4D-XY Quantum Criticality in Underdoped High-T_c cuprates

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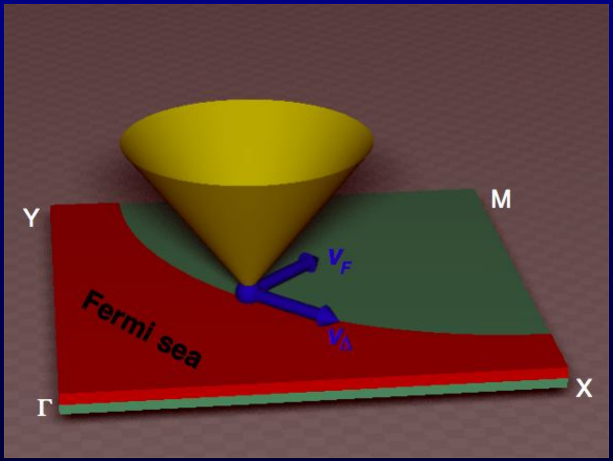


In collaboration with: A.P. Iyengar (theory)

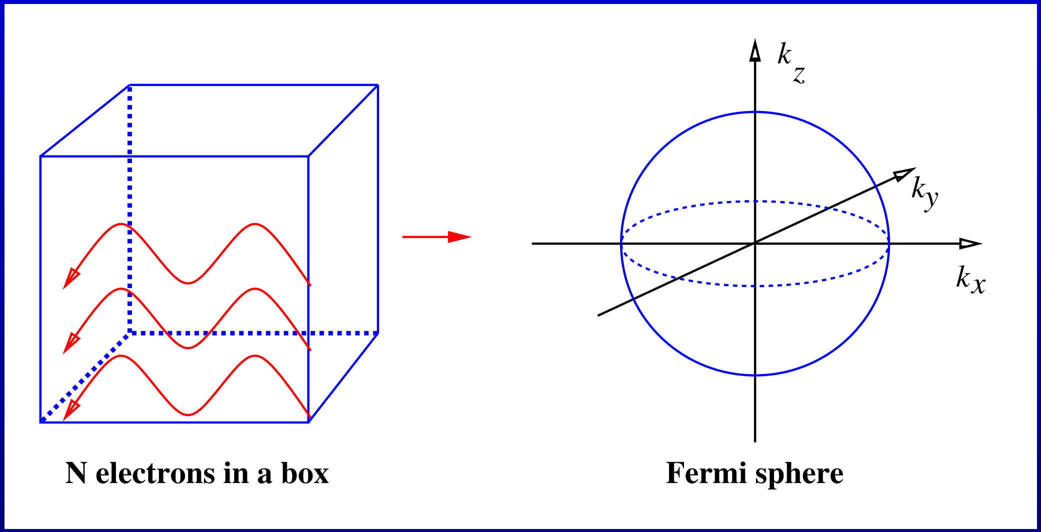
D.P. Broun, D.A. Bonn (experiment)

Landau's Fermi liquid paradigm

Lev Davidovich Landau:

"Electron states in solids are adiabatically connected to the states of noninteracting electron gas"

Despite enormous Coulomb forces ($U_C \sim \frac{e^2}{a_0} \sim 1 - 10$eV) at low energies most metals behave like a free electron gas [Landau, 1957]



$$\epsilon(\mathbf{k}) = \frac{\hbar^2 \mathbf{k}^2}{2m^*}, \quad k_F = (3\pi^2 n)^{1/3}$$

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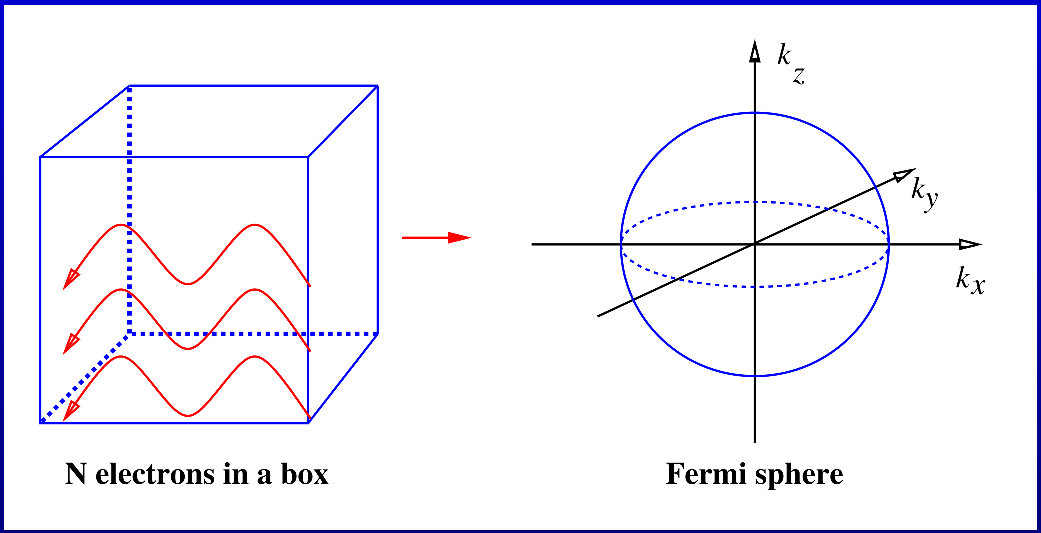

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Root cause: Pauli exclusion principle

$\longrightarrow$ phase space for scattering near FS is severely limited.

Structure of electron propagator in FL

$$G(\mathbf{k},\omega) = \frac{1}{\omega - \epsilon_{\mathbf{k}} - \Sigma(\mathbf{k},\omega)} = \frac{z_{\mathbf{k}}}{\omega - E_{\mathbf{k}} + i\Gamma_{\mathbf{k}}} + G_{\text{incoh}}(\mathbf{k},\omega)$$

with

$z_{\mathbf{k}}^{-1} = [1 - \frac{\partial \text{Re}\Sigma}{\partial \omega}]_{\omega = E_{\mathbf{k}}}$, "quasiparticle weight"

$\tau^{-1} \equiv \Gamma_{\mathbf{k}} \sim (E_{\mathbf{k}} - E_F)^2$, "quasiparticle lifetime"

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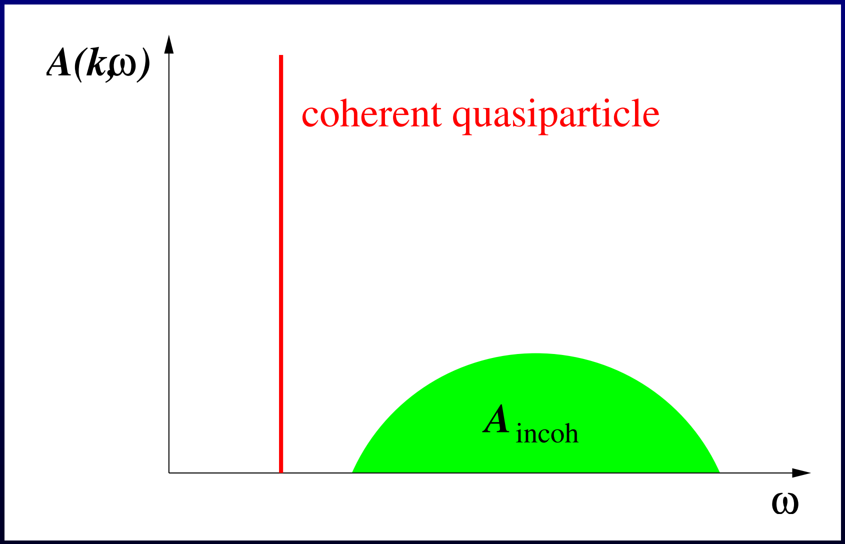
Spectral function:

$$\begin{aligned} A(\mathbf{k},\omega) &= -2\mathrm{Im}G(\mathbf{k},\omega) \\ &\simeq z_{\mathbf{k}}\delta(\omega - E_{\mathbf{k}}) + A_{\rm incoh}(\mathbf{k},\omega) \end{aligned}$$



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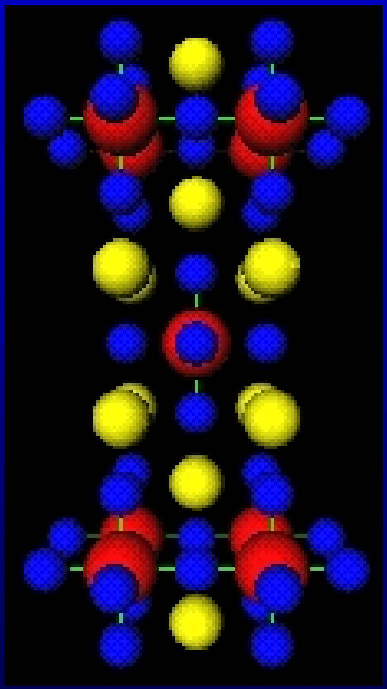
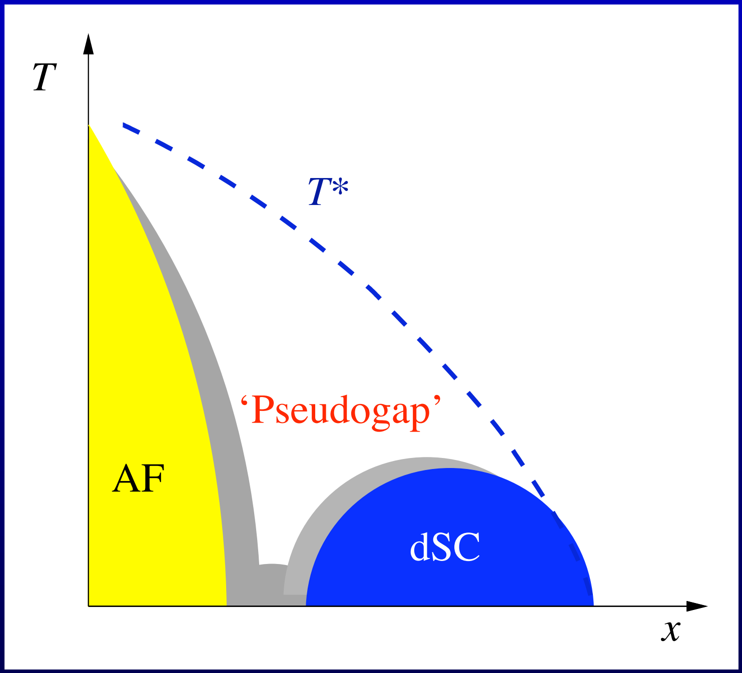
- High-T_c Cuprate Superconductors (?)

Cuprate superconductors

Cuprates are layered quasi-2D materials with electronic properties dominated by the CuO_2 layers.



Crystal structure of $La_{2-x}Sr_xCuO_4$



Phase diagram of cuprates.

d-wave superconductivity in cuprates

Superconducting order parameter in cuprates exhibits d-wave symmetry

$$\Delta_{\mathbf{k}} = \frac{1}{2}\Delta_0(\cos k_x - \cos k_y),$$

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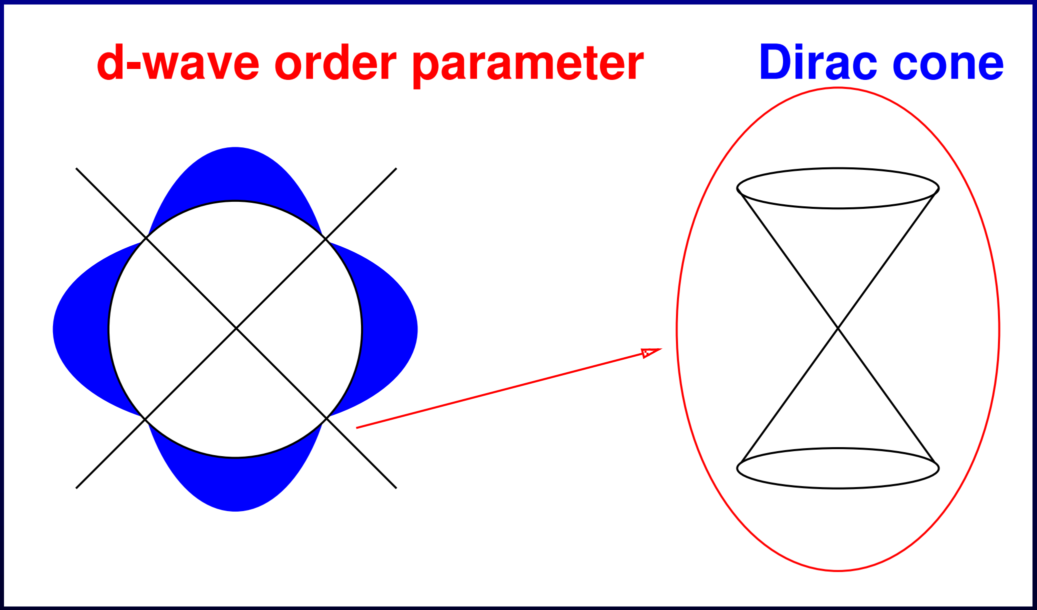

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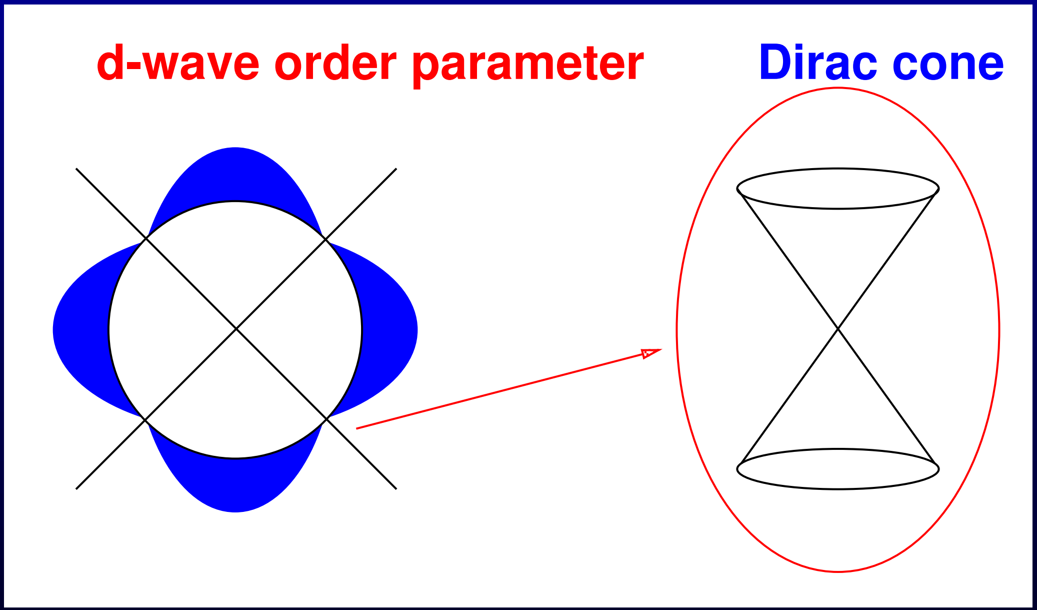

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Superfluid density

Superfluid density ρ_s is a fundamental characteristic of a superconductor reflecting its ability to carry supercurrent in response to applied magnetic field $\mathbf{H} = \nabla \times \mathbf{A}$:

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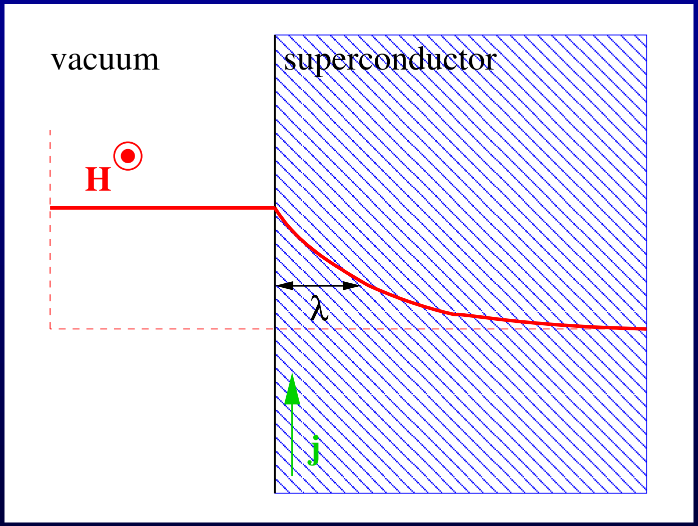
ρ_s is related to London penetration depth λ, a fundamental lengthscale in a superconductor describing penetration of magnetic field $\mathbf{H}$ into the bulk via

$$\rho_s = \frac{\hbar^2 c^2}{16\pi e^2 \lambda^2}$$

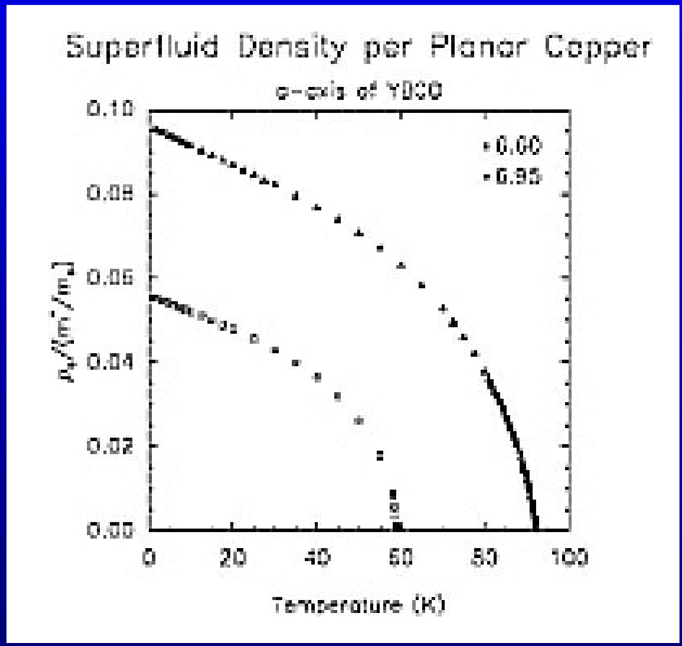


Superfluid density in cuprates, ab-plane: the old story

In the optimally doped and moderately underdoped region experiments show

$$\rho_s^{ab}(x,T) \sim \lambda_{ab}^{-2}(x,T) \simeq ax - bk_B T,$$

with $a \simeq 244$meV and $b \simeq 3.0$ [Lee and Wen, PRL **78**, 4111 (1997)].



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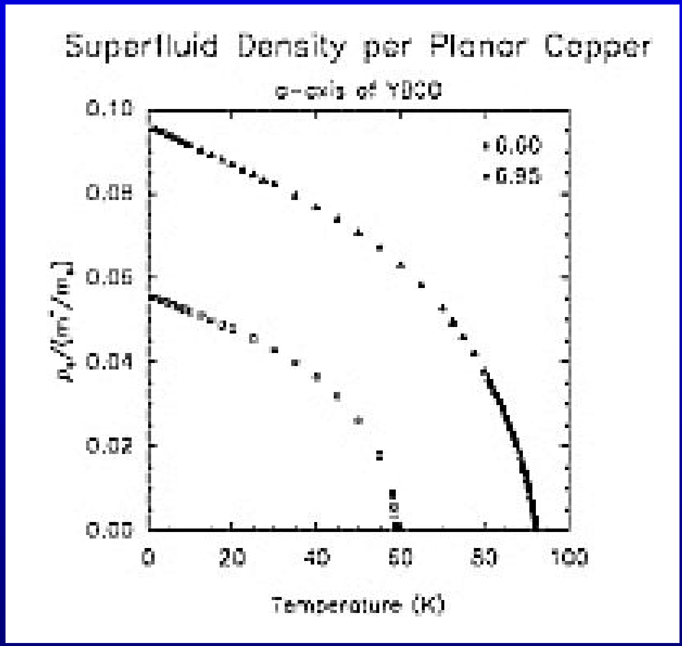
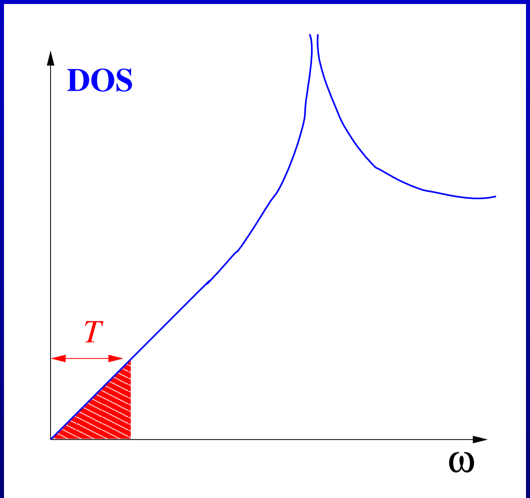


In a BCS d-wave superconductor one would have

$$\rho_s^{ab}(x,T) \simeq a(1-x) - 2\ln 2\frac{v_F}{v_\Delta} k_B T.$$

- The linear T-dependence is known to arise from thermally excited nodal quasiparticles which exhibit *linear* density of states at low energies.



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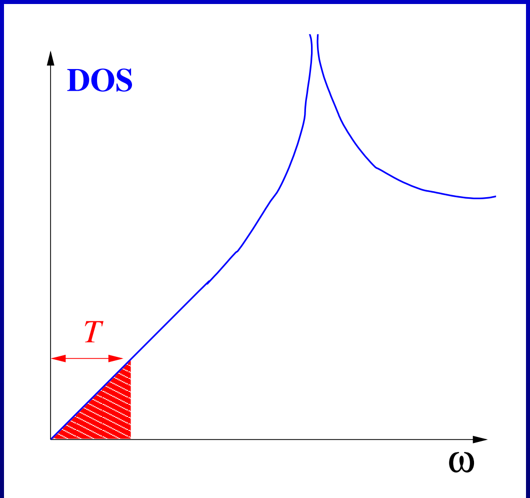

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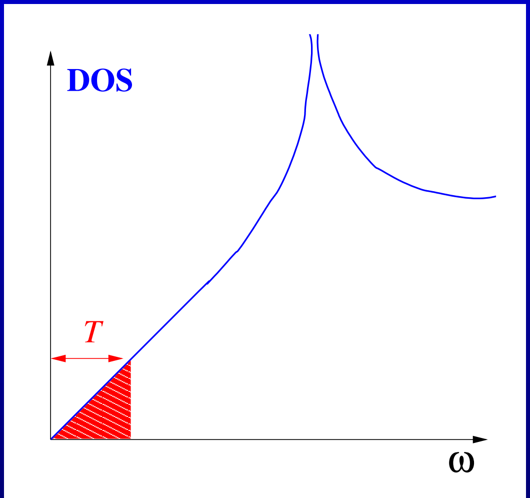

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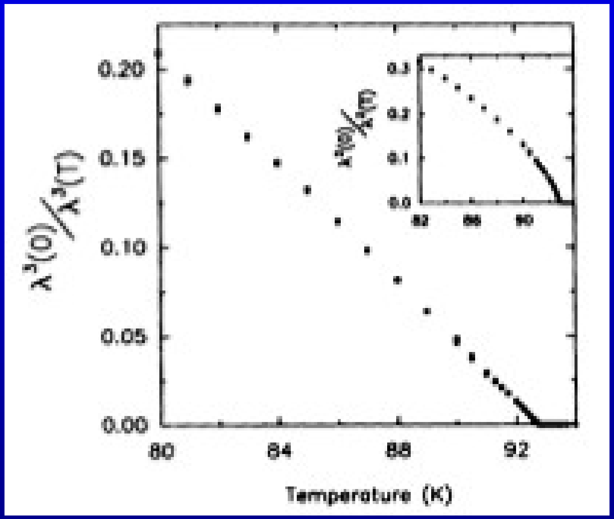
- Problem: models that give correct x-dependence (e.g. RVB-type theories) generally yield strong ($\sim x^2$) dependence of the coefficient b.

3D-XY critical scaling

Optimally doped YBCO shows 3D-XY critical behavior near T_c: $\rho_s(T) \sim (T_c - T)^{2/3}$ with relatively wide critical region $\Delta T \simeq 10$K.

Such critical behavior is characteristic of phase disordering transition to the normal state. In 3d this is known to be caused by vortex loop unbinding.



Kamal et al., PRL **73** 1845 (1994)

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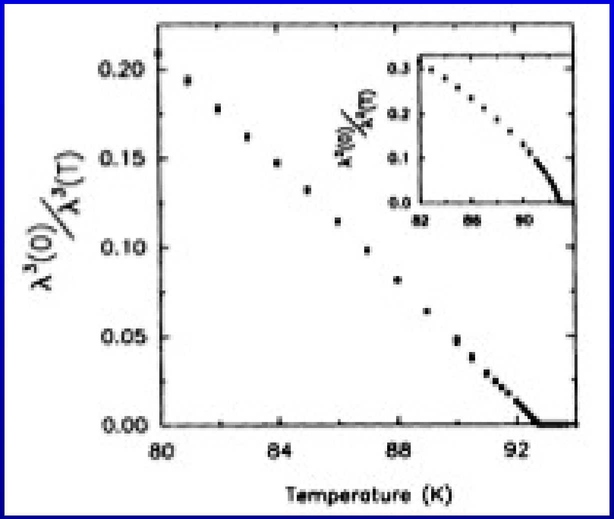

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3D-XY critical behavior has also been observed in other high-T_c compounds using transport and thermodynamic probes. It thus appears to be a fundamental universal feature of cuprates.

Superfluid density in cuprates: the new story

Recent UBC Group data on ultra-pure YBCO single crystals for doping levels as low as $T_c = 5$K show ab-plane show tantalizing *qualitative deviations* from the “old” phenomenology.

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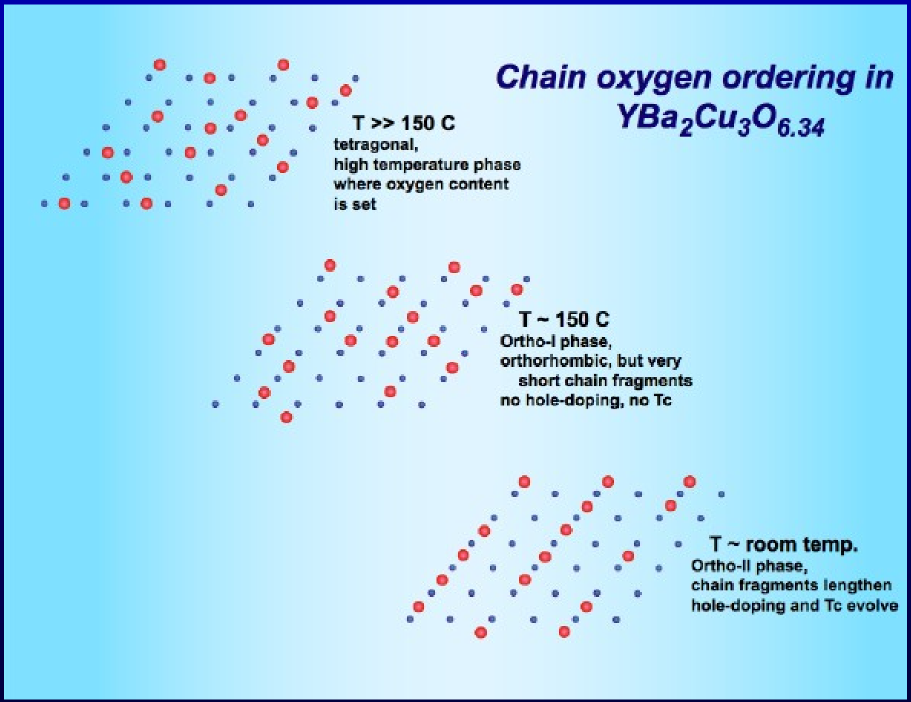

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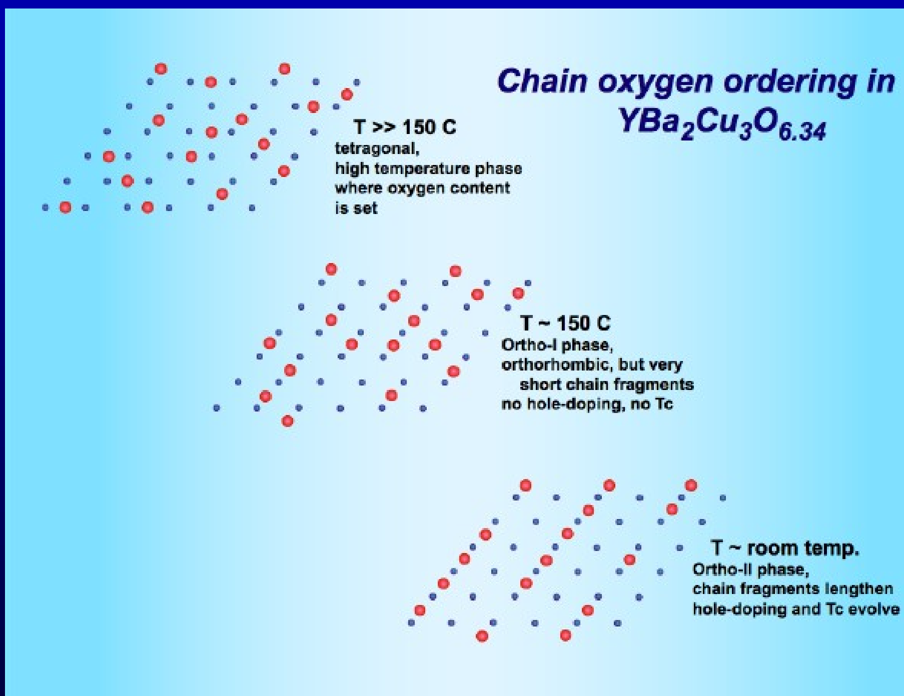
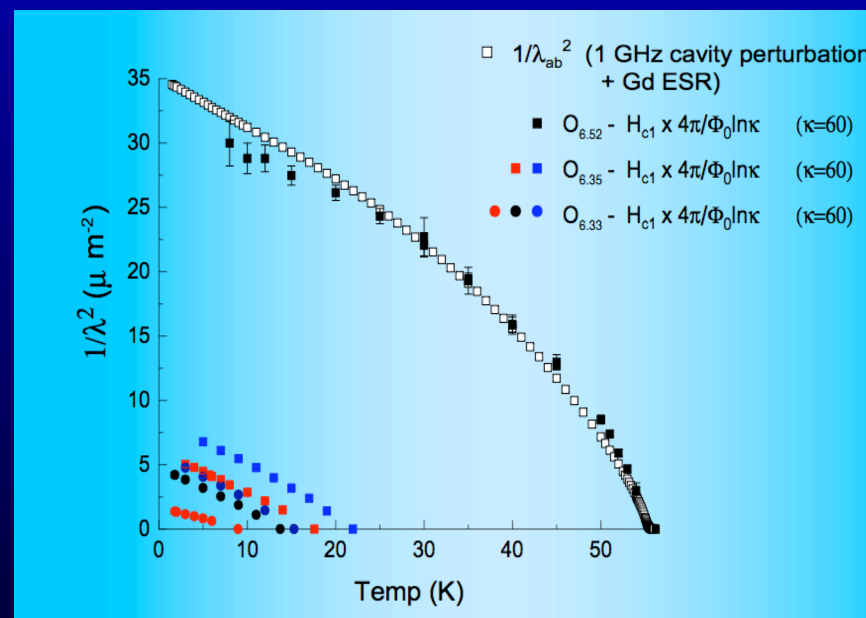
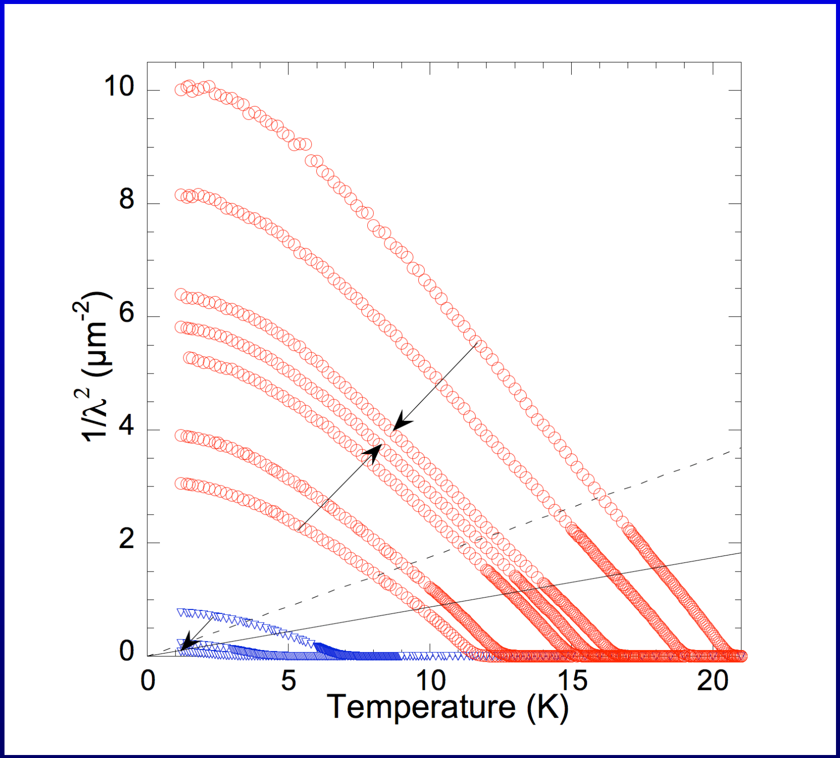




Data from UBC/SFU group [Broun *et al.* unpublished]

New ab-plane phenomenology



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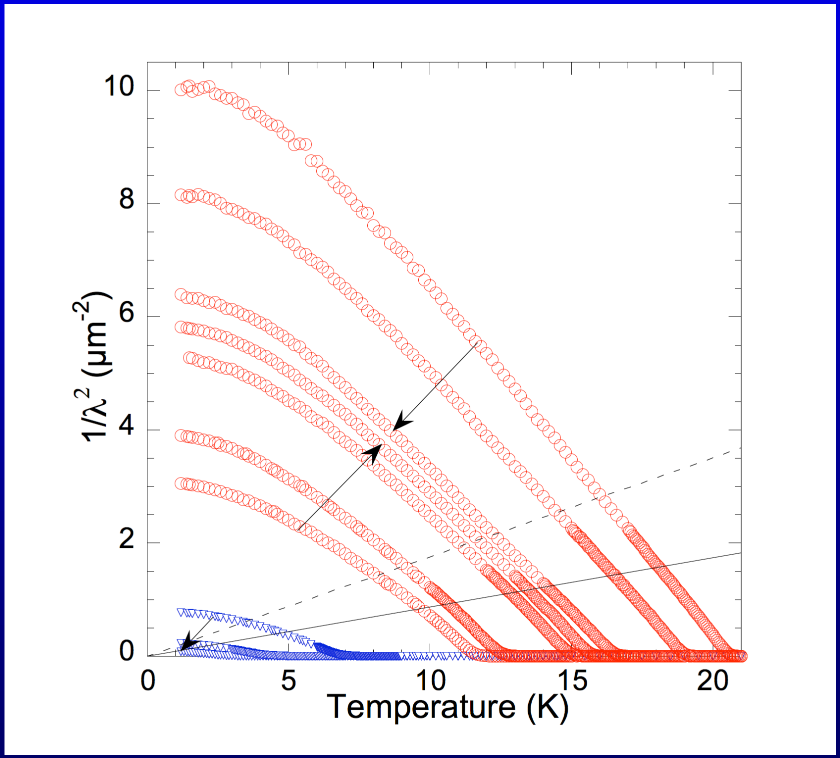

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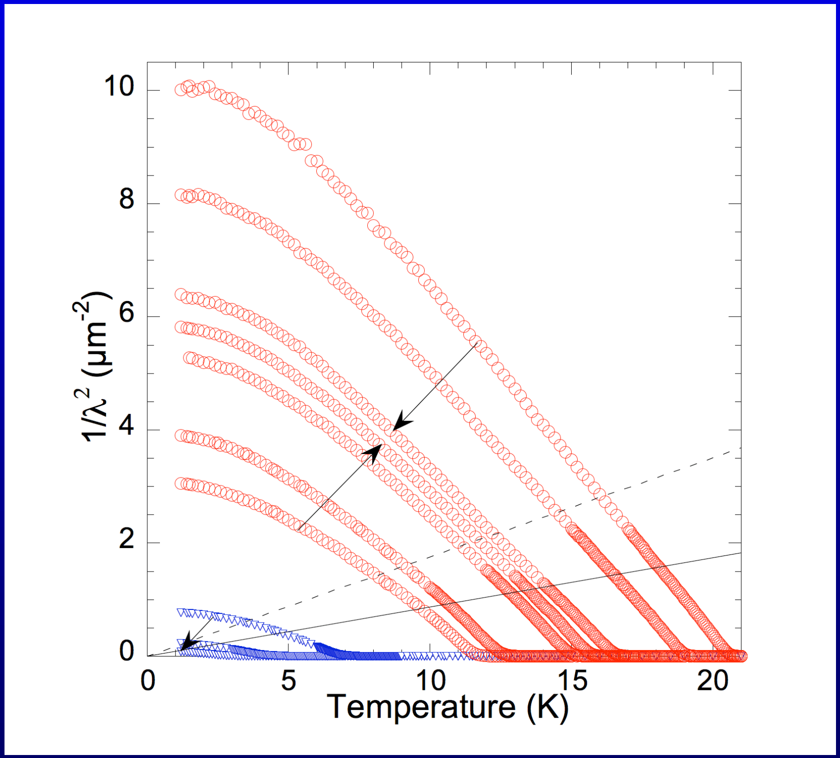

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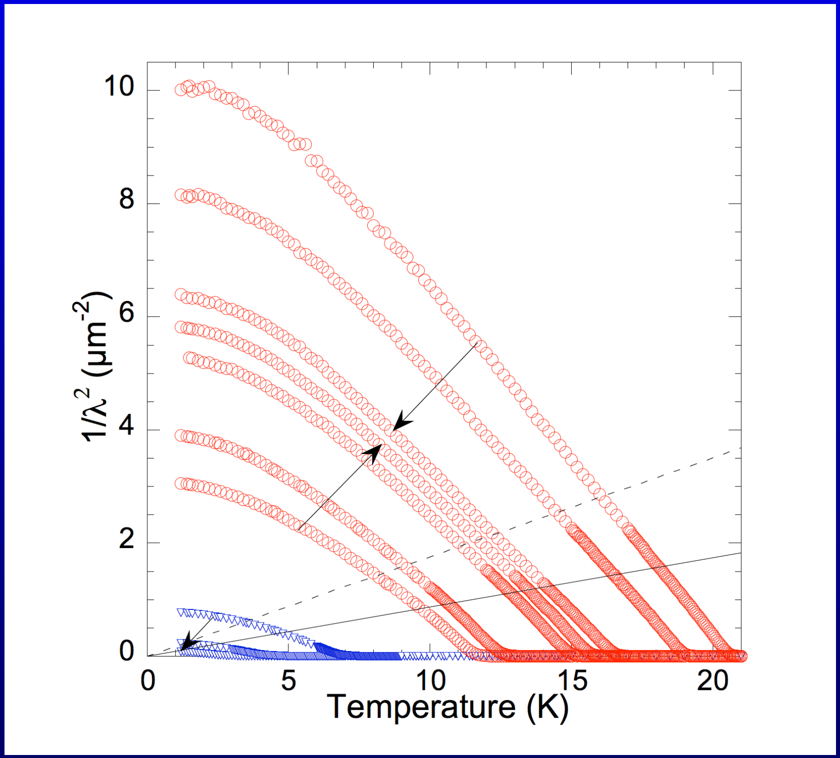

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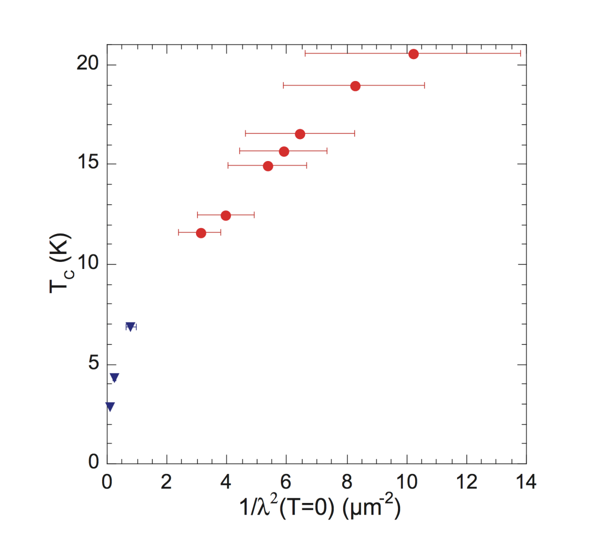

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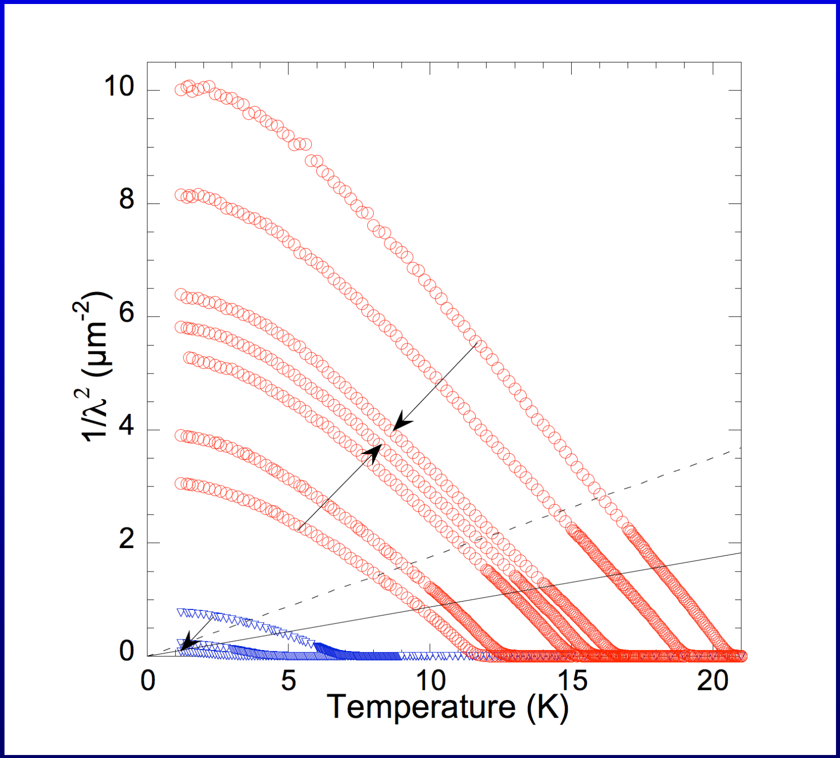

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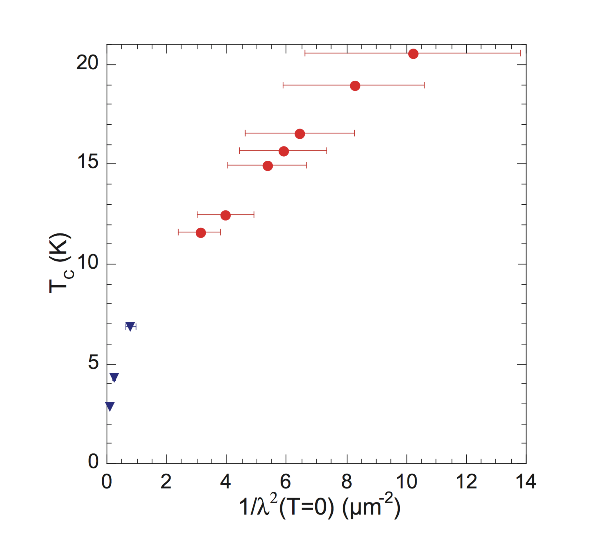
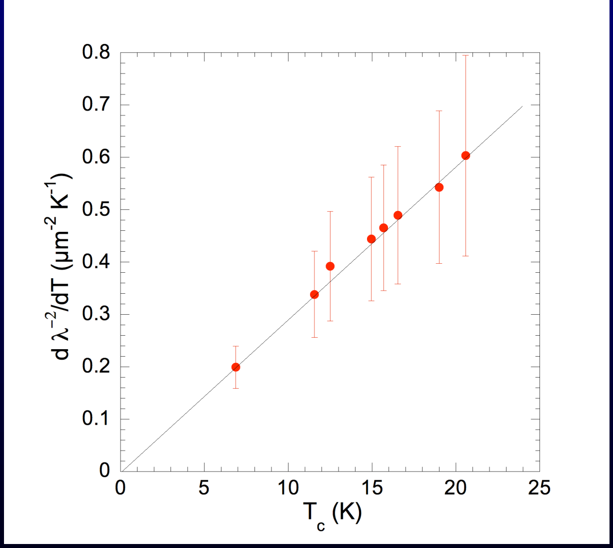


slope



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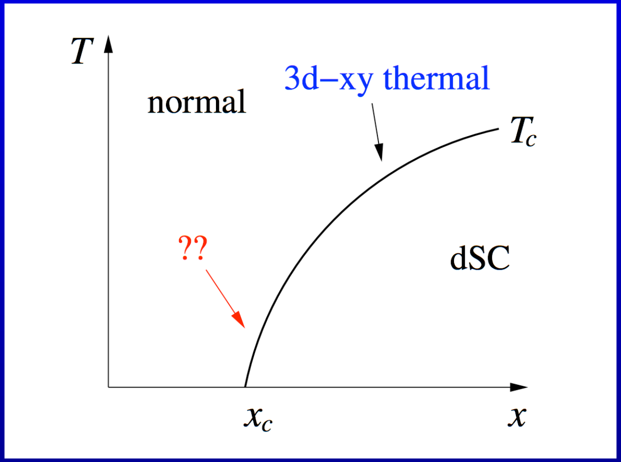

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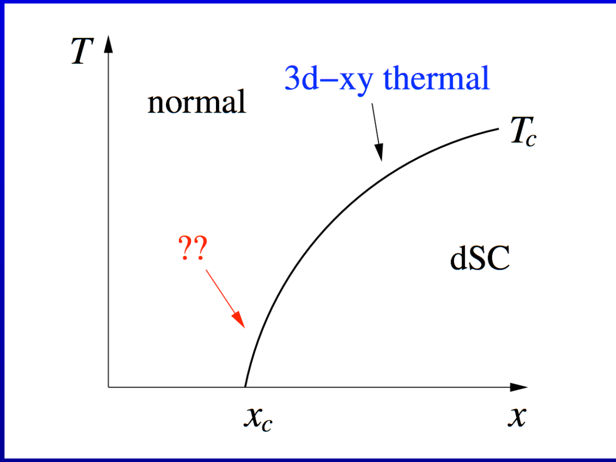

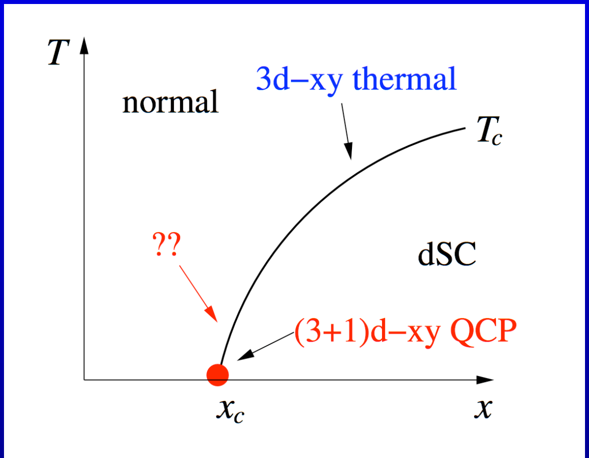

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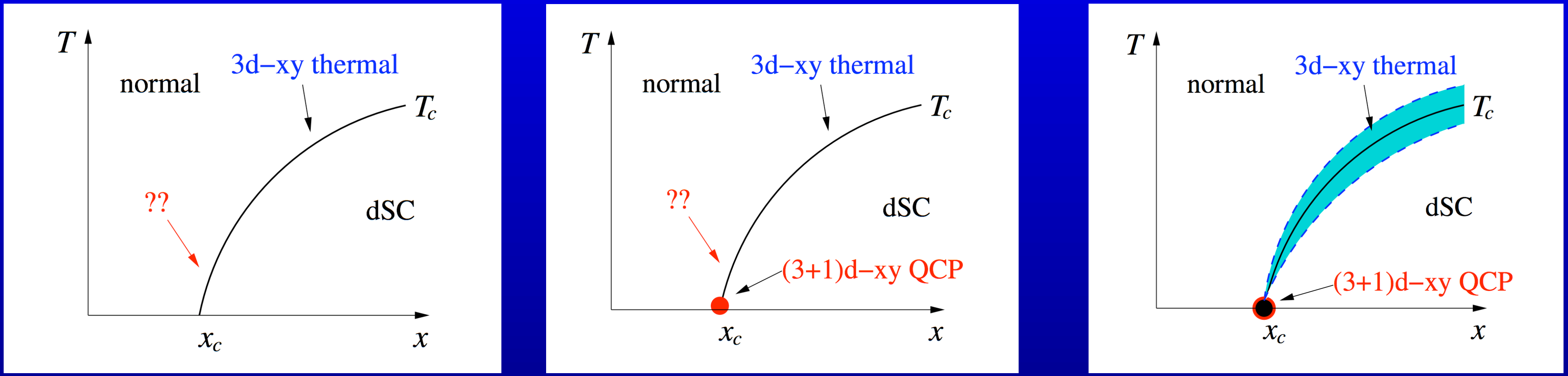

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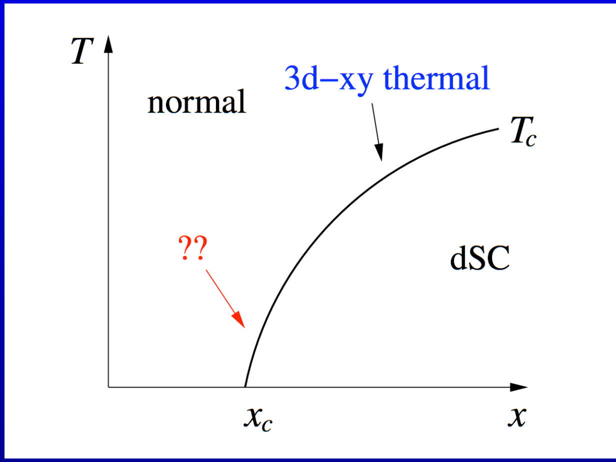

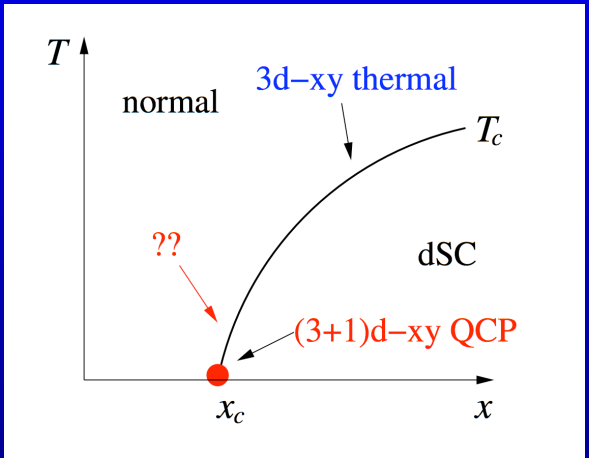

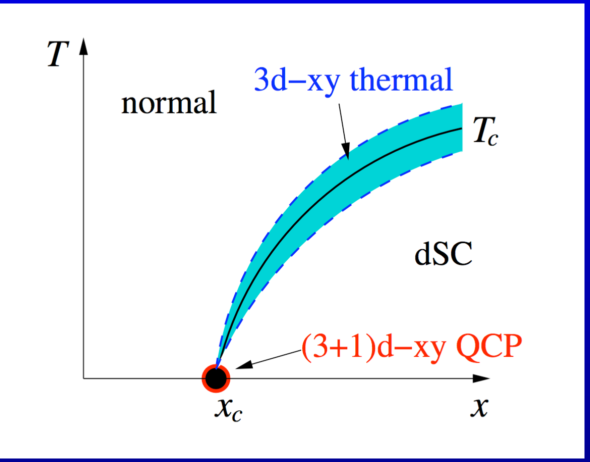

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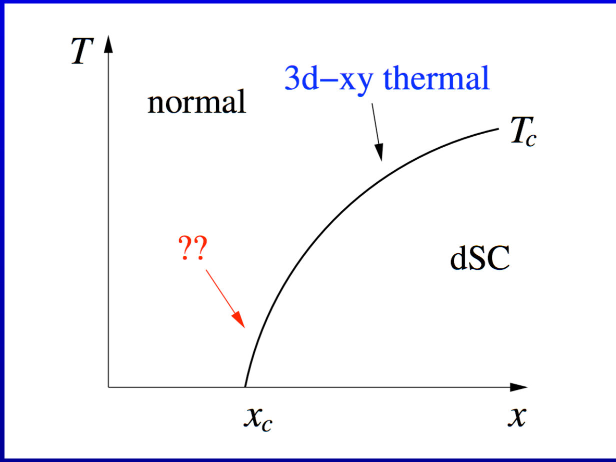

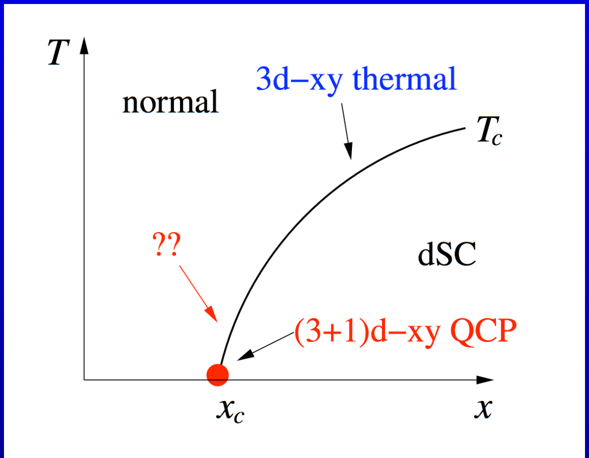

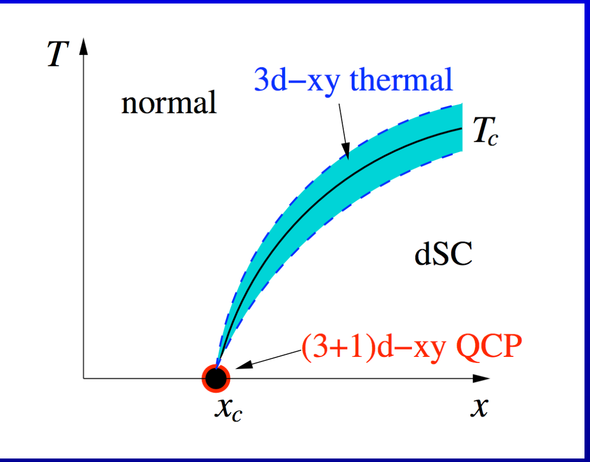

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- For xy-type models 4 is the upper critical dimension; one thus expects mean field critical behavior.

Consistency check

If the underdoped region is controlled by (3+1)D-XY quantum critical point then we should be able to understand the behavior of $\rho_s^{ab}(x,T)$ based on very general scaling arguments.

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For $1 \leq z \leq 2$ this is consistent with experiment!

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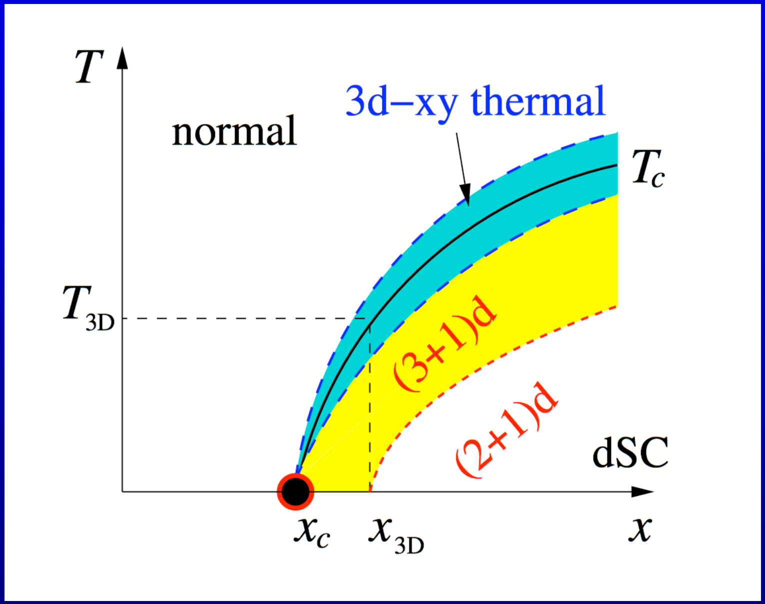
- Cuprates are strongly anisotropic; it is unclear how broad the (3+1)D critical region is.
- There is strong (linear) T-dependence of the “bare” superfluid density coming from quasiparticles which may invalidate the scaling laws.

2D - 3D crossover



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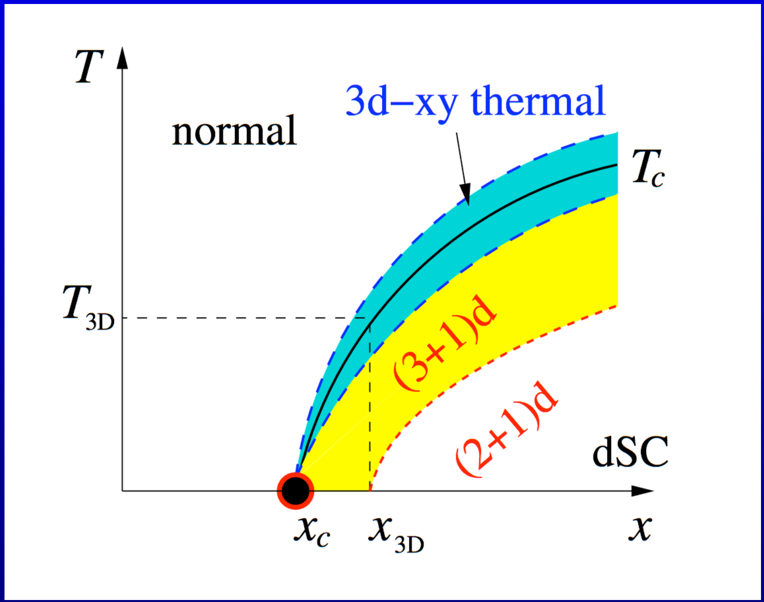

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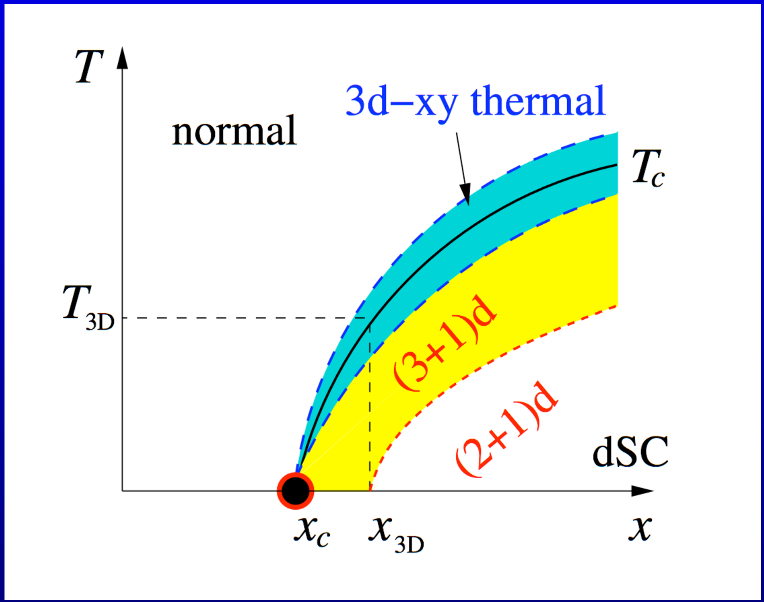
Close to QCP the c-axis coherence length grows, $\xi_c \sim (x - x_c)^{-\nu}$, and ultimately exceeds the interlayer spacing d_0. At that point system starts behaving 3 dimensionally.

2D - 3D crossover



At $T = 0$, away from QCP, fluctuations in CuO_2 layers are decoupled and system behaves 2 dimensionally.

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For YBCO we estimate, using $\xi_c \approx \lambda_{ab}^2/\lambda_c\kappa$,

$$T_{3D} \approx 5 - 10\text{K}.$$

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$$[\hat{n}_i, \hat{\varphi}_j] = i\delta_{ij}.$$

The sites $\mathbf{r}_i$ do not necessarily represent individual Cu atoms; rather one should think in terms of “coarse grained” lattice model valid at long lengthscales where microscopic details no longer matter.

- The first term in H_{XY} describes interactions between Cooper pairs; we take

$$V_{ij} = U\delta_{ij} + (1 - \delta_{ij})\frac{e^2}{|\mathbf{r}_i - \mathbf{r}_j|}.$$

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- In the absence of interactions J clearly must be identified as the superfluid density. We thus take

$$J = J_0 - \alpha T$$

with $\alpha = (2\ln 2)v_F/v_\Delta$, as in the BCS d-wave superconductor.

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It also naturally yields the shrinking classical fluctuation region with decreasing T_c.

Self-consistent harmonic approximation

The idea is to replace the XY Hamiltonian

$$H_{\mathrm{XY}} = \frac{1}{2}\sum_{ij}\hat{n}_i V_{ij}\hat{n}_j - J\sum_{\langle ij\rangle}\cos(\hat{\varphi}_i - \hat{\varphi}_j).$$

by the “trial” harmonic Hamiltonian

$$H_{\mathrm{har}} = \frac{1}{2}\sum_{ij}\hat{n}_i V_{ij}\hat{n}_j + \frac{1}{2}K\sum_{\langle ij\rangle}(\hat{\varphi}_i - \hat{\varphi}_j)^2.$$

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This is just a variational principle which can be extended to $T > 0$ case using the Gibbs-Bogolyubov inequality $F \leq F_{\mathrm{har}} + \langle H - H_{\mathrm{har}} \rangle_{\mathrm{har}}$.

$H_{\rm har}$ is quadratic in $\hat{n}_i$ and $\hat{\varphi}_j$ and can thus be easily diagonalized:

$$H_{\rm har} = \sum_{\mathbf{q}} \hbar\omega_{\mathbf{q}}(a^\dagger_{\mathbf{q}} a_{\mathbf{q}} + \frac{1}{2})$$

with the frequencies

$$\hbar\omega_{\mathbf{q}} = 2\sqrt{K Z_{\mathbf{q}} V_q}, \quad Z_{\mathbf{q}} = \sin^2(q_x/2) + \sin^2(q_y/2) + \sin^2(q_z/2).$$

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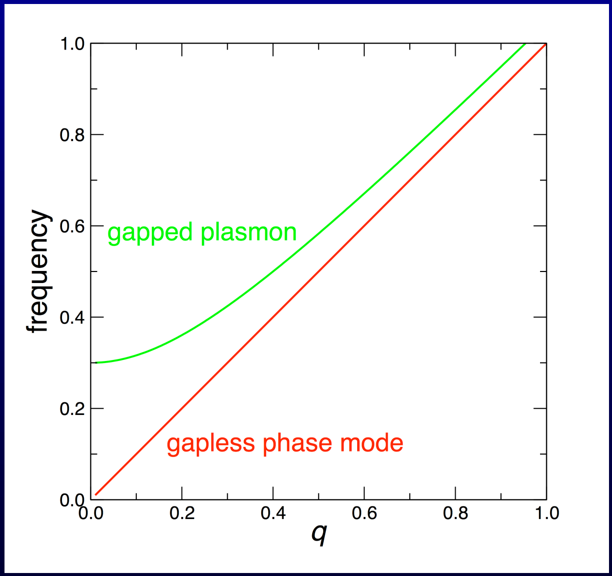
- For short range interactions $V_{\mathbf{q}}$ $\rightarrow$const as $q \rightarrow 0$; we have $\omega_{\mathbf{q}} \sim q$, i.e. acoustic phase mode.
- For Coulomb interactions $V_{\mathbf{q}} \sim 1/q^2$ as $q \rightarrow 0$; we have $\omega_{\mathbf{q}} \rightarrow \omega_{\rm pl}$, i.e. gapped plasma mode.



Simple power counting shows that at low T the contribution from the phase mode to the superfluid density is

$$\delta\rho_s^{\mathrm{ph}} \sim \begin{cases} T^3, & \text{short range interaction} \\ e^{-\omega_{\mathrm{pl}}/T}, & \text{Coulomb interaction} \end{cases}$$

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However, as we shall see, quantum fluctuations will strongly renormalize both the amplitude J_0 and the slope α.

SCHA: the results

Using the identity $\langle\cos(\hat\varphi_i-\hat\varphi_j)\rangle_{\rm har}=\exp\left[-\frac{1}{2}\langle(\hat\varphi_i-\hat\varphi_j)^2\rangle_{\rm har}\right]$, valid for harmonic Hamiltonians, we obtain

$$E_{\rm har}=\langle H_{\rm XY}\rangle_{\rm har}=\sqrt{KS}-Je^{-\sqrt{S/K}},$$

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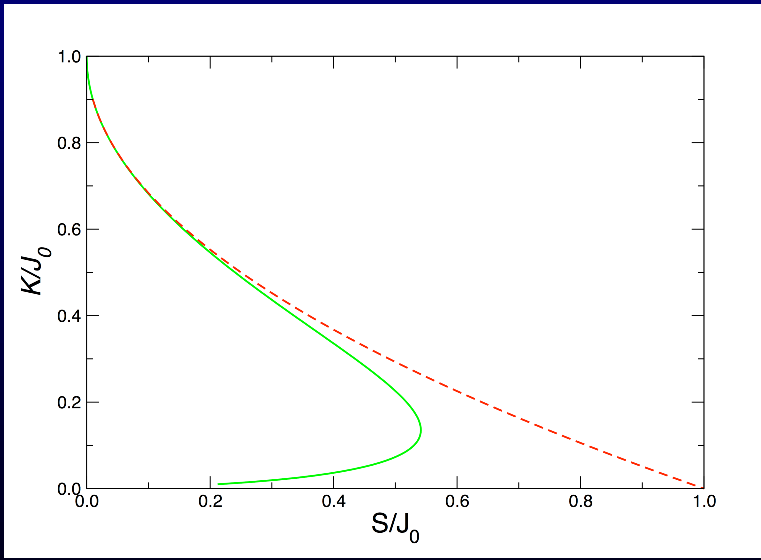
Minimizing $E_{\rm har}$ with respect to K we obtain

$$K = Je^{-\sqrt{S/K}} \simeq J(1 - \sqrt{S/J}).$$



To obtain the leading temperature dependence substitute $J = J_0 - \alpha T$ and expand to leading order in T:

$$\rho_s(x,T) = K \simeq J_0\left(1 - \sqrt{\frac{S}{J_0}}\right) - \alpha T\left(1 - \frac{1}{2}\sqrt{\frac{S}{J_0}}\right).$$

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In particular, for small $\sqrt{S/J_0}$ the above expression is consistent with experimentally observed behavior

$$\rho_s^{ab}(x,T) \simeq J_0 x^2 - \alpha x T,$$

if we identify $x \simeq (1 - \frac{1}{2}\sqrt{\frac{S}{J_0}})$.

How about the region $S \approx J_0$?

In this regime one can construct a “critical” theory of strong phase fluctuations [Doniach, PRB 24, 5063 (1981)] as an expansion in small order parameter $\psi(x,\tau)$. This leads to a quantum Ginzburg-Landau action

$$S = \int_0^\beta d\tau \int d^3x \left\{ r|\psi|^2 + \frac{1}{2}u|\psi|^4 + \frac{1}{2}|\nabla\psi|^2 + \frac{1}{2c^2}|\partial_\tau \psi|^2 \right\},$$

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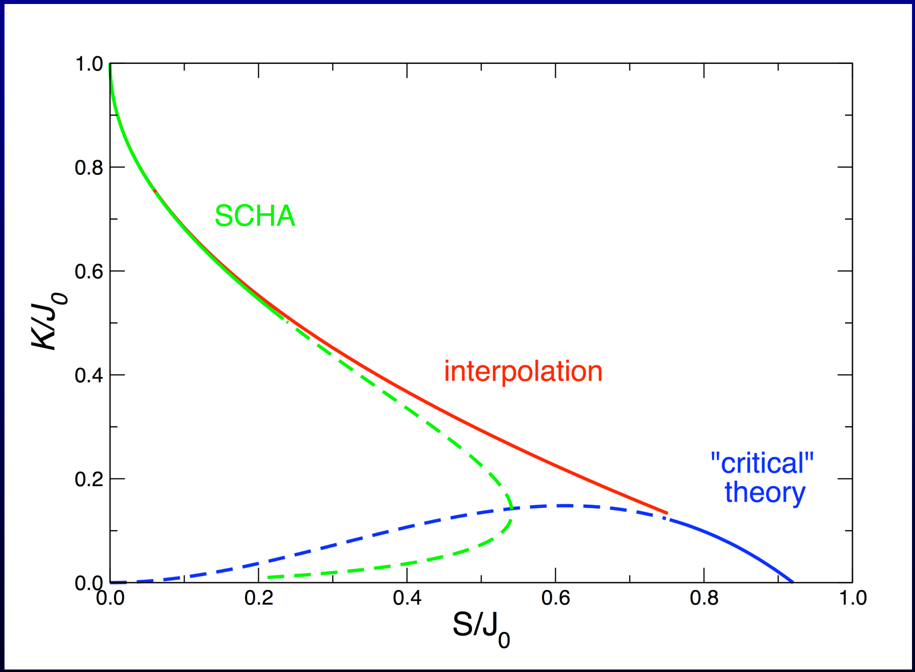
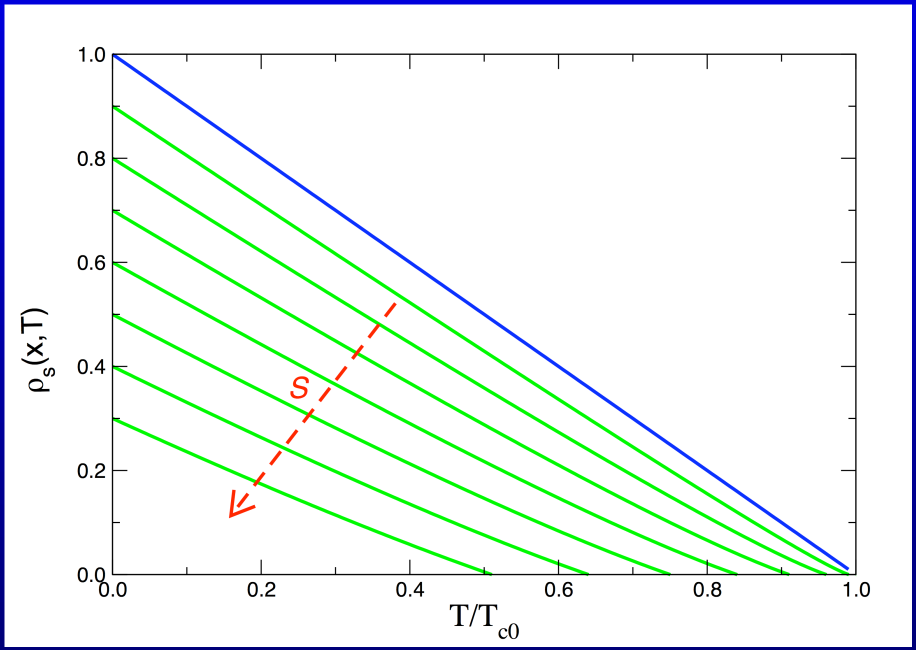
Combining SCHA with this critical theory provides a consistent picture for the suppression of ρ_s by quantum fluctuations.



Summary

The combined SCHA+critical analysis of the Quantum XY model predicts doping and temperature dependence of $\rho_s^{ab}(x,T)$ that is consistent with recent experiments on strongly underdoped YBCO.



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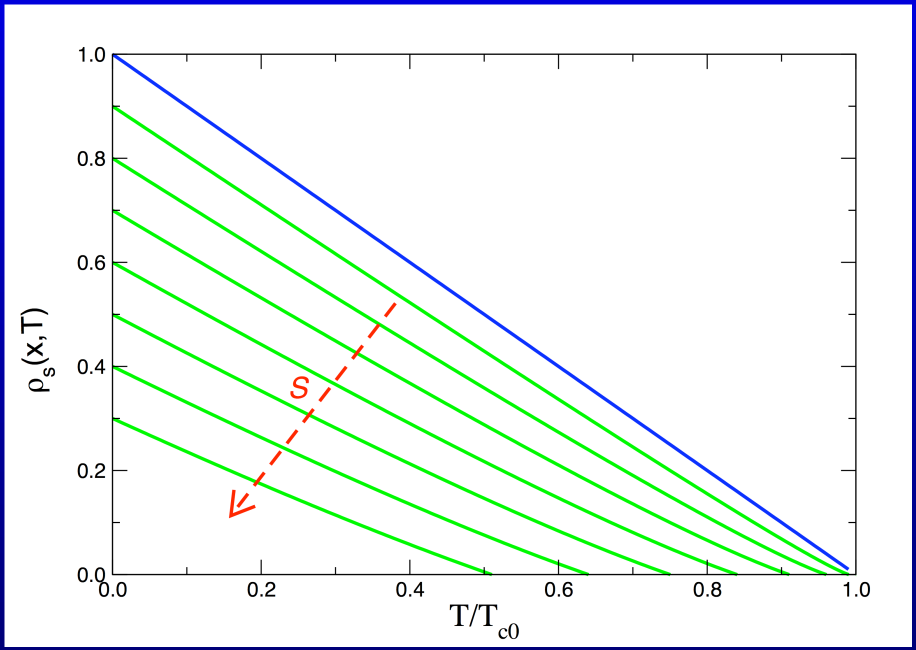

We are currently investigating implications of this model for other physical observables, namely specific heat, c-axis superfluid density, fluctuation diamagnetism, thermal and electrical conductivity.

Conclusions

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- This phenomenology puts severe constraints on microscopic models of underdoped cuprates.