E2 * SQRT(E2^2 + F2^2)$*". Ask for help if you get stuck.*

Problem 4

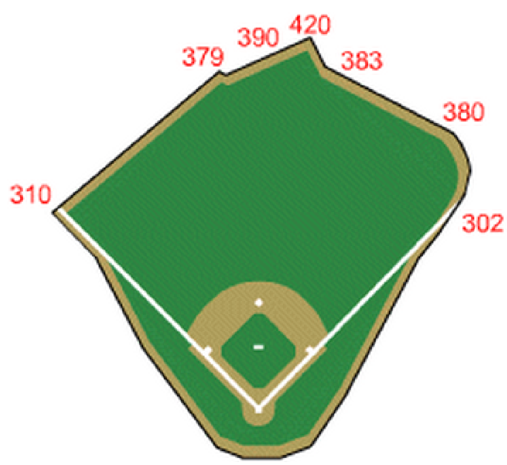
Chris's years of hard work have paid off and he's finally been called up to "The Show" and given the opportunity to umpire professional baseball. Not only that, but his first game is at his favourite stadium: Fenway Park. Fenway is known for its odd field shape that includes "The Triangle" and, more famously, the left field wall charmingly known as the "Green Monster". Though it only lies 97 meters (308 feet) away from home plate, it's a towering 11.3 meters high.

While killing time the day before the game Chris decides to use the radar gun to measure ball velocities of baseballs hit during batting practice. He measures a hit by David Ortiz going 47 m/s (105 miles/hr). We know the magnitude of the drag force on a baseball is given by $F_{\text{drag}} = m_{\text{baseball}} 0.00805 v^2$. Is it possible for Ortiz to hit a home run (i.e., crank one over the Green Monster)?

You will have to play with the angle of the trajectory of the ball. Without drag, the optimal angle is 45 degrees. At what angle can we hit it the furthest when we account for drag?

Hint: you will want to define a box that contains the angle and then have the initial velocity boxes calculate the initial velocities based on this angle and the initial speed. You will also want to use a time step of $\Delta t = 0.005$ *seconds or smaller.*

Fenway Park



Ballpark Diagram & Dimensions