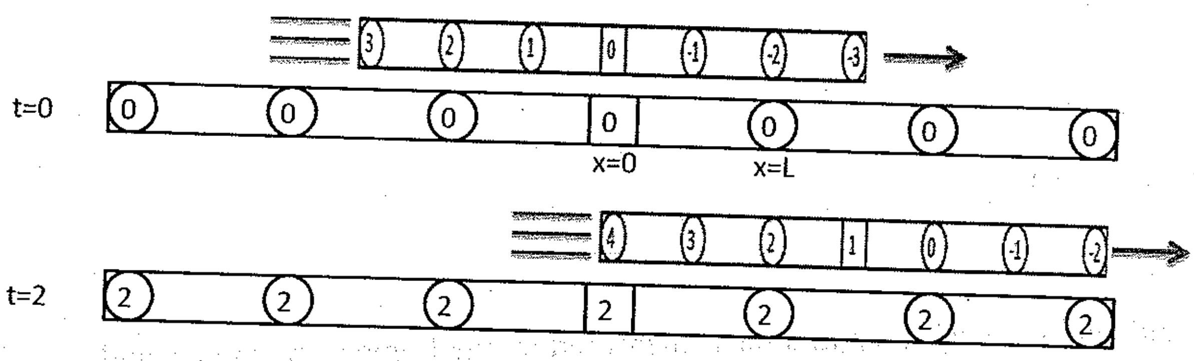
Name: Key.

Physics Tutorial 5: Lorentz Transformations

Question 1

The pictures below show two snapshots from the video that we saw in class.



a) A firecracker explodes at position x=2L and time t=2 in the frame of the lower ruler. Where and when does this event occur in the frame of the upper (moving) ruler?

$x' = L \quad t' = 0$

To get these we look at the lower ruler at $t=2$ and $x=2L$ and read off what the upper ruler reads.

b) A cell phone rings at time t=0 and position x = -L in the frame of the lower ruler. Where and when does this event occur in the frame of the upper (moving) ruler?

Similarly,

$x' = -2 \quad t = 2$

Hopefully, it is now clear that we can figure out the position and time of any event in the frame of the moving ruler if we are given the position and time in the frame of the fixed ruler (assuming that we have pictures of the two rulers and their clocks at all possible times).

Mathematically, this means there is a function that outputs the position and time in the moving frame given the position and time in the fixed frame. Here's what it looks like:

$x' = \gamma\,(x - v\,t)$

$t' = \gamma\,(t - (v/c^2)\,x)$

Here, x and t are the position and time as measured in one frame of reference, and x' and t' are the position and time as measured in a frame of reference moving in the positive x direction at speed v relative to the first frame. We assume (as in the example on the previous page) that both observers agree on what event to call x=t=0. In the previous example, this is when the middle points on the two rulers pass.

(You can apply these formulae to check your results on the previous page, but you need to know that the speed was $v = (3/4)^{1/2}$ c. Also the times were given in units of vL/c^2, so for example, t=2 really meant $t = 2\ vL/c^2$.)

for $x=2L$ $t=2\frac{vL}{c^2}$:

$$x' = \frac{1}{\sqrt{1-\frac{3}{4}}}\left(2L - \left(\frac{3}{4}\right)^{\frac{1}{2}}c\,2\left(\frac{3}{4}\right)^{\frac{1}{2}}c\frac{L}{c^2}\right)$$

$$= 2\left(2L - \frac{3}{4}2L\right) = 4\frac{1}{4}L = L$$

$$t' = 2\left(2\left(\frac{3}{4}\right)^{\frac{1}{2}}c\frac{L}{c^2} - \left(\frac{3}{4}\right)^{\frac{1}{2}}\frac{c}{c^2}2L\right)$$

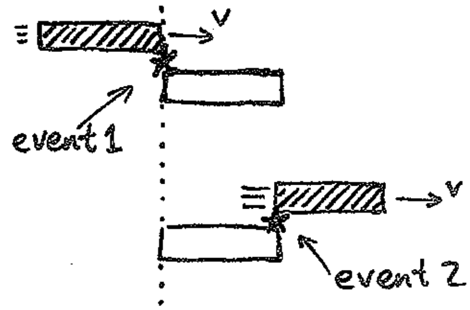
$$= 0$$

It works!

Question 2

As an example of how to use the Lorentz transformation, consider the third homework problem:

A train of length 300m observes another train on a parallel track coming towards it at $v = \sqrt{3/4}\,c$. The other train appears (i.e. is measured by the first train) to have length 300m also. In the reference frame of the second train, how long does it take for the two trains to pass each other (i.e. what is the time between when the fronts align and when the backs align)



This problem is asking for the time between two events in the frame of the moving train. As a first step, what are the times and positions of these two events in the original frame?

Frame of stationary train:

we can choose our coordinates so this is true

EVENT 1: time $t_1 = 0$

position $x_1 = 0$

EVENT 2: time $t_2 = 600\,\mathrm{m}\sqrt{\frac{4}{3}}\,\frac{1}{c} = \frac{1200\,\mathrm{m}\sqrt{3}}{3c} = 400\sqrt{3}\,\frac{\mathrm{m}}{c}$

position $x_2 = 300\,\mathrm{m}$

Now, using the Lorentz transformation, determine the times for the two events in the frame of the moving train

Frame of moving train: $\gamma = \frac{1}{\sqrt{1-\frac{3}{4}}} = 2$

EVENT 1: time $t'_1 = 2(0\,\mathrm{m} - v\,0\,\mathrm{s}) = 0\,\mathrm{s}$

EVENT 2: time $t'_2 = 2\left(400\sqrt{3}\,\frac{\mathrm{m}}{c} - \sqrt{\frac{3}{4}}\,\frac{c}{c^2}\,300\,\mathrm{m}\right) = 2\left(250\sqrt{3}\,\frac{\mathrm{m}}{c}\right) = 500\sqrt{3}\,\frac{\mathrm{m}}{c}$

In the frame of the moving train, how long does it take for the two trains to pass each other?

In the moving frame it takes

$$t'_2 - t'_1 = 500\sqrt{3}\,\frac{\mathrm{m}}{c}$$

for the two trains to pass, which is longer!

Assume that observers in frame S define the event where the ship passes Planet Totem to be at x=0 and t=0.

a) For the event where the comet passes Planet Vanier, what is the position and time as measured in the frame of the planets (frame S)?

$x_V = 1\ cy$

$t_V = 0\ s$

b) For the event where the comet passes Planet Totem, what is the position and time as measured in the frame of the planets (frame S)?

$x_T = 0\ cy$

$t_T = 1\ cy/_4 = 1\,cy/0.8c = \frac{5}{4}\ Y$

c) Using the Lorentz Transformation, determine the positions and times for these events as measured in the frame of the ship. Remember that the v and γ in the Lorentz Transformation refer to the velocity of the frame S' relative to the frame S.

$\gamma = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{1}{\sqrt{\frac{9}{25}}} = \frac{5}{3}$

$x'_V = \frac{5}{3}\left(1cy - \frac{4}{5}c\ 0s\right) = \frac{5}{3}cy$

$t'_V = \frac{5}{3}\left(0s - \frac{4}{5}\frac{c}{c^2}1cy\right) = -\frac{4}{3}Y$

$x'_T = \frac{5}{3}\left(0cy - \frac{4}{5}c\frac{5}{4}Y\right) = -\frac{5}{3}cY$

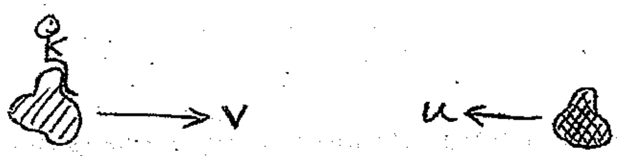
$t'_T = \frac{5}{3}\left(\frac{5}{4}Y - \frac{4}{5}\frac{c}{c^2}0Y\right) = \frac{25}{12}Y$

d) What speed do observers on the ship measure for the comet? Is this greater than c?

$$u' = (x'_T - x'_V)/(t'_T - t'_V) = \frac{\left(-\frac{5}{3}cY - \frac{5}{3}cY\right)}{\left(\frac{25}{12}Y + \frac{16}{12}Y\right)}$$

$$= \frac{-\frac{10}{3}cY}{\frac{41}{12}Y} = -\frac{40}{41}c$$

below c and negative!



e) Derive a general formula for the relative velocity of two objects if the objects are observed to be travelling towards each other with speed v and u respectively. What does your formula give if we take u to be c?

Generally,

$$u' = \frac{(x'_T - x'_v)}{(t'_T - t'_v)}$$

$$= \frac{\gamma(x_T - vt_T) - \gamma(x_v - vt_v)}{\gamma(t_T - \frac{v}{c^2}x_T) - \gamma(t_v - \frac{v}{c^2}x_v)}$$

Now we can set the origin to be $t_v = 0$, $x_T = 0$.
And we know that

$$x_v = x_v \, , \quad t_T = \frac{x_v}{u}$$

Subbing in,

$$u' = \frac{\cancelto{0}{x_T} - \frac{v x_v}{u} - x_v + \cancelto{0}{v t_v}}{\frac{x_v}{u} - \cancelto{0}{\frac{v}{c^2} x_T} - \cancelto{0}{t_v} + \frac{v}{c^2} x_v}$$

$$= \frac{-\left(\frac{v}{u} x_v + x_v\right)}{\frac{x_v}{u} + \frac{v}{c^2} x_v} = \frac{-(v+u)}{1 + \frac{vu}{c^2}}$$

$$\Rightarrow \boxed{u' = -\frac{(v+u)}{1 + \frac{uv}{c^2}}}$$