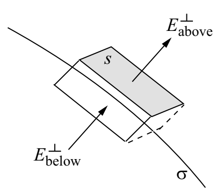
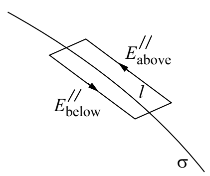
Physics 301, Solutions to Midterm

Problem One

(a)

$$\vec{E} = \frac{Q}{4\pi\epsilon_0}\frac{\vec{r}}{r^3}$$
$$\vec{\nabla}\cdot\vec{E} = \frac{1}{r^2}\frac{\partial}{\partial r}\left(r^2 E_r\right) = 0 \quad \text{if } r \neq 0$$
$$\oint \vec{E}\cdot d\vec{s} = \frac{Q}{\epsilon_0} \;\Rightarrow\; \vec{\nabla}\cdot\vec{E} = \frac{Q}{\epsilon_0}\delta(\vec{r})$$
$$\nabla\times\vec{E} = \frac{1}{r}\left(\frac{1}{\sin\theta}\frac{\partial E_r}{\partial\phi}\right)\hat{\theta} - \frac{1}{r}\frac{\partial E_r}{\partial\theta}\hat{\phi} = 0$$
$E_\theta = E_\phi = 0$, only $E_r \neq 0$ and E_r is independent of θ and ϕ.





(b) Following the Gauss's law,

$$\left(E^{\perp}_{above} - E^{\perp}_{below}\right)s = \frac{\sigma s}{\epsilon_0}$$
$$E^{\perp}_{above} - E^{\perp}_{below} = \frac{\sigma}{\epsilon_0}$$
$$\oint \vec{E}\cdot d\vec{l} = 0$$
$$E^{||}_{above}l = E^{||}_{below}l$$
$$E^{||}_{above} - E^{||}_{below} = 0$$

Problem Two

(a) Using spherical coordinates and with $\vec{P}$ in z direction,

$$\rho = \nabla\cdot\vec{P} = 0$$
$$\sigma(\theta) = \vec{P}\cdot\vec{n} = |\vec{P}|\cos\theta$$

For electric potential,

$$V_{\text{out}}(r,\theta)\Big|_{r\to\infty} = 0$$
$$V_{\text{out}}(r=a,\theta) = V_{\text{in}}(r=a,\theta)$$
$$\frac{\partial V_{\text{out}}}{\partial r} - \frac{\partial V_{\text{in}}}{\partial r} = -\frac{|\vec{P}|\cos\theta}{\epsilon_0}$$

(b)

$$V_{\text{out}} = \sum_{l=0}^{\infty} \frac{B_l}{r^{l+1}} P_l(\cos\theta)$$

$$V_{\text{in}} = \sum_{l=0}^{\infty} A_l r^l P_l(\cos\theta)$$

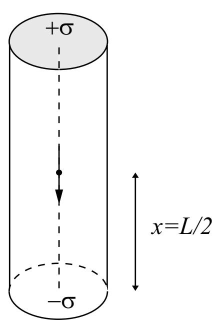
$$B_l = A_l a^{2l+1}, \quad (l+1)\frac{B_l}{a^{l+2}} + l A_l a^{l-1} = \frac{|\vec{P}|}{\epsilon_0}\delta_{1,l}$$

$$\Rightarrow \quad B_1 = A_1 a^3, \quad 2B_1 + A_1 a^3 = \frac{|\vec{P}|}{\epsilon_0} a^3$$

$$\Rightarrow \quad A_1 = \frac{|\vec{P}|}{3\epsilon_0}, \quad B_1 = \frac{|\vec{P}|}{3\epsilon_0} a^3 \quad \text{otherwise } A_l = B_l = 0$$

$$E_{\text{inside}} = -\frac{\vec{P}}{3\epsilon_0}, \quad E_{\text{outside}} = -\nabla V_{\text{dipole}}$$

$$\text{where } V_{\text{dipole}} = \frac{\frac{4\pi a^3}{3}\vec{P}\cdot\vec{r}}{4\pi\epsilon_0 r^3} \quad \text{with dipole moment } = \frac{4\pi a^3}{3}\vec{P}$$



(c) $\sigma = \vec{P}\cdot\vec{n} = |\vec{P}|$ or $-|\vec{P}|$

Electric field in the mid of the cylinder is

$$E \sim \frac{1}{4\pi\epsilon_0}\frac{2|\vec{P}|\pi b^2}{(L/2)^2}$$

$$\sim \frac{|\vec{P}|}{\epsilon_0}\left(\frac{b}{L}\right)^2 \text{const.} \ll \frac{|\vec{P}|}{\epsilon_0}$$

Electric field far away from the cylinder is a dipole field

dipole moment $\vec{p} = L\pi b^2 \vec{P}$

$$|\vec{E}| \sim \frac{|\vec{P}|}{4\pi\epsilon_0}\frac{1}{r^3}\left(2\cos\theta\,\hat{r} + \sin\theta\,\hat{\theta}\right)$$

$$V \sim \frac{\vec{p}\cdot\vec{r}}{4\pi\epsilon_0 r^3}$$