Solutions

Question 1: Energy is distributed over 4 molecules.

If $E = 1$ the configurations are:

(1000) (0100) (0010) (0001)

If $E = 2$ the configurations are:

(2000) (0200) (0020) (0002)
(1100) (1010) (1001) (0110) (0101) (0011)

If $E = 3$ the configurations are:

(3000) (0300) (0030) (0003)
(2100) (2010) (2001) (0210) (0201) (0021)
(1200) (1020) (1002) (0120) (0102) (0012)
(1110) (0111)

If $E = 0$ the configurations are:

(0000)

Question 2:

For M molecules we can find general expressions for the number of configurations N.

For E = 0: There's only one way to do it.

$$N = 1$$

For E = 1: We can take the one energy and put it in a single molecule.

$$N = M$$

For E = 2: We can take both energies and put them in a single molecule. There are M ways to do this.

We can put one energy in one molecule (which there are M of) and we can put the other energy in any other molecule (of which there are M-1 of). The number is M(M-1) but we divide it by 2 because this counts a configuration like (1 1 0) and (1 1 0) as 2 seperate things. So

$$N = M + \frac{M(M-1)}{2}$$

Question 3:

Using the choose formula $\frac{N!}{n!(N-n)!}$ we can identify $N = E + M - 1$ and $n = M - 1$. The number of ways of distributing the energy is

$$\Omega(E) = \frac{(E+M-1)!}{(M-1)!(E+M-1-M+1)!}$$

$$= \frac{(E+M-1)!}{(M-1)!\,E!}$$

Question 4:

The entropy is given by

$$S = k_B \ln \Omega(E)$$

$$= k_B \ln \left(\frac{(E+M-1)!}{(M-1)!\,E!} \right)$$

$$= k_B \ln \left((E+M-1)! \right) - k_B \ln \left((M-1)! \right) - k_B \ln \left(E! \right)$$

$$= k_B (E+M-1) \ln (E+M-1) - k_B \cancel{(E+M-1)}$$

$$- k_B (M-1) \ln (M-1) + k_B \cancel{(M-1)}$$

$$- k_B E \ln E + \cancel{k_B E}$$

$$= k_B (E+M-1) \ln (E+M-1) - k_B (M-1) \ln (M-1)$$

$$- k_B E \ln E$$

Question 5:

The inverse of the temperature is given by

$$T^{-1} = \frac{dS}{dE}$$

$$= k_B \ln(E+M-1) + k_B \frac{\cancel{(E+M-1)}^{1}}{\cancel{(E+M-1)}}$$

$$- k_B \ln E - k_B \frac{\cancel{E}^{1}}{\cancel{E}}$$

$$= k_B \left[\ln(E+M-1) - \ln E \right]$$

$$= k_B \ln\left(\frac{E+M-1}{E}\right)$$

$$T = \frac{1}{k_B \ln\left(\frac{E+M-1}{E}\right)}$$

Question 6:

If $E >> M$ then

$$T = \frac{1}{k_B \ln\left(1 + \frac{M-1}{E}\right)} \simeq \frac{1}{k_B\left(\frac{M-1}{E}\right)}$$

$$= \frac{E}{k_B(M-1)} \simeq \frac{E}{k_B M}$$

Since M is very big usually the temperature is proportional to $\frac{E}{M}$, which is the energy per molecule.

Question 7:

We know that $\Delta E = n c_v \Delta T$. We can use our expression from Q6 to get

$$\frac{T}{E} = \frac{1}{n c_v} = \frac{1}{k_B M} \Rightarrow n c_v = k_B M$$

We also know that $k_B = \frac{R}{N_A}$ and $n = \frac{M}{N_A}$. So

$$c_v = \frac{k_B M}{n} = \frac{R}{N_A} M \left(\frac{N_A}{M}\right) = R \ !$$

The specific heat is R! Compare this to a monotomic gas that has $c_v = \frac{3}{2} R$.