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A Physics Worksheet: Relativistic Energy and Momentum

If we assume that the speed of light is the same in all frames of reference, it's necessary to modify our definition of momentum in order to preserve conservation of momentum as a valid physical law:

$$\vec{p} = m\frac{d\vec{x}}{d\tau} = \gamma m\vec{v}, \quad \text{where} \quad \gamma = \frac{1}{\sqrt{1-\frac{v^2}{c^2}}}$$

and v is the velocity of the object and m is its mass. With this definition, the total momentum of all objects in a system is conserved in any frame of reference.

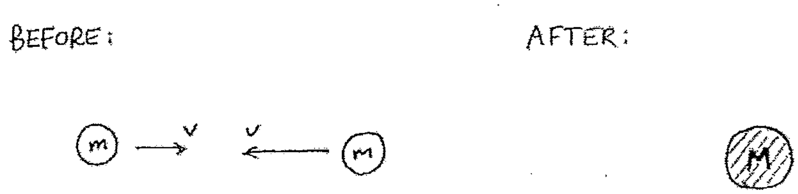
It's also necessary to change our definition of energy:

$$E = \gamma mc^2.$$

In ordinary mechanics we're used to mass being conserved and kinetic energy being conserved only in elastic collisions. In this worksheet we'll reexamine these notions.

Question 1

Suppose we have two objects of mass m that travel towards each other, each with speed $v = 4/5$c.



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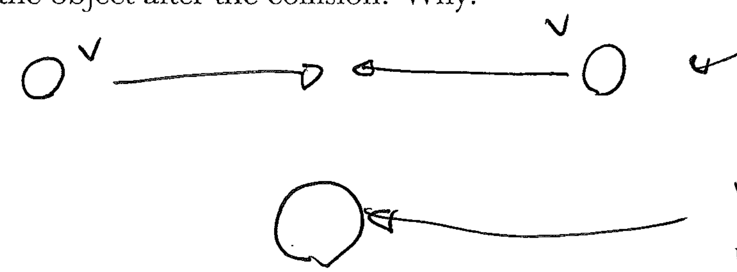
a) What is the total relativistic energy before the objects collide (answer in terms of m and c)?

$$\gamma = \frac{1}{\sqrt{1-\frac{16}{25}}} = \frac{5}{3}$$

$$E = \gamma m c^2 + \gamma m c^2 = 2\gamma m c^2 = \frac{10}{3} m c^2$$

b) If the objects stick together when they collide to form a new object, what is the velocity of the object after the collision? Why?



total momentum before is zero

momentum after is zero, so object doesn't move.

c) If the mass of the new object is M, what is the total relativistic energy after the collision (in terms of M and c)?

Velocity is zero, so $\gamma = 1$.

$$E = \gamma M c^2 = M c^2$$

d) Assuming that energy is conserved what is M in therms of m? Is mass conserved?

$$E_{before} = E_{after}$$

$$\Rightarrow \frac{10}{3} m c^2 = M c^2 \Rightarrow M = \frac{10}{3} m$$

If mass was conserved we'd expect $M = 2m$. So mass isn't conserved!

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Question 2

When we take $v \ll$ c for the momentum equation $x = v^2/c^2$ is much smaller than 1 and $\gamma \sim 1$. We're left with something that looks like the classical expression for momentum. This isn't true for the expression for relativistic energy. We get something that doesn't look anything like energy as we know it.

There is a way of approximating functions called the Taylor expansion. If we expand the energy

$$E(x) = mc^2(1-x)^{-1/2}$$

around x = 0 the first three terms of the expansion are:

$$E(x) = E(0) + E'(0)\,x + \frac{1}{2}E''(0)\,x^2 + \ldots$$

The first two terms look a lot a like a straight line. If x is very small, all you're really doing is approximating the function near a point by a straight line.

Write the first two terms of the approximation. Be sure to rewrite the energy in terms of v and c so you can see something amazing happen.

$$E(0) = mc^2(1-0)^{-1/2} = mc^2$$

$$E'(0) = mc^2(1-x)^{-3/2}\left(+\frac{1}{2}\right)\Big|_{x=0}$$

$$= \frac{mc^2}{2}$$

So

$$E(x) = mc^2 + \frac{1}{2}mc^2 x + \ldots \qquad \frac{v^2}{c^2}$$

$$= mc^2 + \frac{1}{2}mv^2 + \ldots$$

mass energy

kinetic energy appears!

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Question 3

What is the next term in the expression above? How fast does an object have to go before this is 1% as big as the second term?

Question 4

The energy produced by the Sun comes from nuclear fusion reactions, in which lighter elements fuse together into heavier elements (in contrast to fission reactions by which most nuclear reactors operate). Much research has gone into developing nuclear fusion power, since the potential for accidents and runaway reactions is much less. One of the simplest fusion reactions involves two isotopes of Hydrogen: Deuterium, with one proton and one neutron, and Tritium, with one proton and two neutrons).

$$^2H + {}^3H \rightarrow {}^4He + n$$

The masses of these isotopes are:

^2H: 2.0141u , ^3H: 3.0161u, ^4He: 4.0026u, n: 1.0087u, where u = 1.661×10^{-27} kg.

The total electrical power usage in Canada is about 2×10^{18} J. If we want to produce this amount of energy using fusion, what mass of deuterium would we need?

Question 5

The mass of a hydrogen atom is actually less that the individual masses of an electron and a proton. How is this possible?

See clicker question.

Question 3 :

$$E''(0) = \frac{mc^2}{2}\left(-\frac{3}{2}\right)(1-x)^{-5/2}(-1)\Big|_{x=0}$$

$$= \frac{3}{4}mc^2$$

The third term is $\frac{1}{2}E''(0)x^2 = \frac{3}{8}mc^2\left(\frac{v^2}{c^2}\right)^2$

$$= \frac{3}{8}\frac{mv^4}{c^2}$$

So for

$$\frac{\text{third term}}{\text{second term}} = \frac{3}{8}\frac{mv^4}{c^2}\frac{2}{mv^2} = 0.01$$

$$\Rightarrow \frac{3}{4}\frac{v^2}{c^2} = 0.01$$

$$\Rightarrow v = \sqrt{(0.01)c^2\frac{4}{3}}$$

$$\underline{v = 0.12c}$$

Nuclear reactions: Question 4.:

$$^{2}H + {}^{3}H \longrightarrow {}^{4}He + n.$$

In this reaction the initial mass is higher than the products.

The mass difference is $0.0189\,u$.

$$= 3.14 \times 10^{-29}\ \text{kg}.$$

The energy released is this $\times c^2$

$$= 2.82 \times 10^{-12}\ \text{J}$$

To get 2×10^{18} J we need this reaction to happen

$$N = \frac{2.82\times10^{-12}\ \text{J}}{2\times10^{18}\ \text{J}} = 1.41\times10^{30}\ \text{times}$$

Which means we need

$$(1.41\times10^{30})(2.0141)(1.661\times10^{-27}\ \text{kg})$$

$$= 4700\ \text{kg}.$$

About 5 tonnes powers Canada for a year. For comparison, the same energy would take 36 million tonnes natural gas.