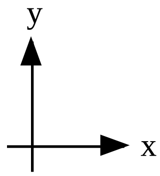
Laplace's Equation: $\nabla^2 V = 0$

Types of solutions – linear combinations of the forms below:

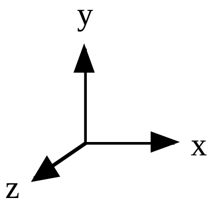
2-D Cartesian



$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

$$V(x,y) \Rightarrow \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} + \left.\begin{matrix} e^{kx} \\ e^{-kx} \end{matrix}\right\} \left.\begin{matrix} \cos ky \\ \sin ky \end{matrix}\right\} + \text{ permutations } \{x \Leftrightarrow y\}$$

3-D Cartesian

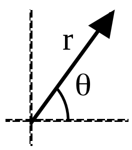


$$\nabla^2 V = \frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(x,y) \Rightarrow \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} y \\ 1 \end{matrix}\right\} \left.\begin{matrix} z \\ 1 \end{matrix}\right\} + \left.\begin{matrix} x \\ 1 \end{matrix}\right\} \left.\begin{matrix} \cos qy \\ \sin qy \end{matrix}\right\} \left.\begin{matrix} e^{qz} \\ e^{-qz} \end{matrix}\right\} + \left.\begin{matrix} e^{px} \\ e^{-px} \end{matrix}\right\} \left.\begin{matrix} \cos qy \\ \sin qy \end{matrix}\right\} \left.\begin{matrix} \cos\sqrt{p^2 - q^2}\, z \\ \sin\sqrt{p^2 - q^2}\, z \end{matrix}\right\}$$

$$+ \text{ all permutations } \{x,y,z\}$$

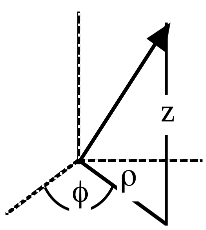
2-D Plane Polar



$$\nabla^2 V = \frac{1}{r}\frac{\partial}{\partial r}\left(r\frac{\partial V}{\partial r}\right) + \frac{1}{r^2}\frac{\partial^2 V}{\partial \theta^2} = 0$$

$$V(r,\theta) \Rightarrow \left.\begin{matrix} \ln r \\ 1 \end{matrix}\right\} + \left.\begin{matrix} r^n \\ r^{-n} \end{matrix}\right\} \left.\begin{matrix} \cos n\theta \\ \sin n\theta \end{matrix}\right\}$$

3-D Cylindrical Polar



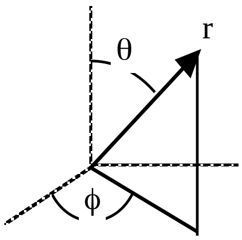
$$\nabla^2 V = \frac{1}{\rho}\frac{\partial}{\partial \rho}\left(\rho\frac{\partial V}{\partial \rho}\right) + \frac{1}{\rho^2}\frac{\partial^2 V}{\partial \phi^2} + \frac{\partial^2 V}{\partial z^2} = 0$$

$$V(\rho,\theta,z) \Rightarrow \left.\begin{matrix} J_n(k\rho) \\ N_n(k\rho) \end{matrix}\right\} \left.\begin{matrix} \cos n\phi \\ \sin n\phi \end{matrix}\right\} \left.\begin{matrix} e^{kz} \\ e^{-kz} \end{matrix}\right\}$$

$J_n(k\rho) \rightarrow$ Bessel functions

$N_n(k\rho) \rightarrow$ Neumann functions

3-D Spherical:



$$\nabla^2 V = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2\frac{\partial V}{\partial r}\right) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial \theta}\left(\sin\theta\frac{\partial V}{\partial \theta}\right) + \frac{1}{r^2 \sin^2\theta}\frac{\partial^2 V}{\partial \phi^2} = 0$$

$$V(r,\theta,\phi) \Rightarrow \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} P_n^m(\cos\theta) \\ Q_n^m(\cos\theta) \end{matrix}\right\} \left.\begin{matrix} \cos m\phi \\ \sin m\phi \end{matrix}\right\}$$

$P_n^m(\cos\theta) \rightarrow$ Legendre polynomials,

$Q_n^n(\cos\theta) \rightarrow$ Legendre polynomials of the second kind

If AXIAL symmetry: $V(r,\theta) \Rightarrow \left.\begin{matrix} r^n \\ r^{-(n+1)} \end{matrix}\right\} \left.\begin{matrix} P_n(\cos\theta) \\ Q_n(\cos\theta) \end{matrix}\right\}$

Janis McKenna, PHYS 354, Sept 2004