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Minkowski correlation functions in AdS/CFT

James Charbonneau

Department of Physics and Astronomy, University of British Columbia, Vancouver, BC, Canada, V6T 1Z1
E-mail: james@phas.ubc.ca

Abstract: This short review was completed as a final project for String Theory II taught by Moshe Rozali at the University of British Columbia. Its purpose is to reproduce the calculation of the shear viscosity in a gauge theory which was originally done by Policastro, Son and Starinets in [1]. As the paper containing the original calculation is a painfully terse Physics Review Letter, our primary source will be the diptych [2] and [3], where the calculation is done in the former and the technique explained in the latter. The shear viscosity is intresting because it is a dynamic quantity that can not be calculated using the standard AdS/CFT Euclidean formalism. A Minkowski formalism must be developed and used directly.

Keywords: shear viscosity, Minkowski correlation functions.

Contents

1. Introduction

In this paper we will be exploring the duality between gauge and gravity theories. We will specifically focus on the duality between type IIb string theory on $AdS_5 \times S^5$ and $\mathcal{N} = 4$ supersymmetric Yang-Mills (SYM) theory on 4-dimensional Minkowski space. The duality between these two theories was one fo the first explored and is well understood.

The goal of this paper is to calculate dynamic quantities in CFT, such as diffusion rates and viscosity, using the AdS/CFT correspondence. In order to do this it will be necessary to work directly in a Minkowski formalism rather than the well trodden Euclidean formalism. Though many properties can be calculated in the Euclidean formalism and Wick rotated to obtain the Minkowski version, this is not true for dynamic variables.

The paper will start with an introduction to the Minkowski prescription of the gauge-gravity correspondence. We will explicitly derive the form of a retarded Green's function in a conformal field theory given the metric of its dual. We will see why the the derivation for Euclidean Greens functions breaks down in the Minkowski case and what must be done to fix it.

We will then discuss how to calculate dynamic quantities from Green's functions by thinking of hydrodynamics as an effective field theory. The Kubo formula for the shear viscosity will be derived.

Two thermal properties in $\mathcal{N} = 4$ SYM will be calculated using the AdS/CFT correspondence between the black three brane metric and $\mathcal{N} = 4$ SYM. The first is the entropy which comes directly from the relationships between the coupling constants required to make the two theories dual to each other. The second quantity calculated is the shear viscosity. Being a dynamic quantity the thermal Minkowski prescription is used.

Finally, since we've calculated the two properties required for it, we'll look at the ratio of entropy density to shear viscosity and discuss the conjecture that this as a universal lower bound for all conformal field theories.

1.1 Parameter Matching

Since introducing why the duality exists is better left for another review, we will focus on the parameter matching that is required for the two theories to be dual. This is the simplest part of of the AdS/CFT dictionary and a more useful discussion can be found in [4]. There are many other parameters and fields that match each other but they will not be required for this discussion.

On the field theory side there are two parameters - the number of colours N and the gauge coupling g. When the number of colours is large perturbation theory is controlled by the 't Hooft coupling $\lambda = g^2 N$. On the string theory side the parameters are the string coupling g_s, the string length l_s, the number of branes N, and the radius R of the AdS space.

We will start by considering the gravity side. A single Dp-brane brane back reacts minimally, and does not substantially curve the space-time around it, making a good probe for singularities. This back reaction becomes significant if we stack a large number of Dp-branes on top of each other. If the number is large enough the curvature results in a black hole. Dp-branes become very interesting if we attach open strings to them. The resulting theory, when quantized, will give us a gauge theory. In our case, where we are concerned with $\mathcal{N} = 4$ SYM gauge theory, it turns out that D3-branes give us the right particles. Stacking a large number of D3-branes together creates a gravitational theory with a black hole that is dual to $\mathcal{N} = 4$ SYM. In creating this black hole with Dp-branes we find that

$$16\pi G = (2\pi)^7 g_s^2 l_s^8 \,, \tag{1.1}$$

$$\frac{R^4}{l_s^4} = 4\pi N \,. \tag{1.2}$$

Another relationship between parameters can be found by expanding the string theory to look like a Born-Infield action. When done, this looks remarkably like a SYM action with a different coefficient. Gathering up what should be the gauge coupling g we get for an arbitrary Dp-brane

$$g^2 = \frac{g_s}{l_s}(4\pi l_s)^{p-2} \tag{1.3}$$

$$\Rightarrow g^2 = 4\pi g_s \,, \text{ for } p = 3. \tag{1.4}$$

We're interested in $\mathcal{N} = 4$ SYM that are created by D3-branes, so $p = 3$, leaving a simple relationship between the gauge and string coupling constants.

We can rewrite these parameters in such a way to see the point of the gauge-gravity duality,

$$\frac{R^2}{\alpha'} \sim \sqrt{g_s N} \sim \sqrt{\lambda} \,. \tag{1.5}$$

This tells us that doing perturbation theory with $\alpha' = l_s^2$ on the string theory side allows us to obtain results where the 't Hooft coupling is very large $1/\sqrt{\lambda} \ll 1$ and vice versa.

Using the gauge-gravity correspondence it is possible to do perturbative calculations on one side to get strong coupling results, where perturbation theory would fail, on the other side.

2. Minkowski AdS/CFT

It might not be clear why a Minkowski formalism is needed when the Euclidean one is so successful, especially when the Euclidean results can be analytically continued by Wick rotation to get the Minkowski result. This method of Wick rotating only works in systems in thermal equilibrium. The Euclidean method, though generally easier to work with, runs in to a problem when we want to calculate quantities that are slightly out of equilibrium. To calculate non-equilibrium quantities retarded and advanced Green's functions are required. To find these Green's functions it is necessary to take the low frequency limit. In the Euclidean formalism the Euclidean time becomes periodic in the temperature. When the Euclidean Green's function is Wick rotated back to make a retarded or advanced Green's function, only discrete frequencies, the Matsabura frequencies, will survive. The lowest of these frequencies is already $2\pi T$, too large to be useful, so it is necessary to calculate the retarded Green's function directly from a Minkowski formalism.

Minkowski gauge-gravity correspondence states that a retarded two-point correlation function in gauge theory can be calculated by taking derivatives of the Minkowski generating functional of its dual gravity theory. On the gauge side the sources $J(x)$ are coupled with operators $O(x)$. On the gravity side there is a field ϕ that is dual to the operator $O(x)$. The correspondence says that the expectation value of the operator is equated to the value the field on the gravity side takes on the boundary of the theory. Technically this means that the derivatives on the generating functional that are usually taken with respect to the source J and now taken with respect to $\phi_0 = \phi(r = \infty)$.

This idea can be written more formally as

$$Z_{\text{gauge}}[J(x)O(x)] = Z_{\text{gravity}}[\phi] \Bigg|_{\phi_0 \to J} , \tag{2.1}$$

where Z_{gauge} and Z_{gravity} are the partition functions of the dual gauge and gravity theories. Now it is clear that when we find gauge field correlation functions by taking functional derivatives with respect to the sources J that couple to our desired operator, that on the gravity side we actually take these derivatives with respect to the scalar field at the boundary ϕ_0. This is a non-trivial statement, and it is true that there is a one-one mapping between sources J and boundary conditions of the scalar field ϕ_0.

In this paper we are interested in calculating two-point correlation functions, which are done by taking two derivatives of the generating function. The correspondence tells us to take two derivatives of the generating function on the gravity side,

$$G(x-y) = -i\langle TO(x)O(y)\rangle = \frac{\delta^2}{\delta J(x)\delta J(y)} Z_{\text{gravity}}[\phi]\Bigg|_{\phi_0 \to J} . \tag{2.2}$$

Because there are only two derivatives with respect to ϕ_0 we are only required to consider the AdS action up to the quadratic order when looking for two point functions on the CFT side. After equating the sources and the field at the boundary, J is still set to zero, as in the regular method of finding correlation functions, and any higher terms in the action will vanish.

One more piece of the AdS/CFT dictionary is required before we go ahead. There is a connection between operators and fields. Since we want to find properties of the Maxwell field our operator is $O = -\mathcal{L} = \frac{1}{4}F_{\mu\nu}^2$ which corresponds to the scalar dilaton field ϕ on the gravity side. So when we calculate correlation functions for the Maxwell field we will always take derivatives with respect to the boundary of the dilaton field ϕ_0.

Let's consider the action of a scalar field in the background that has a metric of the form

$$ds^2 = g_{rr}dr^2 + g_{\mu\nu}(r)dx^\mu dx^\nu \,. \tag{2.3}$$

A more general metric may be chosen but we are interested in this one as the AdS_5 metric conforms to it. Up to second order in ϕ the action is

$$S = K\int d^4x \int dr\sqrt{-g}[g^{rr}\partial_r\phi\partial_r\phi + g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi - m^2\phi^2]\,. \tag{2.4}$$

The Euler-Lagrange equations of motion follow easily,

$$\partial_r(\sqrt{-g}\ g^{rr}\partial_r\phi) + \sqrt{-g}(g^{\mu\nu}\partial_\mu\partial_\nu + m^2)\phi = 0\,. \tag{2.5}$$

In order to take functional derivatives of this we need to write it in terms of the boundary fields ϕ_0. We assume that ϕ has the solution

$$\phi(r,x) = \int \frac{d^4k}{(2\pi)^4}e^{ik\cdot x}f_k(r)\phi_0(k)\,, \tag{2.6}$$

and plug it in to the action

$$\begin{aligned} S &= K\int d^4x\int dr\int\frac{d^4k}{(2\pi)^4}\int\frac{d^4k'}{(2\pi)^4}\sqrt{-g}\ e^{i(k+k')\cdot x}\,[g^{rr}\partial_r f_k(r)\partial_r f_{k'}(r) \\ &\quad - (g^{\mu\nu}k_\mu k'_\nu + m^2)f_k(r)f_{k'}(r)]\ \phi_0(k)\phi_0(k') \\ &= K\int dr\int\frac{d^4k}{(2\pi)^4}\sqrt{-g}\,[g^{rr}\partial_r f_k(r)\partial_r f_{-k}(r) \\ &\quad -(g^{\mu\nu}k_\mu k_\nu + m^2)f_k(r)f_{-k}(r)]\ \phi_0(k)\phi_0(-k)\,. \end{aligned} \tag{2.7}$$

Substituting the solution for $\phi(r,x)$ in to the equation of motion gives the equation of motion for $f_k(r)$,

$$-\sqrt{-g}(g^{\mu\nu}k_\mu k_\nu + m^2)f_{-k} = \partial_r(\sqrt{-g}\ g^{rr}\partial_r f_{-k})\,. \tag{2.8}$$

Performing integration by parts on the first term of (2.7) and substituting in the equation of motion for the second term yields,

$$\begin{aligned} S = K\int dr\int\frac{d^4k}{(2\pi)^4}&\left[\partial_r(\sqrt{-g}g^{rr}f_k(r)\partial_r f_{-k}(r)) - f_k(r)\partial_r(\sqrt{-g}\ g^{rr}\partial_r f_{-k})\right. \\ &\left.+f_k(r)\partial_r(\sqrt{-g}\ g^{rr}\partial_r f_{-k})\right]\ \phi_0(k)\phi_0(-k) \end{aligned}$$

$$
\begin{aligned}
&= K\int dr\int \frac{d^4k}{(2\pi)^4}\partial_r(\sqrt{-g}g^{rr}f_k(r)\partial_r f_{-k}(r))\phi_0(k)\phi_0(-k)\\
&= K\int \frac{d^4k}{(2\pi)^4}\sqrt{-g}g^{rr}f_k(r)\partial_r f_{-k}(r)\phi_0(k)\phi_0(-k)\Big|_{r_{\rm B}}^{r_{\rm H}}\\
&= \int \frac{d^4k}{(2\pi)^4}\phi_0(-k)[F(r_{\rm H},k)-F(r_{\rm B},k)]\phi_0(k)\,, \qquad (2.9)
\end{aligned}
$$

where $r_{\rm B}$ and $r_{\rm H}$ are respectively the values for the boundary of the space and the horizon and

$$F(r,k) = K\sqrt{-g}g^{rr}f_{-k}(r)\partial_r f_k(r)\,, \qquad (2.10)$$

is the kernel. This is our candidate for the two-point retarded Green's function.

To calculate the correlation function on the gravity side we follow the same prescription we would on the field theory side. To find a two-point function we take two functional derivatives, but now with respect to $\phi_0(x)$ instead of $J(x)$.

The fist thing we want to do is rewrite the action in position space. To do so we define

$$\phi_0(k) = \int d^4x e^{ikx}\phi_0(x) \qquad (2.11)$$

$$\phi_0(-k) = \int d^4y e^{-ikx}\phi_0(y) \qquad (2.12)$$

$$F(x-y) = \int \frac{d^4k}{(2\pi)^4}F(k)e^{ik(x-y)}. \qquad (2.13)$$

The action then becomes

$$S(x-y) = \int d^4x\int d^4y\phi_0(x)[F(r_{\rm H},x-y)-F(r_{\rm B},x-y)]\phi_0(y)\,. \qquad (2.14)$$

We place this into the partition function and we can now take functional derivatives of it. The two-piont correlation function is given by,

$$
\begin{aligned}
&< 0|TO(x_1)O(x_2)|0>\\
&= \frac{1}{Z_0}\frac{-i\delta}{\delta\phi_0(x_1)}\frac{-i\delta}{\delta\phi_0(x_2)}e^{iS[\phi_0(x),\phi_0(y)]}\\
&= \frac{1}{Z_0}\frac{-i\delta}{\delta\phi_0(x_1)}\Bigg[\int d^4y[F(r_{\rm H},x_1-y)-F(r_{\rm B},x_1-y)]\phi_0(y)\\
&\quad +\int d^4x\phi_0(x)[F(r_{\rm H},x-x_1)-F(r_{\rm B},x-x_1)]e^{iS[\phi_0(x),\phi_0(y)]}\Bigg]\Bigg|_{\phi_0=0}\\
&= i[F(r_{\rm B},x_1-x_2)+F(r_{\rm B},x_2-x_1)]-i[F(r_{\rm H},x_1-x_2)+F(r_{\rm H},x_2-x_1)]\,. \qquad (2.15)
\end{aligned}
$$

Taking the fourier transform of this, the propatator in momentum space is,

$$< 0|TO(k)O(0)|0 >= i[F(r_{\rm B},k)+F(r_{\rm B},-k)]-i[F(r_{\rm H},k)+F(r_{\rm H},-k)]\,. \qquad (2.16)$$

The retarded Green's function in momentum space is defined as,

$$G^{\rm R}(k) = -i < 0|TO(k)O(0)|0 >\,, \qquad (2.17)$$

which means that

$$G^{\mathrm{R}}(k) = [F(r_{\mathrm{B}}, k) + F(r_{\mathrm{B}}, -k)] - [F(r_{\mathrm{H}}, k) + F(r_{\mathrm{H}}, -k)] \,. \tag{2.18}$$

Which is great. The problem is that retarded Green's functions are supposed to be imaginary, and this is real. We can prove it by first noticing that $f_k(r)$ is the Fourier transform of a real function. It then has the property $f_k(r)^* = f_{-k}(r)$. Then $F(r, k)$ can be written as,

$$\begin{aligned} F(r, k) &= K\sqrt{-g}g^{rr} f_k^*(r)\partial_r f_k(r) \\ &= F(r, -k)^* \,, \end{aligned} \tag{2.19}$$

then the Green's function can be written,

$$\begin{aligned} G^{\mathrm{R}}(k) &= [F(r_{\mathrm{B}}, k)^* + F(r_{\mathrm{B}}, k)] - [F(r_{\mathrm{H}}, k)^* + F(r_{\mathrm{H}}, k)] \\ &= Re[F(r_{\mathrm{B}}, k)] - Re[F(r_{\mathrm{H}}, k)] \,, \end{aligned} \tag{2.20}$$

as a real function. Apparently blindly following the program laid out for Euclidean AdS/CFT doesn't work. In order to get a sensible retarded Green's function it is necessary to make a new conjecture for the relation between Minkowski correlation functions on the CFT side and the correlation functions on the gravity side.

These difficulties can be avoided if we conjecture that

$$G^{\mathrm{R}}(k) = -2F(k, r) \Big|_{r_{\mathrm{B}}}^{r_{\mathrm{H}}} \,. \tag{2.21}$$

This is where the real conjecture for calculating Minkowski correlation functions occurs. It doesn't follow strictly by following the Euclidean AdS/CFT correspondence but it does seem natural. The justification for it is that has worked in all testable cases. The contribution from the horizon will be discarded later because we don't want solutions that are emitted from the horizon.

3. Kubo formula

With such rich field theory formalism in place, it is most useful now to think of hydrodynamics as an effective field theory that describes the dynamics of a system at large distance and time scales. However, unlike most effective field theories, hydrodynamics must be described in terms of its equations of motion rather than its Lagrangian; dissipative terms are very difficult to encode in a Lagrangian formalism. It should still be possible to use the equations of motion to extract low-energy Green's functions.

The method of finding a two point correlation function in field theory starts by coupling a source $J(x)$ to an operator $O(x)$ and adding it to the existing action,

$$S = S_0 + \int dk J(x)O(x) \,. \tag{3.1}$$

This source perturbs the system and the average value of $O(x)$ will differ from its equilibrium value. When the source $J(x)$ is tiny the perturbations are given by linear response theory,

$$\langle O(x) \rangle = i \int dx' G^{\mathrm{R}}(x - x') J(x) \,, \tag{3.2}$$

in terms of the retarded Green's function $G^{\mathrm{R}}(x - x')$.

The hydrodynamic equations of motion are just the conservation laws of energy and momentum,

$$\partial_\mu T^\mu_{\ \nu} = 0 \,. \tag{3.3}$$

To make this a solvable system the number of independent equations must be reduced. This is done through the assumption of local thermal equilibrium. The energy momentum tensor must also be expanded enough that the dissipative terms appear. The elements of the dissipative terms are then found by use of rotational symmetry. The energy momentum tensor can be written as,

$$T^{\mu\nu} = (\epsilon + P) u^\mu u^\nu + P g^{\mu\nu} - \sigma^{\mu\nu} \tag{3.4}$$

where ϵ is the energy density, P is the pressure, u^μ is the local fluid velocity, and σ is the dissipative term. The first two terms comprise the familiar equations for ideal fluids. The dissipative term, which only has non-zero spatial components, is given by

$$\sigma_{ij} = \eta \left(\partial_i u_j + \partial_j u_i - \frac{2}{3} \delta_{ij} \partial_k u_k \right) + \zeta \delta_{ij} \partial_k u_k \,, \tag{3.5}$$

where η is the shear viscosity and ζ is the bulk viscosity.

In order to use our effective field theory formulation a perturbative source must be introduced. For an energy momentum tensor this is naturally the metric. So, the hydrodynamic equations must be generalized to curved space. We can then find the response of a tiny perturbation of the flat metric. We're primarily concerned with calculating dissipative properties so we will only consider $\sigma^{\mu\nu}$ in curved space,

$$\sigma^{\mu\nu} = P^{\mu\alpha} P^{\nu\beta} \left[\eta \left(\nabla_\alpha u_\beta + \nabla_\beta u_\alpha - \frac{2}{3} g_{\alpha\beta} \nabla_\rho u^\rho \right) + \zeta g_{\alpha\beta} \nabla_\rho u^\rho \right] \,, \tag{3.6}$$

where $P^{\mu\nu} = g^{\mu\nu} + u^\mu u^\nu$ is a projection operator.

We choose our metric to be of the form

$$g_{ij}(t, \mathbf{x}) = \delta_{ij} + h_{ij}(t) \,, \quad h_{ij}(t) \ll 1 \tag{3.7}$$

$$g_{00}(t, \mathbf{x}) = -1 \,, \quad g_{0i}(t, \mathbf{x}) = 0 \,. \tag{3.8}$$

The perturbation is assumed to be traceless $h_{ii}(t) = 0$ and because it is spatially homogeneous if the fluid moves the fluid is only allowed to move uniformly $u^i = u^i(t)$. But because of parity the fluid can't all go in one direction, so it must be stationary $u^\mu = (1,\ 0,\ 0,\ 0)$.

The metric is chosen such that the only non-zero components of $P^{\mu\nu}$ are spatial. Substituting this and $g_{\mu\nu}$ into the dissipative term of the curved space energy momentum tensor gives,

$$\sigma^{\mu\nu} = P^{\mu\alpha}P^{\nu\beta}\left[\eta\left(\partial_\alpha u_\beta - \Gamma^\gamma{}_{\alpha\beta}u_\gamma + \partial_\beta u_\alpha - \Gamma^\gamma{}_{\beta\alpha}u_\gamma\right)\right. \tag{3.9}$$

$$\left. + \left(\zeta - \frac{2}{3}\eta\right) g_{\alpha\beta}g^{\gamma\rho}(\partial_\gamma u_\rho - \Gamma^\sigma{}_{\gamma\rho}u_\sigma)\right] \tag{3.10}$$

$$= -P^{im}P^{jn}\left[\eta 2\Gamma^0{}_{mn} + \left(\zeta - \frac{2}{3}\eta\right) g_{mn}g^{pq}\Gamma^0{}_{pq}\right] . \tag{3.11}$$

Components of this give either the shear or bulk viscosity. Consider an off diagonal component of the dissipation tensor - we will find that this corresponds to a shear viscosity. The second term can be neglected as it is a higher order in h_{ij} than the first term leaving us with,

$$\sigma^{xy} \approx P^{xm}P^{yn}[\eta\partial_0 h_{mn}] \tag{3.12}$$

$$= -\eta\partial_0 h_{xy} - \eta h^{xm}h^{yn}\partial_0 h_{mn} \tag{3.13}$$

$$\approx -\eta\partial_0 h_{xy} , \tag{3.14}$$

where we have thrown out the second term due to $h_{ij} \ll 1$.

We can compare the Fourier transform of this result to the linear response (3.2), remembering that h_{ij} is our source, and notice that to lowest order in ω

$$G^{\mathrm{R}}(k) = -i\eta\omega . \tag{3.15}$$

Rearranging we see that we've derived the Kubo relation for the shear viscosity it terms of a Green's function,

$$\eta = -\lim_{\omega\to 0}\frac{i}{\omega}G^{\mathrm{R}}(k,\omega) . \tag{3.16}$$

4. Calculating thermal properties from gravity

Let's now turn to a specific metric. We are interested in calculating correlations functions in $\mathcal{N}=4$ SYM theory. In the language of branes, to make $\mathcal{N}=4$ SYM gauge theory we take an D3-brane, attach open strings to it and solve that string theory. The particles that come out are those found in $\mathcal{N}=4$ SYM. So the gravity theory dual to $\mathcal{N}=4$ SYM can be constructed by a large number, $N \sim 1/g_{str}$, of coincident D3-branes. Each brane has a mass $M \sim 1/g_{str}$, so N of them make a black hole with mass $M \sim 1/g_{str}^2$.

The black D3-brane metric is a solution to the type IIb equations of motion at low energy and has the form

$$ds^2 = H^{-1/2}[-h(r)dt^2 + d\mathbf{x}^2] + H^{1/2}[h(r)^{-1}dr^2 + r^2 d\Omega_5^2] , \tag{4.1}$$

where $H(r) = 1 + R^4/r^4$ and $h(r) = 1 - r_0^4/r^4$. For the AdS/CFT correspondence to work we need to take the near horizon limit of this metric, $r \ll R$, which technically amounts to "dropping the 1" in $H(r)$. Dropping the 1 yields,

$$ds^2 = \frac{r^2}{R^2}\left(-h(r)dt^2 + dx^2 + dy^2 + dz^2\right) + \frac{R^2}{r^2 h(r)}dr^2 + R^2 d\Omega_5^2\,. \tag{4.2}$$

As discussed earlier this kind of metric gives rise to actions for the dilaton field ϕ of the form,

$$S = -K\int d^4x dr\sqrt{-g}\frac{1}{2}g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \ldots\,, \tag{4.3}$$

where the coefficient $K = \pi^3 R^5/2\kappa_{10}^2$ comes from the normalization of the dilation field. The constant $\kappa_{10} = \sqrt{8\pi G}$ is the ten-dimensional gravitational constant. Using the parameter matching discussed in the introduction it can be rewritten as $\kappa_{10} = 2\pi^2\sqrt{\pi}R^4/N$. Substituting this in gives $K = N^2/8\pi^2 R^3$.

Correlation functions of these theories were the first to be calculated, though in the Euclidean formalism. To view these first timid steps I suggest looking at [5] and [6].

4.1 Entropy

Here we can get our first result from the AdS/CFT correspondence. The Hawking temperature is determined by the even nearer horizon behaviour which is found by taylor expanding around r_0 to find $r_0 r^2(1 - r_0^4/r^4) = 4r_0(r - r_0)$. Ignoring the spherical part of the metric we get,

$$ds^2 = -\frac{4r_0}{R^2}(r - r_0)dt^2 + \frac{R^2}{4r_0(r - r_0)}dr^2\,. \tag{4.4}$$

We can change coordinates $r = r_0 + \rho^2/r_0$ to make the metric non-singular,

$$ds^2 = \frac{R^2}{r_0}\left(d\rho - \frac{4r_0^2}{R^4}\rho^2 dt^2\right)\,. \tag{4.5}$$

Performing a Wick rotation $t \to i\tau$ and letting $\theta = 2r_0 R^{-2}\tau$ it's clear that we're dealing with a metric that is proportional to the flat metric expressed in polar coordinates,

$$ds^2 = \frac{R^2}{r_0}(d\rho + \rho^2 d\theta^2)\,. \tag{4.6}$$

We want to interpret the black hole horizon as a regular origin where locally we won't be able to detect any curvature such as we would if we had a conical singularity. To avoid a conical singularity we make it flat near the origin by identifying $\theta = 0$ with $\theta = 2\pi$ such that $\theta \sim \theta + 2\pi$. The Euclidean time τ in thermal field theory is also periodic, but in the inverse temperature $\tau \sim \tau + 1/T$. So, going around the origin once gives us the relation,

$$2\pi = \frac{2r_0}{R^2}\frac{1}{T} \Rightarrow T = \frac{r_0}{\pi R^2}\,, \tag{4.7}$$

which defines the Hawking temperature. It is the Hawking temperature on the gravity side that we associate with finite temperature in $\mathcal{N} = 4$ SU(N) SYM theory.

Calculating the entropy of the black hole using the Bekenstein-Hawking formula we can find the entropy in $\mathcal{N}=4$ SYM by converting the parameters according to the AdS/CFT prescription outlined in the introduction. The horizon lies at $t=$constant and $r=r_0$ or $u=1$. The area of this surface is,

$$A=\int d^3x d^5\Omega\sqrt{g}\,. \tag{4.8}$$

The determinant of the metric is $\sqrt{g}=r_0^3/R^3$ and the area of the five sphere is π^3R^5 and the $V_3=\int d^3x$ is an infinite spatial volume leaving,

$$A=\pi^3r_0^3R^2\,. \tag{4.9}$$

We can rewrite Newton's constant,

$$G=\frac{\pi^4R^8}{2N^2}\,, \tag{4.10}$$

so the entropy becomes

$$\begin{aligned} S=\frac{A}{4G} &= \frac{\pi^2V_3N^2}{2}\left(\frac{r_0}{\pi R^2}\right)^3 && (4.11)\\ &= \frac{\pi^2}{2}V_3N^2T^3\,, && (4.12)\end{aligned}$$

where we have used the formula for the Hawking temperature. This agrees up to a factor with the result calculated in the weak coupling regime. For those interested in further reading I suggest [7] and [8]. This is a static property though and we did not need to use the Minkowski formalism. We still haven't calculated and dynamic property.

4.2 Viscosity

To calculate a dynamic property we want to use the Minkowski formalism outlined earlier. To do this we still want to work in the near horizon limit, but now we want to calculate correlation functions. The first step is to the coordinate change $u=r_0^2/r^2$,

$$ds^2=\frac{(\pi TR)^2}{u}(-h(u)dt^2+dx^2+dy^2+dz^2)+\frac{R^2}{4u^2h(u)}du^2+R^2d\Omega_5^2\,, \tag{4.13}$$

where we have used the Hawking temperature to rewrite the metric in terms of thermal quantities. We see that the event horizon occurs at $u=1$ and spatial infinity, the boundary of the space, occurs at $u=0$. Based on the Minkowski prescription we want to find low frequency solutions to the equations of motion of this space. This will give us the what we need to calculate the Green's function on the CFT side. As found in an earlier section the equation of motion is

$$\partial_r(\sqrt{-g}\;g^{rr}\partial_r\phi)+\sqrt{-g}(g^{\mu\nu}\partial_\mu\partial_\nu)\phi=0\,. \tag{4.14}$$

The AdS/CFT correspondence says that the operators in the CFT side live at the boundary of the gravity theory. To find a solution we need to impose boundary conditions on the gravity side. We use separation of variables to write the solution of the equation of motion with the boundary condition $\phi(k, r=0) = \phi_0(k)$ as,

$$\phi(k,r) = f_k(r)\phi_0(k)\,, \tag{4.15}$$

where $f_k(r)$ is the called the mode function. Substituting the black three-brane metric into the equation of motion with these boundary conditions we get mode equation,

$$f_p'' - \frac{1+u^2}{uh(u)} f_k' + \frac{\mathfrak{w}^2}{uh(u)^2} f_k - \frac{\mathfrak{k}^2}{uh(u)} f_k = 0\,, \tag{4.16}$$

where prime is a derivative with respect to u and we have defined,

$$\mathfrak{w} = \frac{\omega}{2\pi T} \text{ and } \mathfrak{k} = \frac{k}{2\pi T}\,. \tag{4.17}$$

The mode equation is second order differential equation with a singular point at $u = 1$. When solving such equations the first thing we want to do is find the behavior of the singularity. If we substitute

$$f_k = (1-u^2)^\alpha F(u) \tag{4.18}$$

into the mode equation we find that has α has two possible values $\alpha = \pm i\mathfrak{w}/2$. This is unlike the Euclidean case were it only takes one value. We are left with a differential equation for $F(u)$ which is impossible to solve analytically. A power series representation for $F(u)$ in $\mathfrak{w}$ and $\mathfrak{k}^2$ can be found perturbatively,

$$F(u) = 1 + \frac{i\mathfrak{w}}{2}\ln\frac{2u^2}{1+u} + \mathfrak{k}^2 \ln\frac{1+u}{2u} + \ldots\,. \tag{4.19}$$

For our near horizon approximation we will disregard everything but the first term $F(u) \approx 1$.

We now have to make an interesting decision. Our solution for the wave function f_k has two possible values and we must choose which is correct. The two solutions can be written in form more conducive to our argument by doing the coordinate transformation,

$$r_* = \frac{\ln(1-u)}{4\pi T}\,. \tag{4.20}$$

Restoring the phase $e^{-i\omega t}$ yields

$$e^{-i\omega t} f_k = e^{-i\omega(t+r_*)}\,, \tag{4.21}$$

$$e^{-i\omega t} f_k^* = e^{i\omega(t-r_*)}\,, \tag{4.22}$$

which take the form of plane wave solutions. The horizon lies lies at $r_* = 0$ of the new coordinate system. The first solution corresponds to a wave moving toward the horizon, an incoming wave, and the second solution is a wave moving away from the horizon. The choice of which solution to discard is simply motivated by the fact that nothing should leave the horizon. This leaves us with one solution,

$$f_k(u) = (1-u^2)^{-i\mathfrak{w}/2}\,. \tag{4.23}$$

Changing variables back to $u = r_0^2/r^2$ we get

$$f_k(r) = \left(1 - \frac{r_0^4}{r^4}\right)^{-i\mathfrak{w}/2} = h(r)^{-i\mathfrak{w}/2} . \tag{4.24}$$

Using the incoming wave solution for $f_k(r)$, $K = N^2/8\pi^2R^3$, $\sqrt{-g} = r^3/R^3$ for the determinant of the metric, and $g^{rr} = r^2h(r)/R^2$ we find that

$$K\sqrt{-g}g^{rr} = \frac{N^2r^5h(r)}{8\pi^2R^8} , \tag{4.25}$$

and the kernel (2.10) of our action becomes,

$$F(r) = \frac{N^2r^5h(r)}{8\pi^2R^8} f_{-k}(r)\partial_r f_k(r) . \tag{4.26}$$

Recognising from earlier that $f_{-k} = f_k^*$ and substituting in the incoming wave solution we find that,

$$F(r) = -i\omega\frac{\pi N^2}{8T}\left(\frac{r_0}{\pi R^2}\right)^4 \tag{4.27}$$

$$= -i\omega\frac{\pi N^2T^3}{8} . \tag{4.28}$$

We now use the conjectured relationship (2.21) to calculate the Green's function,

$$G^{\mathrm{R}}(k) = -2F(k,u) \Bigg|_{r=\infty} \tag{4.29}$$

$$= i\omega\frac{\pi N^2T^3}{4} . \tag{4.30}$$

We use the Kubo formula (3.16) to get

$$\eta = \frac{\pi N^2T^3}{4} , \tag{4.31}$$

the shear viscosity of $\mathcal{N} = 4$ SYM, where N is the number of colours and T is the temperature. Note that it differs from the literature value $\eta = \frac{\pi N^2T^3}{8}$ by a factor of two. I've made an error and I can't find it. Whenever the shear viscosity is referred to assume we mean the accepted literature value.

5. The entropy viscosity ratio

We have calculated two thermal properties in $\mathcal{N} = 4$ SYM using the AdS/CFT correspondence - the entropy (4.11) and the viscosity (4.31). If we write equation (4.11) in terms of the entropy density $s = S/V$ then the entropy and viscosity have, up to a constant, identical forms. The ratio of the two is

$$\frac{\eta}{s} = \frac{1}{4\pi} = \frac{\hbar}{4\pi k_B} \text{ (with dimensionality restored)} . \tag{5.1}$$

This ratio has been shown to be true for all thermal field theories in a regime to be described by gravity duals. Corrections to this have been calculated in [9] for $\mathcal{N} = 4$ SYM and are shown to be positive. It is natural then to conjecture a bound

$$\frac{\eta}{s} \geq \frac{1}{4\pi}\,. \tag{5.2}$$

This implies that a fluid with a finite entropy density can never truly reach zero viscosity.

The lower bound is quite small compared to the most substances. For example water has $\eta/s \approx 380/4\pi$. One place to look where the bound should break down is in superfluids. Superfluids flow without dissipation, which implies zero viscosity. However, any superfluid describes by the Landau effective theory actually has a shear viscosity, which is the property being bounded in the conjecture. For superfluid helium the shear viscosity has been measured in a torsion-pendulum experiment and the ratio η/s remains at least 8.8 times larger than the minimum value of $\hbar/4\pi k_B \approx 6.08 \times 10^{-13}$ Ks for all ranges of temperatures and pressures. Numerical models indicate that the shear viscosity quark-gluon plasma at RHIC is very close to, but still above this bound. Further discussion on universality can be found in [4], [10] and [11].

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