Introduction to the Ginzburg-Landau Equations

James Charbonneau*
Department of Physics & Astronomy,
University of British Columbia,
Vancouver, B.C. Canada, V6T 1Z1
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I. INTRODUCTION

In 1950 Landau and Ginzburg proposed a theory which phenomenologically describes much of the behavior seen in superconductors. Not only does it encapsulate the work done by F. London and H. London in explaining the Meissner effect, but was used to postulate some very remarkable phenomena. The focus of this paper is Abrikosov's prediction of the vortex, a line defect in the superconductor which carries quantized magnetic flux. It is important to note that Ginzburg and Landau derived this theory phenomalogically, before the BCS theory of superconductivity was introduced, and that many years Gorkov showed that it comes from BCS naturally.

We will start with the Landau-Ginzburg free energy and a derivation of the equations of motion [1]. In part III the equations of motion will then be used to show that the theory contains the Meissner effect [2]. Part IV will discuss cylindrically symmetric solutions which lead to vortices and the quantization of magnetic flux [1] [3]. Also, the equations of motion will be solved to investigate the structure of the condensate in a vortex [4]. Using this, the energy of a single vortex [1] will be discussed in part V and the interaction energy between two vortices will be found [5] in part VI. This will give us insight into the stability of vortices in type I and type II superconductors. Finally, in part VII, the critical magnetic fields for type I [3] and type II [1] superconductors will be found.

II. THE GINZBURG-LANDAU ENERGY

The Ginzburg-Landau energy is based on the work of Gorter and Casimir who introduced the idea of an order parameter $|\psi|^2$ proportional to the density of superconducting electrons to describe the state of a superconductor. They postulated a free energy for a superconductor near critical temperature T_c.

$$E = -v\,|\psi|^2 + \frac{u}{2}\,|\psi|^4 \tag{1}$$

Landau noticed this idea could be expanded by considering a complex order field $\psi\left(\mathbf{x}\right)$ which could be used to describe fluctuations in the order parameter by adding a gradient to Gorter and Casimir's guess of the free energy. He and Ginzburg could then write the free energy of a superconductor near the critical temperature T_c. To investigate at a superconductors in magnetic fields, similar to F. London and H. London, they added the field energy and a guage invariant derivative to arrive at,

$$E = \int dx^2 \left\{ \frac{\hbar^2}{2m} \left| \left(\nabla - \frac{iq\mathbf{A}\left(\mathbf{x}\right)}{\hbar c} \right) \psi\left(\mathbf{x}\right) \right|^2 - v\,|\psi\left(\mathbf{x}\right)|^2 + \frac{u}{2}\,|\psi\left(\mathbf{x}\right)|^4 + \frac{1}{8\pi}\left(\nabla \times \mathbf{A}\left(\mathbf{x}\right)\right)^2 \right\} \tag{2}$$

This energy can be minimized to yield the Landau-Ginzburg equations. Minimizing with respect to the vector potential $\mathbf{A}$ gives us,

$$\frac{1}{4\pi}\left(\nabla^2\mathbf{A} - \nabla\left(\nabla\cdot\mathbf{A}\right)\right) = -\frac{\hbar q}{2m}\frac{1}{2i}\left(\psi^\dagger\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)\psi - \left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)\psi^\dagger\psi\right) \tag{3}$$

* james@phas.ubc.ca

The right hand side can be written in terms of a Noether current[6],

$$\mathbf{j} = \frac{1}{2i}\left(\psi^\dagger\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)\psi - \left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)\psi^\dagger\psi\right) \tag{4}$$

$$= \frac{1}{2i}\left(\psi^\dagger\nabla\psi - \psi\nabla\psi^\dagger\right) - \frac{q}{\hbar c}\mathbf{A}\left|\psi\right|^2 \tag{5}$$

This identification of the currect is critical and will later lead to the result that flux is quantized inside a vortex.

Minimization of the free energy with respect to the order field ψ yields,

$$\frac{\hbar^2}{2m}\left(\nabla - \frac{iq}{\hbar c}\mathbf{A}\right)^2\psi = u\left|\psi\right|^2\psi - v\psi \tag{6}$$

This will be used to determine the structure of the flux tube.

Note that the Ginzburg-Landau equations are invariant under the guage transformation

$$\mathbf{A}\left(\mathbf{x}\right) \to \mathbf{A}\left(\mathbf{x}\right) + \nabla\varphi\left(\mathbf{x}\right) \tag{7}$$

$$\psi\left(\mathbf{x}\right) \to e^{\frac{iq}{\hbar c}\varphi(\mathbf{x})}\psi\left(\mathbf{x}\right) \tag{8}$$

This transformation can be used to remove the phase of the order parameter.

Now that we have established the field equations we can begin to apply them. The first will be a demonstration of the Meissner effect and a derivation of the London pepetration depth.

III. THE MEISSNER EFFECT

The Meissner effect follows from equation 3 quite nicely. Consider it in cartesian coordinates for now, where a superconducting state exists for $x > 0$ and a normal state for $x < 0$. Using curl identities and taking a rewriting the current we can write equation 3 as,

$$\frac{1}{4\pi}\nabla\times\nabla\times\mathbf{A} = \frac{\hbar q}{mc}\left(\frac{1}{2i}\left(\psi^\dagger\nabla\psi - \psi\nabla\psi^\dagger\right) - \frac{q}{\hbar c}\mathbf{A}\left|\psi\right|^2\right) \tag{9}$$

We will use the polar decomposition of ψ as a ansatz.

$$\psi\left(x\right) = \sqrt{\frac{v}{u}}\rho\left(x\right)e^{i\varphi(x)} \tag{10}$$

For now $\rho(x) = [0, 1]$ where 0 indicates a normal state and 1 indicates a superconducting state. The states between are called mixed states. Substititing this ansatz into equation 9 yields,

$$\nabla\times\nabla\times\mathbf{A}(x) = \frac{\hbar q}{mc}\nabla\varphi\left(x\right)\frac{v}{u}\rho(x) - \frac{4\pi q^2}{mc^2}\mathbf{A}(x)\frac{v}{u}\rho(x) \tag{11}$$

Let's assume we are looking in a region of the superconductor without many disturbances. This is the same as setting $\rho(x) = 1$.

$$\nabla\times\nabla\times\mathbf{A}(x) = \frac{\hbar qv}{mcu}\nabla\varphi(x) - \frac{4\pi q^2 v}{mc^2 u}\mathbf{A}(x) \tag{12}$$

Taking the curl of both sides gives us the London equation,

$$\nabla\times\nabla\times\mathbf{B}(x) = -\frac{4\pi q^2 v}{mc^2 u}\mathbf{B}(x) \tag{13}$$

with a penetration depth $\lambda = \sqrt{\frac{mc^2}{4\pi q^2\frac{v}{u}}}$. The solution is $B(x) = e^{-\frac{x}{\lambda}}$ which indicates that the magnetic field penetrates λ past the surface of the superconductor. Comparing this to the London equation we notice that $\frac{v}{u}$ indeed matches up with the density of superconducting electrons. It is also instructive to note that $q = 2e$ and $m = 2m_e$. This is consistent with the picture that Cooper pairs are responsible for the condensate. Having made sure that the theory contains the fundamental results of the London equations we can now see what new phenomena the theory predicts.

IV. VORTEX LINES

The vortex line solution comes from solving the equations of motion in cylindrical coordinates. A vortex is a cylindrcally symetric line defect which exists in an otherwise undisturbed superconductor. It is similar to a the fluid vortices that are formed when water goes down a drain. In a superconductor the electrons rotate around a core where the density of superconducting electrons drops to zero. We'll first investigate how this structure leads to the quantization of magnetic flux.

A. Flux Quantization

An indication that somthing interesting is happening comes form our definition of the current $\mathbf{j}$ given by 4. If we solve equation 4 for the vector potential we get,

$$\mathbf{A} = -\frac{\hbar c}{q}\frac{\mathbf{j}}{|\psi|^2} + \frac{\hbar c}{q}\frac{1}{2i}\frac{1}{|\psi|^2}\left(\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger\right) \tag{14}$$

In cylindrical coordinates, equation 6 incidates that as $r \to 0$, $\rho \to 0$. If the superconductor is not in a voltage potential then current can only be produced by disturbances in the superconductor. Far away from $r = 0$, the superconductor is in an undisturbed state so $\mathbf{j} = 0$.

$$\mathbf{A} = \frac{\hbar c}{q}\frac{1}{2i}\frac{1}{|\psi|^2}\left(\psi^\dagger \nabla \psi - \psi \nabla \psi^\dagger\right) \tag{15}$$

Substituting the ansatz for ψ gives,

$$\mathbf{A} = \frac{\hbar c}{q}\nabla\varphi(\mathbf{x}) \tag{16}$$

Integrating on a closed contour around the vortex leads to a quantization condition,

$$\oint \mathbf{A}\cdot dl = \frac{\hbar c}{q}\oint \nabla\varphi(\mathbf{x})\cdot dl = \frac{hc}{q}2\pi n \tag{17}$$

where n is an integer. We can use Stokes theorem to see that the contour integral of $\mathbf{A}$ is also the flux through the surface.

$$\oint \mathbf{A}\cdot dl = \int \nabla\times\mathbf{A}\cdot dS = \int \mathbf{B}\cdot dS = \Phi \tag{18}$$

Equating the two expressions indicates that the flux is quantized with quantum number n.

$$\Phi = n\Phi_o \text{ with } \Phi_o = \frac{2\pi hc}{q} \tag{19}$$

The integer n is called the winding number and is an indication of the strength of the vortex. It can be shown that in certain situations a single vortex of winding $n = N$ will decay into N vorticies each with a winding number of $n = 1$ [7]. Also notice that the flux away from the vortex is independent of the radius of the loop we integrate around. These will be considerations in choosing the ansatz in the next section.

Right now it is unclear where the flux actually is. As it has been derived it looks like it penetrates the entire plane. To find where this might be localized we have to solve the other equation of motion governing the density of the superconductor.

B. The Structure of the Vortex

The problem with solving the field equations in cylindrical coordinates is that they are coupled, non-linear differential equations. We are looking for defects so we will no longer assume that $\psi(x)$ is constant. To make it easier to decouple and linearize these equations it is convenient to define,

$$\psi = \sqrt{\frac{v}{u}}\rho\left(r\right)e^{i\phi} \tag{20}$$

$$\mathbf{A} = \frac{\hbar q}{c}\frac{a\left(r\right)}{r}\ \hat{\phi} \tag{21}$$

where $\rho\left(r\right),\ a\left(r\right) \to 1$ as $r \to \infty$ and $\rho\left(r\right),\ a\left(r\right) \to 0$ as $r \to 0$ and r and ϕ the cylindrical coordinates. Following the discussion in part **A** the phase of $\psi(x)$ is chosen to mimic a vortex with winding number $n = 1$.

We can further define,

$$\rho\left(r\right) = 1 + \sigma\left(r\right) \tag{22}$$

$$a\left(r\right) = 1 + r\alpha\left(r\right) \tag{23}$$

such that $\sigma\left(r\right),\ \alpha\left(r\right) \to 0$ as $r \to \infty$. Let's start by subsituting 20 and 21 into equation 3.

$$\frac{\hbar c}{4\pi q}\nabla\times\left(\nabla\times\frac{a(r)}{r}\ \hat{\phi}\right) = \frac{\hbar\pi q^2}{mc}\left(\frac{v}{u}\frac{\sigma^2}{r} - \frac{v}{u}\sigma^2\frac{a(r)}{r}\right) \tag{24}$$

Writing the cross products in cylindrical coordinates and using 22 and 23 we get,

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \frac{1}{r^2} - \frac{\alpha}{r^2} = \frac{\hbar\pi v q^2}{mcu}\alpha(1+\sigma)^2 \tag{25}$$

We can linearize this equation by taking $r \to \infty$. We keep only the terms linear in α and σ and the lone $\frac{1}{r^2}$ goes to zero. This leaves us with a modified bessel equation of the first order.

$$\frac{\partial^2\alpha}{\partial r^2} + \frac{1}{r}\frac{\partial\alpha}{\partial r} - \left(\frac{1}{r^2} + \frac{1}{\lambda^2}\right)\alpha = 0 \tag{26}$$

Where λ is the London peretration depth we derived earlier. We want a solution that goes to zero as $r \to \infty$ so we choose the solution to be a modified bessel function of the second kind, $\alpha = \frac{1}{\lambda}K_1\left(\frac{r}{\lambda}\right)$. Going back through all the substitutions and then using a guage transformation we find that the vector potential is,

$$\mathbf{A} = \frac{\hbar c}{q\lambda}K_1\left(\frac{r}{\lambda}\right)\ \hat{\phi} \tag{27}$$

This is the Meissner effect in cylindrical coordinates. As we move from the core of the vortex into the superconducting material the magnetic field decays.

We will now look at the structure of $|\psi|^2$ as a function of the radius. All we know know is that it starts at zero in the core and goes to $\frac{u}{v}$ at infinity. We start by substituting 20 and 23 into 6.

$$\frac{\hbar^2}{2m}\sqrt{\frac{v}{u}}\left(\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\rho}{\partial r} - \frac{\rho}{r^2} - \frac{a^2\rho}{r} + \frac{2a\rho}{r}\right) = (-\rho+\rho^3)\frac{v^{\frac{3}{2}}}{u^{\frac{1}{2}}} \tag{28}$$

Substituting in 22 and 23 and linearizing yields,

$$\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial\sigma}{\partial r} = \frac{4mv}{\hbar^2}\sigma \tag{29}$$

This is a modified bessel equation of the zeroth order. The solution is,

$$\sigma(x) = K_0\left(\frac{\sqrt{2}}{\xi}r\right) \tag{30}$$

The quantity $\xi = \sqrt{\frac{\hbar^2}{2mv}}$ is called the coherence length. It gives a length scale for the change in density $|\psi|^2$ from the non-superconducting core at $r = 0$ to the undisturbed superconductor $r = \mathcal{O}(\xi)$.

We can start to understand what is happening in a vortex. It is a disturbance which has a non-superconducting core of radius ξ it carries quanta of magnetic flux. Because the superconductor displays the Meissner effect this flux can only be carried along the core of the vortex, where superconductivity is destroyed. Essentially the vortex is a tube of magnetic flux allowing a magnetic field to penetrate the superconductor.

V. THE ENERGY OF A VORTEX

Having solved the field equations for $\psi(x)$ and $\mathbf{A}(x)$ it is now possible to get a rough estimate of the energy of a vortex. If we take the field equation 6 and substitute it into equation 1 we get an expression for the free energy.

$$E = \int dx^2 \left\{\frac{v^2}{u}\rho^4 + \frac{(\nabla\times\mathbf{A})^2}{8\pi}\right\} \tag{31}$$

We know that zeroth order Bessel functions have the following property,

$$\frac{dK_0(\mu r)}{dr} = -\mu K_1(\mu r) \tag{32}$$

and, in cylindrical coordinates, is a Green's function for

$$(\nabla^2 - \mu^2)K_0(\mu r) = -2\pi\delta^2(r) \tag{33}$$

Using these on 27, when $r \neq 0$, we get,

$$\nabla\times\mathbf{A} = \frac{1}{r}\frac{\partial}{\partial r}r\mu K_1(\mu r) = -\frac{1}{r}\frac{\partial}{\partial r}r\frac{\partial}{\partial r}K_0(\mu r) = -\mu^2 K_0(\mu r) \tag{34}$$

It can be shown that at large r the leading order of ρ^4 is $\frac{\xi^2}{2r^2}$ [1]. Using all of this the energy becomes,

$$E = \int_0^\infty r dr d\phi \left\{\frac{v^2}{u}\frac{\xi^2}{2r^2} + \frac{\left(\frac{\hbar c}{q\lambda^2}K_0\left(\frac{r}{\lambda}\right)\right)^2}{8\pi}\right\} \tag{35}$$

The first term is easily integrated, but is unbounded at its limits. Instead of $r = \infty$ we use a cutoff $r = \Lambda$ which is the size of the container holding the superconductor. We remove the singularity at $r = 0$ by neglecting the core of the vortex $r < \xi$. The second term is evaluated by using $\int_0^\infty rK_0^2(r) = \frac{1}{2}$. The energy per unit length of the vortex becomes,

$$E = \frac{\pi\hbar^2}{m}\frac{v}{u}\log\left(\frac{\Lambda}{\xi}\right) + \frac{1}{8}\left(\frac{\hbar c}{q\lambda}\right)^2 \tag{36}$$

The fact that a cutoff is required indicates that a vortex can only exist in a container of finite size. Now that we have calculated the energy for a single vortex it will be interesting to look at two vorticies and the interaction energy between them.

VI. INTERACTION ENERGY BETWEEN TWO VORTICES

The philosophy behind calculating the intervortex force is to find the energy of the entire system and then subtract off the energy of the individual vorticies as originally outlined by Kramer [8]. The technique we will use was introduced by Speight [5]. The same philosophy is used but the actual calculation becomes much less cumbersome. We will reduce the theory to a non-interacting, linear one and model the vorticies as point sources. The interaction energy is then easily calculated from this linear theory. Start by linearizing the theory, only keeping terms that are quadratic in σ, where σ is defined as earlier.

$$E_{free} = \int dx^2 \left\{ \frac{v}{u}\frac{\hbar^2}{2m}(\nabla\sigma)^2 + \frac{1}{8\pi}\left((\nabla \times \mathbf{A})^2 + \frac{\mathbf{A}}{\lambda^2} \right) + 2\frac{v^2 2}{u}\sigma \right\} \tag{37}$$

And the source terms are,

$$E_{source} = \int dx^2 \left\{ \tau\sigma + \mathbf{j}\cdot\mathbf{A} \right\} \tag{38}$$

where τ and $\mathbf{j}$ are the sources for the fields σ and $\mathbf{A}$. Minimizing this we get the equations of motion,

$$\left(\nabla^2 - \frac{2}{\xi^2} \right)\sigma = \frac{u}{\hbar^2}\frac{u}{v}\tau \tag{39}$$

$$\left(\nabla^2 - \frac{1}{\lambda^2} \right)\mathbf{A} = 4\pi\mathbf{j} \tag{40}$$

We want to solve for the sources $\mathbf{j}$ and τ such that they have the same asymtotic solutions we obtained earlier in 27 and 30. Using 33 and the derivative of 33 we can solve for the sources.

$$\tau = -\frac{\hbar^2}{m}\frac{v}{u}2\pi\delta^2(\mathbf{x}) \tag{41}$$

$$\mathbf{j} = \frac{\hbar c}{2q}\frac{\partial\delta^2(\mathbf{x})}{\partial x}\ \hat{\phi} \tag{42}$$

The interaction energy is found by substituting $\mathbf{j} = \mathbf{j}_1 + \mathbf{j}_2$, $\mathbf{A} = \mathbf{A}_1 + \mathbf{A}_2$, $\tau = \tau_1 + \tau_2$ and $\sigma = \sigma_1 + \sigma_2$ into the total energy $E = E_{free} + E_{source}$ and subtracting of the energies of the vortices, leaving only cross terms. The subscripts 1 and 2 refer to two seperate vorticies and positions $\mathbf{x}_1$ and $\mathbf{x}_2$ respectivly. The cross terms left over are interpreted as the interaction energy.

$$E_{interaction} = \int dx^2 \left\{ \tau_1\sigma_2 + \mathbf{j}_1\cdot\mathbf{A}_2 \right\} \tag{43}$$

Using 21, 22, 41 and 42 the interaction energy can be written,

$$\begin{aligned} E_{interaction} &= \int dx^2 \left\{ -\frac{\hbar^2}{m}\frac{v}{u}2\pi\delta^2(\mathbf{x}-\mathbf{x}_1)K_0\left(\frac{\sqrt{2}}{\xi}(\mathbf{x}-\mathbf{x}_2)\right) - \frac{\hbar c}{2q}\frac{\partial\delta^2(\mathbf{x}-\mathbf{x}_1)}{\partial x}\frac{\hbar c}{q\lambda}K_1\left(\frac{\mathbf{x}-\mathbf{x}_2}{\lambda}\right) \right\} &(44)\\ &= \left(\frac{\hbar c}{q\lambda}\right)^2 \frac{1}{2}K_0\left(\frac{d}{\lambda}\right) - \frac{2\pi\hbar^2 v}{mu}K_0\left(\frac{\sqrt{2}d}{\xi}\right) &(45)\\ &= \frac{2\pi\hbar^2 v}{mu}\left(K_0\left(\frac{d}{\lambda}\right) - K_0\left(\frac{\sqrt{2}d}{\xi}\right) \right) &(46) \end{aligned}$$

where $d = \mathbf{x_1} - \mathbf{x_2}$.

There are two terms working to oppose each other. The first term is a repulsive force similar to the force between two wires with currents in opposite directions. The current in this case is caused by the electrons rotating around the vortex. Twovortices placed side by side will have currents running in the opposite direction and be repelled. We can see from equation 42 that the current is in the $\hat{\phi}$ direction around the vortex. The second term is an attractive force caused by the superconductor prefering to be in a state with no defects and attemping to restore order by making only one vortex. When the first term is larger $E_{interaction}$ is positive and the vorticies repel. When the second term is larger the vortices attract. What governs this is the relative size of λ and $\frac{\sqrt{2}}{\xi}$. For vortices to repel,

$$\frac{d}{\lambda} < \frac{\sqrt{2}d}{\xi} \tag{47}$$

rearranging yields,

$$\kappa > \frac{1}{\sqrt{2}} \quad \text{where} \quad \kappa = \frac{\lambda}{\xi} \tag{48}$$

The dimensionless quantity κ is the famous Ginzburg-Landau parameter used to determine whether a superconductor is type I or type II. A type II superconductor is one which allows partial penetration of a magnetic field. A type I superconductor is one which fully displays the Meissner effect. If $\kappa > \frac{1}{\sqrt{2}}$ then vortices repel from each other and they will form a triangular lattice [9] [10], each vortex carrying a quanta of flux Φ_0. This accounts for the partial penetration of the magnetic field exhibited by type II superconductors. If $\kappa < \frac{1}{\sqrt{2}}$ then all the vortices attract each other and collapse. The superconductor now has no mechanism to carry flux and exhibits the Meissner effect, behaving like a type I superconductor.

VII. CRITICAL MAGNETIC FIELDS

Type I and type II supercondunductors have another distinguishing feature, the magnetic fields at which the Meissner effect is destroyed. A type I superconductor will display the Meissner effect until a critical external field B_{c_I} destroys the superconducting state.

A type II superconductor will display the Meissner effect until a critical field $B_{c_{II}}$ when vortices start to form and allow part of the field to penetrate it. Increasing the magnetic field strength further will create more and more vorticies until there are so many superconductivity is destroyed.

Let's first consider a type I superconductor. The density ψ is uniform and there is no magnetic field inside so equation 1 quickly becomes,

$$E_{condensate} = \frac{Vu}{2}|\psi|^4 - Vv|\psi|^2 \tag{49}$$

where V is the volume of the superconductor. This has a minima at $|\psi|^2 = \frac{v}{u}$. Applying an external magnetic field changes the energy by $-\frac{B^2}{8\pi}$. If we set $E_{condensate} = 0$ the condenstate has been destroyed and we get a critical magnetic field.

$$B_{c_I} = \sqrt{\frac{4\pi v^2}{u}} \tag{50}$$

Now consider a type II superconductor. There are both energy gradients and magnetic fields inside the superconductor. We use the energy of a vortex we calculated earlier and this time the magnetic field inside the vortex B_{int} couples with the external filed B_{ext} through the interaction term.

$$E_{vortex} = \frac{\pi\hbar^2}{m}\frac{v}{u}\log\left(\frac{\Lambda}{\xi}\right) + \frac{1}{8}\left(\frac{\hbar c}{q\lambda}\right)^2 - \int dx^2 \frac{\mathbf{B}_{int}\cdot\mathbf{B}_{ext}}{8\pi} \tag{51}$$

The last term can be simplified as this integral is the flux quantization, $\int dx^2\mathbf{B}_{int} = 2\pi\frac{\hbar c}{q}$. The energy $E = 0$ is when a vortex will first form inside the superconductor.

$$B_{c_{II}} = \frac{4\pi\hbar qv}{mcu}\left(\frac{1}{4} + \frac{1}{\pi}\log(\frac{\Lambda}{\xi})\right) \tag{52}$$

Comaparing the two critical fields we see that $B_{c_{II}}$ is much smaller than B_{c_I}. This is expected because a type II superconductor only has to let one quantum of flux through at $B_{c_{II}}$, where B_{c_I} has the energy to destroy the entire state.

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