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INTRODUCTION TO MATHEMATICAL PHYSICS

PHYSICS 312

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Course web site: http://www.physics.ubc.ca/~birger/n312toc/index.html
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1 Introduction

1.1 About this course

Many of the important ideas on how to solve differential equations originate in physics. Modeling of physical systems (or for that matter most scientific modeling) requires an understanding of the mathematics of ordinary and partial differential equations. The main topic of this course is the *application of differential equations to physics*, with emphasis on boundary value problems. We will find that a proper understanding often requires knowledge of some areas of mathematics that are not covered in the standard set of courses taken by UBC Physics Majors, so we will also have to stray beyond the confines of the main topic. The wide availability of software environments such as Maple, Mathematica and Mathlab has caused a profound change the way physicists approach problems with mathematical difficulties, and I will try to let the course reflect this. The main areas covered are:

- Introductory material (Complex algebra summation of series)
- Ordinary and partial differential equations.
- Fourier methods.
- Bessel and Legendre functions.
- Fourier transforms.

The material in these notes was last presented in class as part of the course PHYS 312 at UBC January-April 2003. Included are also a number problems that mostly constitute assignments and exams from the winter sessions 1999–2003, but also some that go further back. The problems are very much part of the course, and without doing them you cannot expect to fully understand the material. Solutions to the problems are available from the course web site, as Maple worksheets.

You are encouraged to use soft-ware such as Maple or Mathematica to solve the assignments! My own personal preference is Maple, and a .html version of a number of Maple worksheets complementing the lectures are available from the course web site. I recommend that you instead, download the corresponding .mws version e.g to use as templates.

The Maple worksheets are generally produced using Maple 6, most of them work well with Maple V. Whenever I am aware of a problem I have tried to indicate where the instructions have to be modified to run on Maple V. I have also posted a list of the Maple Worksheet indicating where different Maple commands are first introduced.

1.2 Texts

If you need a general math reference I recommend purchasing Riley, Hobson and Bence[10]. In preparing these notes I have used this book quite a bit, their explanations, although concise, tend to be more complete than the other books listed below. The second edition of the Riley *et al.* text contains over 1200 pages, most of which are not required in this course. It does, however, address a common problem for majors in undergraduate physics courses, namely that you often are taught the necessary mathematics after it is required in physics. Also, Physics Majors do not always have room in their program for as many math courses as would be desirable. The Riley *et al.* book may help to bridge the gap - it contains most of the math required for an undergraduate degree in physics and I recommend it as a general mathematical reference for physics students. You are likely to find it useful even after you graduate. A similar book, which could serve almost as well, but which you may find a bit harder to read is Arfken and Weber[1]. This text also contains about 1000 pages, most of which are not required in this course.

Much of the course is well covered by Powers[8]. which for many years was the assigned text for PHYS 312. However, this book does not cover any theory of complex variables, has very little on numerical methods, and it is not useful as a general mathematics reference. The same can be said about Boyce and DiPrima[2] which some of you purchased for MATH 215 and which was until recently the text for MATH 316 at UBC. There is a great deal of overlap between PHYS 312 and MATH 316. If you go to Richard Froese's home page http://edziza.math.ubc.ca/rfroese/ you will find links to both a PDF and PS version of a set of lecture notes for MATH 316. Another text is the slim (in size not in price) and quite elegant book by Logan [1998]. The text is a bit advanced for this course, but would be very useful if you wish to go into the material in more depth. Other references are given as we move along. Some useful web links can be found by going to the course web site.

A few of the lectures hours was set aside as Maple tutorials. In preparing these tutorial I have made use of the introductory tutorial of Lynch[6]. When learning to use Maple I found the book by Koefler[5] very useful. Another useful Maple reference is Monagan[7]. When it comes to projects using Maple, the book by Enns and McGuire[3] offers many fine examples. If you are comfortable using Maple or Mathematica, you will find this most helpful in other courses and labs! For Mathematica users Wolfram[11] is the standard reference.

1.3 Complex Algebra

The main topic of PHYS 312 is the study of boundary value problems arising from the partial differential equations of physics. Many of the methods used to do this become much more transparent and straightforward if one can appeal to the *theory of complex variables.* For this reason I will start the course by a brief introduction to complex algebra. Time restrictions prevent me to go into the theory in depth. To do this you need to take courses such as MATH 300 and 301.

COMPLEX VARIABLES AND NUMBERS

A *complex number* is an *ordered pair* (a, b) of real numbers a, b written as

$$a + ib; \quad i = \sqrt{-1}$$

Similarly a *complex variable* can be written

$$z = x + iy$$

with x, y real variables.

Perhaps the first place where you have encountered complex numbers is in solving *polynomial equations.* For example the quadratic equation

$$z^2 + 2az + b = 0$$

has two roots

$$z_1 = -a + \sqrt{a^2 - b}; \quad z_2 = -a - \sqrt{a^2 - b}$$

If $a^2 \geq b$ the roots are *real*, if not they are *complex.* In general *the fundamental theorem of algebra* states a n'th order polynomial can be written

$$P(z) \equiv a_n z^n \cdots a_2 z^2 + a_1 z + a_0 = a_n(z - z_n)(z - z_{n-1}) \cdots (z - z_1)$$

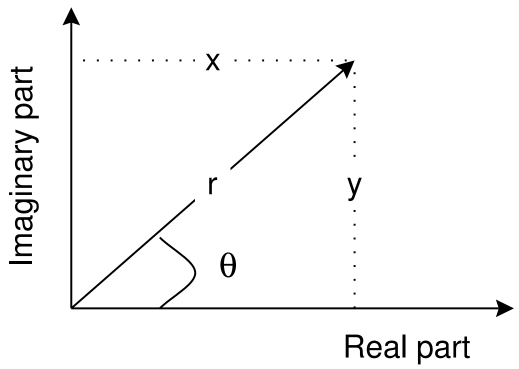
where

$$z_1, z_2, \cdots z_n$$

are the n (possibly complex) roots of

$$P(z) = 0$$

Complex numbers can be presented graphically:



Argand diagram
of complex plane

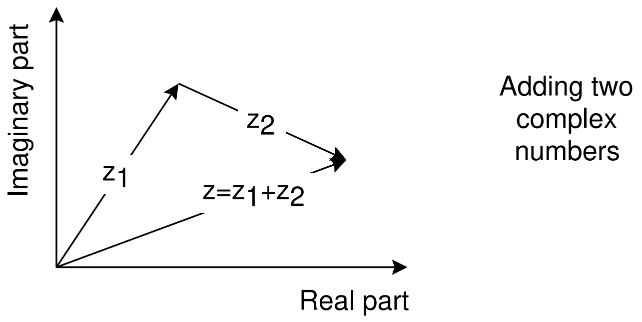
In polar coordinates:

$$x = r\cos\theta; y = r\sin\theta$$

$$\theta = \arg(z) = \tan^{-1}\frac{y}{x}; \quad r = |z| = \sqrt{x^2 + y^2}$$

The rule for addition of two complex numbers is the same as for vector addition

$$z_1 + z_2 = x_1 + x_2 + i(y_1 + y_2)$$



The rule for multiplication is

$$z_1 z_2 = (x_1 + iy_1)(x_2 + iy_2) = x_1 x_2 - y_1 y_2 + i(x_1 y_2 + x_2 y_1)$$

In polar coordinates we have

$$z_1 z_2 = r_1 r_2 \exp i(\theta_1 + \theta_2)$$

Using the polar representation the laws of multiplication of complex numbers can be written

$$z = r_1 \exp(i\theta_1) r_2 \exp(i\theta_2) = r_1 r_2 \exp(i(\theta_1 + \theta_2))$$

Suppose $z = x + iy$ We will often need the *complex conjugate*

$$z^* = x - iy$$

the *real part*

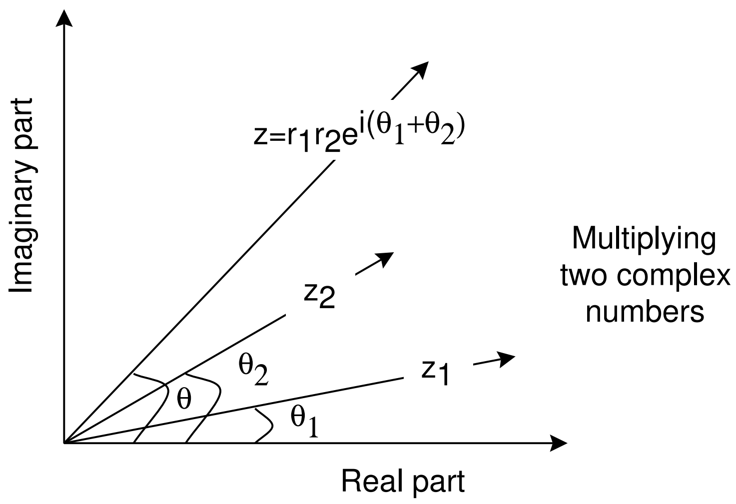
$$\mathrm{Re} z = x = \frac{1}{2}(z + z^*)$$

the *imaginary part*

$$\mathrm{Im} z = \frac{1}{2i}(z - z^*)$$

the and the *absolute value*

$$|z| = \sqrt{zz^*} = \sqrt{x^2 + y^2}$$



FUNCTIONS OF COMPLEX VARIABLES

The elementary functions of real variables can be generalized to allow complex arguments. The most important such function is the *exponential*. The exponential with a complex arguments can be defined from the power series

$$\exp(z) = 1 + z + \frac{z^2}{2!} + \cdots \frac{z^n}{n!} + \cdots$$

With this definition the property

$$\exp(z_1 + z_2) = \exp(z_1)\exp(z_2)$$

can be shown to be preserved. The power series expansions can also be used to extend the definitions of the trigonometric functions to the complex domain

$$\sin z = z - \frac{z^3}{3!} + \frac{z^5}{5!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n+1}}{(2n+1)!}$$

$$\cos z = 1 - \frac{z^2}{2!} + \frac{z^4}{4!} + \cdots = \sum_{n=0}^{\infty} \frac{(-1)^n z^{2n}}{(2n)!}$$

We have

$$\exp(i\theta) = 1 + i\theta + \frac{(i\theta)^2}{2!} + \cdots \frac{(i\theta)^n}{n!} + \cdots$$

$$= 1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} + \cdots + i(\theta - \frac{\theta^3}{3!} + \cdots)$$

Hence

$$\exp(i\theta) = \cos\theta + i\sin\theta$$

The sine and the cosine of a complex variable can thus be expressed as

$$\sin(z) = \frac{1}{2i}[\exp(iz) - \exp(-iz)]$$

$$\cos(z) = \frac{1}{2}[\exp(iz) + \exp(-iz)]$$

The *hyperbolic functions* sinh, cosh and tanh are closely related to the trigonometric functions. The definitions are

$$\cosh(x) \equiv \frac{\exp x + \exp(-x)}{2}$$

$$\sinh(x) \equiv \frac{\exp x - \exp(-x)}{2}$$

$$\tanh(x) \equiv \frac{\sinh(x)}{\cosh(x)} = \frac{\exp x - \exp(-x)}{\exp x + \exp(-x)}$$

The trigonometric and hyperbolic functions are related by

$$\cosh(iz) = \cos(z)$$

$$\sinh(iz) = i\sin(z)$$

The inverse of the exponential is the *natural logarithm.* We require that

$$\ln z = \ln(r\exp(i\theta)) = \ln r + i\theta$$

We have that

$$\exp(i\theta) = \exp(i[\theta + 2\pi n])$$

where n is an arbitrary integer. The value of the imaginary part of the logarithm is thus undetermined up to a multiple of $2\pi in$ (or we say that $\ln z$ is a *multivalued function*). We still have

$$\exp(\ln z) = z$$

no matter which value of the integer n we use when evaluating the logarithm. The *principal value* of the logarithm restricts the phase angle θ so that

$$-\pi < \theta \leq \pi$$

Similar considerations apply to the power function. We define

$$b^z = \exp(z \ln b)$$

where both z and b may be complex. Thus b^z can take on any value

$$b^z = \exp(z[\ln b + 2\pi ni])$$

We will find that using software such as Maple can be very useful in this course. I will complement these notes occasional Maple worksheets which can serve as examples or templates the first of these is at:
http://www.physics.ubc.ca/˜birger/n312l1.mws or .html.

SUMMARY

- We began our discussion of functions of complex variables by discussing
- Complex algebra
- Complex functions
- We have shown how complex numbers, variables and functions can be manipulated in Maple.

PROBLEMS

Problem 1.3:1
Show that

$$\sin(x + iy) = \sin x \cosh y + i \cos x \sinh y$$

$$
\begin{aligned}
\sinh(x+iy) &= \sinh x \cos y + i \cosh x \sin y \\
\cos(x+iy) &= \cos x \cosh y - i \sin x \sinh y \\
\cosh(x+iy) &= \cosh x \cos y + i \sinh x \sin y
\end{aligned}
$$

Problem 1.3:2
Two complex numbers z and w are

$$z = 1 + i; \quad w = 1 - 2i$$

Find

a: z+w

b: wz

c: z/w

Problem 1.3:3
By considering the real and imaginary parts of the product

$$e^{i\phi} e^{i\theta}$$

prove that

$$
\begin{aligned}
\sin(\phi+\theta) &= \sin\phi\cos\theta + \cos\phi\sin\theta \\
\cos(\phi+\theta) &= \cos\phi\cos\theta - \sin\phi\sin\theta
\end{aligned}
$$

Problem 1.3:4
Use the definition

$$\tanh(x) = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

to show that

$$\tanh^{-1}(z) = \frac{1}{2}\ln\frac{1+z}{1-z}$$

Assume that z and x are real in the above formulas (the inverse hyperbolic tangent is sometimes written arctanh).

Problem 1.3:5
In this problem $z = 1 + i$, $w = 2 - i$. Evaluate the real and imaginary parts and the absolute value of
a: $\sqrt{1 + z * w}$.
b: $1/(z - w)$
c: $\sin(z^2 - w)$

Problem 1.3:6
For what values of the complex variable z will
a: $\mathcal{R}z^3 > 0$
b: $z^3 = 1$
c: $\cos(z) = 2$

1.4 Series expansions.

Many of the methods for solving differential equations involve expansions in series of different kinds. Let us therefore recapitulate some of the properties of series expansions that you may have forgotten from previous math courses. Most series that we will encounter are infinite sums, but we may be able to obtain a good approximation by performing a partial sum

$$S_N = s_1 + s_2 + \cdots s_N$$

If the limit

$$S = \lim_{N \to \infty} S_N < \infty$$

exists we say that the series is *convergent*.

Suppose we have a series on the form

$$S = \sum_{n=1}^{\infty} u_n$$

If the series

$$\sum_{n=1}^{\infty} |u_n|$$

is convergent, the series S is *absolutely convergent.*
If a series contain only positive terms, it is *not* enough that

$$u_N \to 0 \; as \; N \to \infty$$

for the series to converge.

EXAMPLE
The *harmonic series*

$$1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} \cdots$$

is divergent. This can be seen by grouping the terms

$$(1) + (\frac{1}{2} + \frac{1}{3}) + (\frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7}) + (\frac{1}{8} \cdots \frac{1}{15}) + \cdots$$

Each expression inside a ( ) is greater than $\frac{1}{2}$ and the series

$$\frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \cdots$$

is clearly divergent.

There are many *tests for absolute convergence* (a number of them are listed in Riley, Hobson and Bence[10], and also in Arfken and Weber[1]). The test which will be most useful to us is the *D'Alembert ratio test*:
Suppose

$$S = \sum_{n=1}^{\infty} u_n$$

with *all the terms* u_n *of the same sign.* Also assume that the limit

$$\rho = \lim_{n \to \infty} \frac{u_{n+1}}{u_n}$$

exists. If

- $\rho < 1$ the series is *absolutely convergent.*
- if $\rho > 1$ the series is *divergent.*
- if $\rho = 1$ the series could be *either* convergent or divergent.

EXAMPLE:
The series

$$S = 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \cdots \frac{1}{n!} + \cdots$$

is absolutely convergent because

$$\frac{u_{n+1}}{u_n} = \frac{n!}{(n+1)!} = \frac{1}{n+1} \to 0 \; as \; n \to \infty$$

ALTERNATING SERIES
If the terms u_n in our series have alternating signs then the condition for convergence is weaker $\Rightarrow$ it is enough that

$$\lim_{n\to\infty} u_n \to 0$$

An alternating series that is convergent, but not absolutely convergent, is said to be *conditionally convergent* The terms can be summed up *in any order* if the series is absolutely convergent. However, if the series is only conditionally convergent, the result of summing the series will depend on the order in which the terms are summed.

EXAMPLE
The *alternating harmonic series*

$$S = \sum_{n=1}^{\infty} \frac{(-1)^{n+1}}{n} = 1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots$$

is conditionally convergent (it can be shown to sum to $\ln 2$). If, however, for some reason we choose to sum two even terms for each odd term, we can group the terms

$$\begin{aligned} S' &= (1 - \frac{1}{2} - \frac{1}{4}) + (\frac{1}{3} - \frac{1}{6} - \frac{1}{8}) + \cdots \\ &= \sum_{n=0}^{\infty} (\frac{1}{2n+1} - \frac{1}{4n+2} - \frac{1}{4n+4}) \\ &= \frac{1}{2} \sum_{n=0}^{\infty} (\frac{1}{2n+1} - \frac{1}{2n+2}) \end{aligned}$$

$$= \frac{1}{2}(1 - \frac{1}{2} + \frac{1}{3} - \cdots) = \frac{S}{2}$$

and the series converges to half of what it did before! Clearly, conditionally convergent series can be treacherous.

There are two special series which I expect you to be able to recognize:

ARITHMETIC SERIES
The distinguishing feature is that the difference between successive terms is constant

$$S_N = a + (a+d) + (a+2d) + \cdots + (a + (N-1)d)) = \frac{N}{2}(a + a + (N-1)d)$$

$$= \frac{N}{2}(first + last\ term))$$

Infinite arithmetic series diverge, so only the finite version makes sense.

GEOMETRIC SERIES
The ratio of successive terms is a constant for a geometric series:

$$S_N = a + ar + ar^2 \cdots + ar^{N-1}$$

This series can be summed noting that

$$S_N - rS_N = a - ar^N$$

so that

$$S_N = \frac{a(1-r^N)}{1-r}$$

This series converges as $N \to \infty$ to

$$S = \frac{a}{1-r}$$

if $|r| < 1$. If $|r| > 1$ it diverges.

SERIES OF FUNCTIONS
We next include the possibility that each term u_n in a series is not a number but some *function* of a variable z. The partial sum

$$S_n(z) = u_1(z) + u_2(z) \cdots + u_n(z)$$

is then a function of z. If the series converges for all z in some interval $a \geq z \geq b$ (or some region of the complex plane if z is complex) the series is said to be uniformly convergent in the interval, and defines a function $S(z)$. An important case is the *Taylor* expansion

$$f(z-a) = f(a) + (z-a)\left.\frac{df}{dz}\right|_{z=a} + \frac{(z-a)^2}{2!}\left.\frac{d^2 f}{dz^2}\right|_{z=a} \cdots + \frac{(z-a)^n}{n!}\left.\frac{d^n f}{dz^n}\right|_{z=a} + \cdots$$

A special case is the *power series*, or *Maclaurin series* which is a Taylor series with $a = 0$

$$f(z) = f(0) + z\frac{df(0)}{dz} + \frac{z^2}{2!}\frac{d^2 f(0)}{dz^2} \cdots + \frac{z^n}{n!}\frac{d^n f(0)}{dz^n} + \cdots$$

We made use of the idea of defining a function from its power series (section 1.3), when we defined the exponential with complex exponent from its power series. We will later consider other types of series of functions such as the cases where negative powers of z may occur (Laurent series), or where $u_n(z)$ is a trigonometric function (Fourier series), Bessel function or polynomial.

Convergent series can generally be integrated term by term, but differentiation can give bad results when the series is only conditionally convergent.

We finally note that the work associated with carrying out partial sums and other operations with series can be quite tedious if done by hand. In the worksheet *Summing series with Maple*
http://www.physics.ubc.ca/~birger/n312l2.mws (or .html)
I give examples on how one can manipulate series on the computer.

SUMMARY
We have discussed some properties of series expansions and concepts such as

- partial sums
- absolute, conditional and uniform convergence
- series of functions including
- Taylor and Maclaurin series

PROBLEMS

Problem 1.4:1
In the theory of polarization of dipoles one encounters the Langevin formula

$$P(x) = \mu(\coth x - \frac{1}{x})$$

where x is $E\mu/k_BT$, μ is the dipole moment, E is the field, k_B the Boltzmann constant and T the temperature. Make a McLaurin expansion of this expression for small x (low field, high temperature)up to and including x^3.

Problem 1.4:2
You invest \$ 1000 on the first day of each year and interest is paid at 5% on your balance at the end of each year. How much money do you have at the end of 25 years, assuming you pay no taxes,and that you havent yet made the 26th payment.

Problem **1.4:3**
Find the Maclaurin series for

$$\frac{1}{1+x^2}$$

Problem 1.4:4
In the special theory of relativity two velocities v_1 and v_2 add according to the formula

$$v = \frac{v_1 + v_2}{1 + \frac{v_1 v_2}{c^2}}$$

Work out the first few terms in a Taylor expansion of $v(v_1, v_2)$ about

a: $v_1 = 0, v_2 = 0$

b: $v_1 = c, v_2 = 0$

c: $v_1 = c, v_2 = c$

Problem 1.4:5
a: Find the Maclaurin series for

$$\ln \frac{1+x}{1-x}$$

b: Show that the series in **a:** is convergent for $x = 1/2$ and divergent for $x = 2$.

Problem 1.4:6
Find the limit as $x \to 0$ of

$$\frac{\sin x - x \cosh x}{\sinh x - x}$$

2 Ordinary differential equations

2.1 Different kinds of differential equations.

Having disposed of some mathematical preliminaries, we now proceed to the main topic of this course, namely the solution of differential equations. The main effort will be on *partial differential equations*, but, as we shall see, many methods of solving such equations rely on converting the problem to one of ordinary differential equations. This and the following subsections are mainly a review with the motivation of preparing the ground for studying partial differential equations.

TERMINOLOGY:
Ordinary versus partial differential equations.
In a partial differential equation such as

$$\frac{\partial T}{\partial t} = k \frac{\partial^2 T}{\partial x^2}$$

$T =$ *dependent variable.*
$t, x =$ *independent variables.*
Ordinary differential equations have only one independent variable e.g x in

$$\frac{du}{dx} = kxu$$

Partial differential equations have more than one independent variable. It doesn't matter for this classification if there are more than one dependent variable e.g.

$$\frac{dx}{dt} = v$$

$$\frac{dv}{dt} = -x$$

is a *system* of *ordinary* differential equations.

LINEARITY

A differential equation (partial or ordinary) is *linear* if it is linear in the *dependent variable(s)* e.g.

$$\frac{\partial T}{\partial t} = kx^3 \frac{\partial^2 T}{\partial x^2}$$

is *linear* (it doesn't matter that it is nonlinear in the independent variable!). The ordinary differential equation

$$\frac{du}{dx} = kxu$$

is also linear.

An example of a *nonlinear* differential is the Navier-Stokes equation

$$\frac{\partial \vec{v}}{\partial t} + (\vec{v} \cdot \nabla)\vec{v} = -\frac{1}{\rho}\nabla p + \nu \nabla^2 \vec{v} + \vec{f}$$

The non-linearity comes from the term

$$(\vec{v} \cdot \nabla)\vec{v}$$

which is nonlinear in the dependent variable. Non-linear equations are much harder to solve than linear equations, and we will not have much to say about them in this course.

HOMOGENEOUS EQUATIONS

If a differential equation has the property that putting the dependent variable(s) to zero gives a solution, then the equation is *homogenous.*

E.g.

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

is homogenous only if $\rho = 0$, otherwise the equation is *non-homogeneous.*

An important property of linear homogeneous differential equations is the *superposition principle*:

Suppose u_1 or u_2 are solutions to a linear and homogeneous ordinary or partial differential.equation. Then

$$u = c_1 u_1 + c_2 u_2$$

is also a solution, with c_1 and c_2 arbitrary constants!

Software such as Maple or Mathematica can be invaluable in solving differential equations. We give some examples in the Maple worksheet http://www.physics.ubc.ca/˜birger/n312l3.mws (or .html).

SUMMARY
We have

- distinguished *independent* and *dependent* variables. Ordinary differential equations have only one independent variable. Partial equations have more than one such variable.
- defined *linear* and *homogeneous differential equations* and established the superposition principle for linear homogeneous equations.
- showed examples of the solution of differential equations using Maple.

PROBLEMS

Problem 2.1:1
Some chemicals A, B and C undergo the reaction

$$A + B \to C$$

The concentrations satisfy the differential equation (t is time).

$$\frac{dC(t)}{dt} = kA(t)B(t)$$

Initially ($t = 0$) the concentration of A is A_0, the concentration of B is B_0 and the concentration of C is zero. Find $C(t)$ if

a $A_0 = B_0$. Plot the result assuming $A_0 = k = 1,\ t = 0..10$.

b $A_0 \neq B_0$. Plot the result assuming $A_0 = k = 1,\ B_0 = 2,\ t = 0..10$.

Problem 2.1:2
Solve numerically the differential equation

$$\frac{d^2y}{dt^2} + y^3 = 0$$

with initial condition $y(0) = 0$, $\frac{dy(0)}{dt} = 1$ in the range $0 < t < 10$ and plot the result!

Problem 2.1:3
Big fish sometimes eat little fish. If they don't find little fish they starve, but if they find them the population prospers. The little fish reproduce at a certain rate, but when too many are eaten by the big fish the population declines. Vito Volterra modeled this by the differential equations

$$\frac{dN_B}{dt} = -aN_B + bN_BN_L$$

$$\frac{dN_L}{dt} = cN_L - dN_BN_L$$

For numerical illustration put $a = 2,\ c = 1,\ b = d = .01$. Plot the number of big fish N_B vs the number of little fish N_L for suitable initial conditions. Also make a plot of N_L and N_B vs. time for the same initial conditions.

Problem 2.1:4
The concentrations of the chemicals A,B and C undergoing the reaction

$$A + B \rightleftarrows C$$

satisfy

$$\frac{dA}{dt} = -k_fAB + k_rC$$
$$\frac{dB}{dt} = -k_fAB + k_rC$$
$$\frac{dC}{dt} = k_fAB - k_rC$$

where k_f and k_r are the forward and reverse reaction rates. Solve the equations and plot the time dependence of the concentrations if in appropriate units

a: $k_f = k_r = 1$, $A(0) = B(0) = 1$, $C(0) = 0$.

b: $k_f = 1$, $k_r = 0$, $A(0) = B(0) = 1$, $C(0) = 0$.

Problem 2.1:5
Solve numerically the differential equation

$$\frac{d^2x}{dt^2} + tx^3 = 0$$

for $t > 0$ with the initial value

$$x(0) = 0, \frac{dx}{dt} = 1$$

and plot the result. Locate the 2 lowest values of t for which the solution returns to $x = 0$.

Problem 2.1:6
It is desired to explore how the solution of the differential equation

$$\frac{dy}{dx} = \cos(\pi xy), y(0) = a$$

depends on the initial condition a. Solve the equation numerically in the range $x = 0..4$ and plot the result for $a = 1, 2, 3$

Problem 2.1:7
A certain species of bacteria exists in a "normal" form and as one of two mutant variants. Let N be the relative concentration of the normal form. A "killer" mutant (concentration K) excretes a toxin which is harmful to the normal bugs, but the killer mutant has developed a resistance. This toxin can only be produced at a certain metabolic cost and soon a second, "cheater", mutant (concentration C) develops that is resistant to the toxin, but is "too lazy" to produce it. However the resistance also carries a cost and once the

cheater becomes dominant there is no longer a need to be protected against the toxin, and the normal form exhibits a resurgence. Assume that the above situation can be described by the set of differential equations

$$\frac{dN}{dt} = N(\beta C - \delta K)$$

$$\frac{dK}{dt} = K(\delta N - \alpha C)$$

$$\frac{dC}{dT} = C(\alpha K - \beta N)$$

a: Assume that in appropriate units, initially $N = K = C = 1/3$, and that $\alpha = 0.2,\ \beta = 0.35, \delta = 0.45$. Integrate the differential equations numerically, and plot the result for a few periods of the natural oscillations of the system.
b: Show that

$$N + K + C = const$$

and verify numerically that this hold for your solution.
c: Show that

$$\frac{d}{dt}(\alpha \ln N + \beta \ln K + \delta \ln C) = 0$$

and verify numerically that this holds for your solution.

2.2 Review of ordinary differential equations

LAST TIME
Started to discuss differential equations

- We made a distinction between *dependent* and *independent* variables.
- Differential equations were *partial* if there were more than one independent variable.
- *Ordinary* differential equations allowed only one independent variable.
- Differential equations were *linear* if the were linear in the dependent variable.
- Differential equation *homogeneous* if putting dependent variable(s) to zero gave rise to trivial solution.

- We showed that linear, homogenous equations satisfy *superposition principle.*

Today we will consider some important special cases of ordinary differential equations.

LINEAR DIFFERENTIAL EQUATIONS
Most of the ordinary differential equations in this course will be *linear*
We write for the *n'th order* equation:

$$\frac{d^n u}{dt^n} + k_{n-1}(t)\frac{d^{n-1}u}{dt^{n-1}} \cdots k_1(t)\frac{du}{dt} + k_o(t)u = k_0(t)$$

putting the coefficient in front to the highest derivative equal to unity. If the coefficients

$$k_{n-1}(t), \cdots k_1(t), k_0(t)$$

are *continuous* the differential equation is *non-singular.* There is then a *unique* solution if *initial values* of the dependent variable and its $n-1$ first derivatives are specified for some t. In *initial value problems* the independent variable is very often time. Often some of the coefficients k_i are *not* continuous. The equation is then *singular* and as we shall see "restrictions do apply". For example the differential equation

$$\frac{d^2 y}{dt^2} + \frac{1}{t}\frac{dy}{dt} + y = 0$$

is singular at $t = 0$.

First some special cases!

FIRST ORDER LINEAR EQUATIONS
The homogeneous linear equation

$$\frac{du}{dt} + k(t)u = 0$$

with the initial condition $u = u_0$ at $t = 0$ has the solution

$$\boxed{u_h(t) = u_0 \exp[-\int_0^t k(\tau)d\tau]}$$

If $u_0 \neq 0$ we can obtain a solution to the inhomogeneous equation

$$\frac{du}{dt} + k(t)u = f(t)$$

by *variation of parameters*:
Substitution of

$$u(t) = v(t)u_h(t)$$

into the inhomogeneous equation yields

$$u_h(t)\frac{dv}{dt} = f(t)$$

$$v(t) = 1 + \int_0^t \frac{f(\tau)}{u_h(\tau)}d\tau$$

and finally

$$\boxed{u(t) = u_h(t)(1 + \int_0^t \frac{f(\tau)}{u_h(\tau)}d\tau)}$$

If $u_0 = 0$ we have to use a slightly different procedure to avoid dividing by zero in the boxed expression above. Let

$$u_1(t) = \exp[-\int_0^t k(\tau)d\tau]$$

be the solution of the homogeneous equation with $u_h(0) = 1$. Then

$$u(t) = u_1(t)\int_0^t \frac{f(\tau)}{u_1(\tau)}d\tau$$

is the solution of the problem with $u(0) = 0$
EXAMPLE:
Solve

$$\frac{du}{dt} + u = t$$

with $u(0) = 1$.
SOLUTION:

$$u_h(t) = \exp[-\int_0^t d\tau] = \exp(-t)$$

$$v(t) = 1 + \int_0^t \tau e^{\tau} d\tau = 2 + e^t(-1+t)$$

$$\boxed{u(t) = 2e^{-t} + t - 1}$$

If instead we have the initial condition $u(0) = 0$ the solution becomes

$$u(t) = t - 1 + e^{-t}$$

LINEAR SECOND ORDER EQUATIONS
Homogeneous case
The general solution to

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = 0$$

can be written on the form

$$u(t) = c_1 u_1(t) + c_2 u_2(t)$$

where c_1, c_2 are constants and $u_1(t), u_2(t)$ are *linearly independent* . Linear independence implies that one cannot find a non-zero solution c_1, c_2 to the set of equations

$$c_1 u_1(t) + c_2 u_2(t) = 0$$

$$c_1 \frac{du_1}{dt} + c_2 \frac{du_2}{dt} = 0$$

for any t. This means that the Wronskian

$$W = \begin{vmatrix} u_1(t) & u_2(t) \\ \frac{du_1(t)}{dt} & \frac{du_2(t)}{dt} \end{vmatrix} \neq 0$$

Inhomogeneous case

$$\frac{d^2u}{dt^2} + k(t)\frac{du}{dt} + p(t)u = f(t)$$

The general solution to the homogeneous problem $f(t) = 0$ can be written

$$c_1u_1(t) + c_2u_2(t)$$

The general solution to the corresponding inhomogeneous equation is then

$$c_1u_1(t) + c_2u_2(t) + u_p$$

where u_p is any *particular solution* to the inhomogeneous equation.

Suppose we have an initial value problem in which

$$u(0) = c_1u_1(0) + c_2u_2(0)$$

and

$$\left.\frac{du}{dt}\right|_{t=0} = c_1\left.\frac{du_1}{dt}\right|_{t=0} + c_2\left.\frac{du_2}{dt}\right|_{t=0}$$

It can be shown by substitution that the correct particular solution to the inhomogeneous problem is

$$u_p(t) = \int_0^t G(t,\tau)f(\tau)d\tau$$

where

$$G(t,\tau) = \frac{u_1(\tau)u_2(t) - u_1(t)u_2(\tau)}{u_1(\tau)\frac{du_2(\tau)}{d\tau} - u_2(\tau)\frac{du_1(\tau)}{d\tau}}$$

is called a *Green's function.* If we can solve the homogeneous problem we can also in principle solve the inhomogeneous problem!

EXAMPLE

$$\frac{d^2u}{dt^2} + u = f(t)$$

with u and du/dt specified for $t = 0$

The general solution to the homogeneous equation

$$\frac{d^2u}{dt^2} + u = 0$$

can be written

$$u_h(t) = c_1 u_1(t) + c_2 u_2(t)$$
$$u_1(t) = \sin(t)$$
$$u_2(t) = \cos(t)$$

Substituting the initial conditions gives two equations that can be solved for c_1 and c_2

The Wronskian is particularly simple

$$W(\tau) = u_1(\tau)\frac{du_2(\tau)}{d\tau} - u_2(\tau)\frac{du_1(\tau)}{d\tau}$$
$$= -\sin^2(t) - \cos^2(t) = -1$$

Then

$$G(t, \tau) = \cos(\tau)\sin(t) - \sin(\tau)\cos(t) = \sin(t - \tau)$$

and

$$u(t) = c_1 \sin(t) + c_2 \cos(t) + \int_0^t \sin(t - \tau) f(\tau) d\tau$$

To proceed further must specify $f(\tau)$.

Unless $f(\tau)$ is simple the integral may need to be evaluated numerically.

There are some more special cases that occur frequently enough that it may be worthwhile to learn to recognize them

SEPARABLE DIFFERENTIAL EQUATIONS
Sometimes differential equations can be integrated by *separating the variables.* As an example consider

$$\frac{dy}{dx} = h(y)g(x)$$

we find

$$\int \frac{dy}{h(y)} = \int g(x)dx$$

Example
Biological populations are often modeled by the *logistic equation*

$$\frac{dP}{dt} = \beta P(\kappa - P)$$

where $P(t)$ represents population size, the constants β and $\kappa$are respectively the net birth rate and the carrying capacity. We find

$$\int \frac{dP}{P(\kappa - P)} = \beta \int dt$$

The integrals can easily be carried out to yield

$$\ln \left| \frac{P}{\kappa - P} \right| = \kappa\beta t + const.$$

where C is a constant. We can solve this equation for P

$$P = \frac{\kappa}{1 + \exp(-\kappa\beta t + C)}$$

where the constant C can be determined from an initial condition. We note that in this example the population will always saturate at the value κ after a long time.

EULER'S DIFFERENTIAL EQUATION

$$a_2 t^2 \frac{d^2y}{dt^2} + a_1 t \frac{dy}{dt} + a_0 y = 0$$

is a second order Euler equation. It can by solved by the trial function

$$y = t^{\lambda}$$

Substituting into the differential equation yields the second order algebraic equation

$$\lambda(\lambda - 1)a_2 + \lambda a_1 + a_0 = 0$$

If the two roots λ_1, λ_2 are different the general solution will be

$$y = c_1 t^{\lambda_1} + c_2 t^{\lambda_2}$$

where c_1 and c_2 are constants. If the two roots coincide the solution is

$$y = t^{\lambda}(c_1 + c_2 \ln t)$$

SUMMARY

We have discussed

- linear
- singular and non-singular
- first and second order
- homogeneous and non-homogeneous ordinary differential equations.

We also introduced the

- Wronskian
- Green's function method to solve inhomogeneous equation when solution to homogeneous equation was known.
- Finally we considered the special cases of Euler's differential equation and separable differential equations.

PROBLEMS

Problem 2.2:1
Solve
$$\frac{du}{dt} + tu = t$$
a: with $u(0) = 1$.
b: with $u(0) = 0$.

Problem 2.2:2
The differential equation
$$t^2 \frac{d^2u}{dt^2} + \frac{1}{4}u = 0$$
has
$$u = \sqrt{t}$$
as a solution. Find a second linearly independent solution.

Problem 2.2:3
Find the general solution to
$$\frac{d^4u}{dx^4} - 2a^2\frac{d^2u}{dx^2} + a^4u = 0$$

Problem 2.2:4
An unstable isotope A can decay to another unstable isotope B or to a stable isotope C. B decays to C. The number of atoms of each species satisfy the differential equations
$$\frac{dA}{dt} = -(\lambda_1 + \lambda_2)A$$
$$\frac{dB}{dt} = \lambda_1 A - \lambda_3 B$$
$$\frac{dC}{dt} = \lambda_2 A + \lambda_3 B$$
Assuming that there are initially N A-atoms, and no B- or C-atoms, find the number of C-atoms at later times.

Problem 2.2:5
a: Find two linearly independent solutions to

$$\frac{d^2u}{dt^2} - u = 0$$

b: What is the associated Wronskian?
c: Use your results under **a:** and **b:** to find the general solution to

$$\frac{d^2u}{dt^2} - u = f(t)$$

d: Verify by substitution in the differential equation that your solution actually solves it!

Problem 2.2:6
Find the general solution to the differential equations:

a:

$$\frac{d^2y}{dx^2} - 2\frac{dy}{dx} + y = 0$$

b:

$$\frac{d^2y}{dx^2} - \frac{dy}{dx} + y = 0$$

c:

$$x^2\frac{d^2y}{dx^2} - x\frac{dy}{dx} = 0$$

Problem 2.2:7
The drag force on an object falling through air can often to a good approximation be taken to be proportional to the square of the speed. Solve the differential equation

$$m\frac{dv}{dt} = mg - kv^2$$

with the initial condition $v(0) = 0$ (m, g and k are constants)

Problem 2.2:8
a: Solve the differential equation

$$\frac{d^2u}{dt^2} + u = \sin t$$

with initial conditions

$$u(0) = 0, \frac{du}{dt} = 0$$

b: Describe qualitatively how the system behaves for large t.

2.3 Boundary value problems for ordinary differential equations. Finite difference method

LAST TIME

- Reviewed first and second order differential equations that were
  - linear
  - homogeneous
  - non-homogeneous
  - separable
  - Euler's differential equation
- Defined Wronskian
- Solved a general inhomogeneous problem using Green's functions.

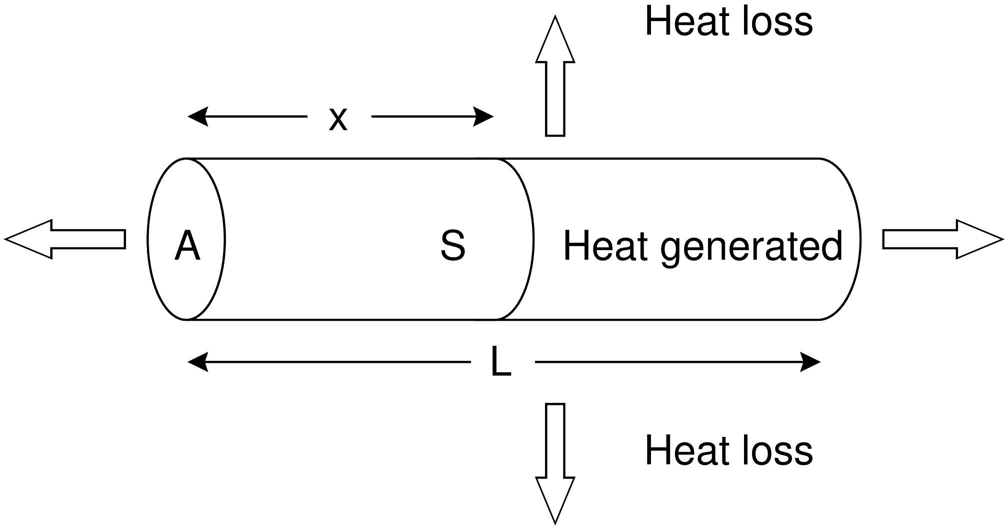
TODAY I wish to distinguish between two different types of differential equation problems:

Initial value problem: The values of the dependent variable(s) and a sufficient number of derivatives are specified for a single value of the independent variable. In this type of problem the independent variable is commonly the *time*, hence the term initial value problem.

Boundary value problem: Conditions on the solution are specified for two or more values of the independent variable. Typically the values are specified on the boundary of a *spatial region* inside which the solution is sought. Hence the term *boundary value problem.*

- I want to start our discussion of *boundary value problems* by considering a heat conduction example.

HEAT CONDUCTION IN THIN ROD
Heat is generated uniformly in the bulk (e.g. by an electric current).
There are heat losses the ends and at the cylindrical surface of the rod.
Wanted: steady state temperature along rod.



I will now make the simplifying assumption that the temperature depends only on x and is constant over cross-sectional area A. Towards the end of the lecture I will come back to this assumption and explore under what conditions it is justified.

$L =$ length of rod

$A =$ cross sectional area

$S =$ area of cylinder

$0 < x < L =$ distance along rod

T_0 = ambient temperature

$T(x)$ = temperature along rod

$\dot{Q}$ = bulk heat generated per unit volume and time

j_x = heat per unit time and area flowing axially

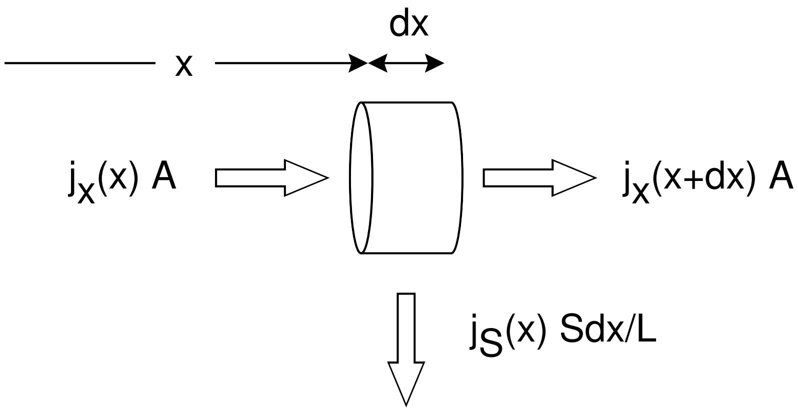
j_S = heat per unit time and area flowing through outer cylindrical surface

$C_V = \frac{dQ}{dT}$ bulk heat capacity of rod

$\lambda = \frac{j_S}{T-T_0}$ = coefficient of heat loss

$\kappa = -\frac{j_x}{dT/dx}$ = thermal conductivity

Heat balance: $\dot{Q}\, dx\, A = (-j_X(x) + j_X(x+dx))\, A + j_S(x)\, Sdx/L$



We assume that the heat current density (heat current per unit area) is proportional to the temperature gradient

$$j_x(x) = -\kappa \frac{dT}{dx}$$

so that

$$j_x(x+dx) - j_x(x) = dx\frac{dj_x}{dx} = -dx\kappa\frac{d^2T}{dx^2}$$

We divide the heat balance equation by $\dot{Q}A\,dx$ and define

$$\alpha = \frac{\lambda S}{AL\dot{Q}}; \quad \beta = \frac{\kappa}{\dot{Q}}; \quad u = T - T_0$$

to obtain the differential equation

$$\beta\frac{d^2u}{dx^2} - \alpha u = -1 \tag{1}$$

with *boundary condition* $u = 0$ for $x = 0$ and $x = L$.

DIMENSION-LESS TEMPERATURE AND LENGTH
In (1) α has dimension 1/temperature β has dimension length^2/temperature
We introduce the dimensionless effective "length" z

$$z = x\sqrt{\frac{\alpha}{\beta}}$$

and the dimension-less effective temperature t

$$t = \alpha u$$

In these units the differential equation has no free parameters!

$$\frac{d^2t}{dz^2} - t = -1 \tag{2}$$

The dimensionless coordinate on the rod where the boundary condition applies is

$$a = L\sqrt{\frac{\alpha}{\beta}} = \sqrt{\frac{\lambda SL}{A\kappa}}$$

The boundary conditions are thus $t = 0$ for $z = 0$ and $z = a$. The general solution to (2) is

$$1 + c_1e^z + c_2e^{-z}$$

The boundary conditions $t = 0$ at $z = 0, z = a$ allow us to determine c_1 and c_2 to get

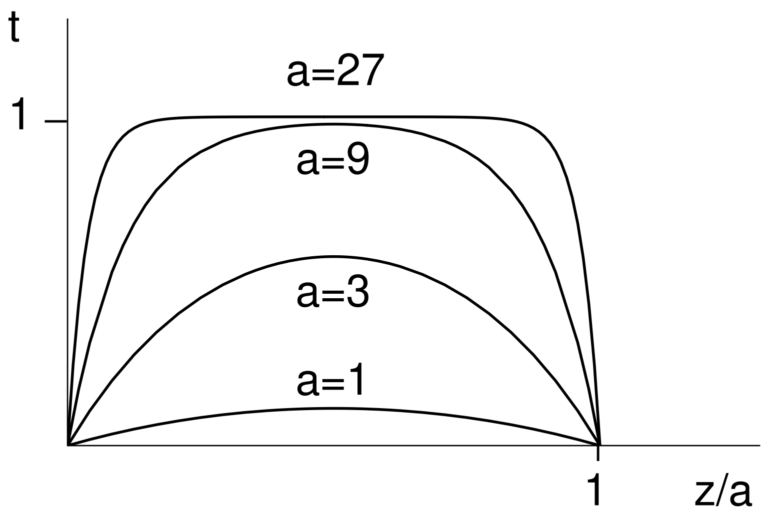
$$\boxed{t = 1 - \frac{\cosh(z - \frac{a}{2})}{\cosh(\frac{a}{2})}}$$

For a circular cross-section with radius R

$$S = 2\pi RL; \quad A = \pi R^2$$

$$a = \sqrt{\frac{2\lambda L}{R\kappa}}$$

The result is plotted below for different values of a



Conclusions:

- if a large $\Rightarrow$ most of the heat loss flows through the sides of the cylinder. The temperature is then approximately constant in the middle.
- if a small $\Rightarrow$ most heat-loss occurs at the ends.

RADIAL FLOW

We must now check the assumption that the temperature variation across a radial cross section is indeed negligible! To this we look at the radial heat balance at radius r

$$\pi r^2 \dot{Q} = -2\pi r \kappa \frac{dT}{dr}$$

and combine with the balance at R

$$\pi R^2 \dot{Q} = 2\pi R \lambda (T - T_0)$$

When studying the effect of radial flow we neglect the axial flow. For reasons of symmetry, this is completely justified in the middle of the rod where the axial flow is zero . This is also the spot where the temperature difference between the rod and the environment is the largest. So if the approximation works there it will work everywhere!

As before:

$$u = T - T_0; \quad \beta = \frac{\kappa}{\dot{Q}}$$

Heat balance in steady state gives differential equation for radial flow

$$r = -2\beta \frac{du}{dr}$$

with general solution

$$u = const - \frac{1}{4\beta} r^2$$

while the condition at R gives

$$u(R) = \frac{R\dot{Q}}{2\lambda}$$

from which we find

$$u = \frac{R\dot{Q}}{2\lambda} + \frac{R^2 - r^2}{4\beta}$$

The temperature difference between the surface and axis of the cylinder is thus in dimension-less units

$$\delta t = \frac{\alpha R^2}{4\beta} = \frac{\lambda R}{2\kappa}$$

We are justified in neglecting the radial temperature profile if δt is *small* compared to the middle to end temperature difference calculated earlier. For large values of the parameter a this difference is just unity (in dimensionless units), and the we are allowed to put the temperature to be approximately constant over the cross section A if

$$2\kappa >> \lambda R$$

For small values of the parameter a the condition is more complicated. We are justified in neglecting the radial temperature dependence if

$$\frac{\lambda R}{2\kappa} << 1 - \frac{1}{\cosh \frac{a}{2}}$$

If this condition is not valid we need to treat *both* x and r as independent variables and the boundary value problem will involve a partial differential equation. We will be solving such problems later in the course.

FINITE DIFFERENCE METHOD

In the example discussed above we were able to find a solution analytically. In practical problems, this is usually not possible, and we have to find an approximate numerical solution. I have already noted that the numerical option of *dsolve* in Maple only works for initial value problems, not for boundary value problems. A favorite numerical alternative is the finite difference method. Let us assume that we wish to find an approximate solution for $a < x < b$. The finite difference method seeks an approximate solution on a discrete set of $N + 1$ points (including the end points). The distance between the points is

$$\Delta = \frac{b-a}{N}$$

The first and last points are called *exterior points*, at these points the function has to be determined by the boundary conditions. The remaining points are the *interior points* where we wish to find a numerical solution.

We need to find an approximate form for the derivatives in terms of the values of the unknown function at neighboring points. To do this we make use of the Taylor expansions

$$u(x-\Delta) = u(x) - \Delta \frac{du}{dx}\bigg|_x + \frac{1}{2}\frac{d^2u}{dx^2}\bigg|_x$$

$$u(x+\Delta) = u(x) + \Delta \frac{du}{dx}\bigg|_x + \frac{1}{2}\frac{d^2u}{dx^2}\bigg|_x$$

Solving these equations for the derivatives gives

$$\frac{du}{dx}\bigg|_x = \frac{u(x+\Delta) - u(x-\Delta)}{2\Delta}$$

$$\frac{d^2u}{dx^2}\bigg|_x = \frac{u(x+\Delta) + u(x-\Delta) - 2u(x)}{\Delta^2}$$

The finite difference method consists of replacing the derivatives occuring in the differential equations by the finite difference expressions above. This reduces the equations to a set of algebraic equations, which are easier to solve.

It remains to consider the boundary conditions. These commonly come in the following forms

- The value of the function u is specified at the boundary. *Dirichlet boundary condition.*
- The derivative of u is specified at the boundary. *Neumann boundary condition.*
- An equation relating the function u and its derivative is specified at the boundary. We will encounter this type of boundary condition in heat conduction problem later as *convective boundary condition.*

The Dirichlet boundary conditions is handled trivially by the finite difference method. In the case of the two latter types of conditions we need to find finite difference expressions for the derivatives which only values of u in regions where the function inside the region where it is defined. Writing

$$u(a+\Delta) = u(a) + \Delta \, \frac{du}{dx}\bigg|_a + \frac{1}{2} \, \frac{d^2u}{dx^2}\bigg|_a$$

$$u(a+2\Delta) = u(a) + 2\Delta \, \frac{du}{dx}\bigg|_a + 2 \, \frac{d^2u}{dx^2}\bigg|_a$$

Solving for the derivative we find

$$\left.\frac{du}{dx}\right|_a = \frac{4u(a+\Delta) - 3u(a) - u(a+2\Delta)}{2\Delta}$$

Similarly at the other end

$$u(b-\Delta) = u(b) - \Delta \left.\frac{du}{dx}\right|_b + \frac{1}{2} \left.\frac{d^2u}{dx^2}\right|_b$$

$$u(b+2\Delta) = u(a) - 2\Delta \left.\frac{du}{dx}\right|_b + 2 \left.\frac{d^2u}{dx^2}\right|_b$$

$$\left.\frac{du}{dx}\right|_b = \frac{-4u(a-\Delta) + 3u(a) + u(a-2\Delta)}{2\Delta}$$

Thus, a Neumann or convective boundary condition give rise to an additional algebraic equation, that needs to be solved. This is not a serious problem, but, as we shall see later in an example, sometimes this problem can be gotten around by considerations of symmetry.

A work-sheet implementing the finite difference method in Maple can be found at
http://www.physics.ubc.ca/~birger/n312l6.mws (or .html).

SUMMARY

- We have analyzed a heat flow boundary value problem.
- To solve the problem using ordinary differential equations we made an approximation about the nature of the flow.
- This lead to a differential equation for the steady state obtained by writing down an equation for the heat balance.
- We started out with a lot of parameters, but showed that in appropriate units only one parameter, the effective length a was needed to describe the behavior.
- Conditions for the validity of our approximation.

- Finally, we have described the *finite difference method* for solving boundary value problems numerically.

PROBLEMS

Problem 2.3:1 Heat is produced uniformly at the rate H (energy per time per volume) inside a sphere of radius c. The thermal conductivity of the sphere is κ. At the surface of the sphere heat loss per unit surface area to the surroundings is

$$\lambda(u(c) - T)$$

Find the steady state temperature $u(\rho)$ at radius ρ inside the sphere.

Problem 2.3:2
A large object has a spherical hole of radius c.The thermal conductivity of the object is κ. At the surface of the hole the temperature is 0. Far away from the hole the temperature is T. Find the steady state temperature at a distance $r > c$ from the center of the hole.

Problem 2.3:3
In a heat conduction problem the steady state temperature T_S satisfies the differential equation

$$\frac{d^2 T_S}{dx^2} = h^2(T_S - T_0 - bx)$$

where T_0 and b are constants and the boundary conditions are

$$T_S(0) = T_S(a) = 0$$

find T_S.

Problem 2.3:4
The rate of evaporation from a spherical drop (with constant density) is proportional to its surface area. Define a parameter λ to describe this proportionality and find a formula for the radius of the drop as a function of time.

Problem 2.3:5
You are called to the scene of a murder and charged with finding out when

the victim died. The temperature of the corpse is found to be 25^oC and one hour later it has dropped to 23.2^oC. The room is at 20^oC. Assume the victim was at 37.5^oC when alive, and that the rate of cooling of the corpse is proportional to the difference in temperature between the body and the room. When did the victim die?

Problem 2.3:6
In a heat conduction problem with position dependent thermal conductivity, the steady state is given by the differential equation

$$\frac{d}{dx}\left((b - fx)\frac{du}{dx}\right) = 0$$

$$0 < x < a$$

$$u(0) = 0; \ u(a) = T_1$$

b, f, k are constants.
a: Find the steady state temperature $u(x)$.
b: Can the constant f be arbitrarily large?

Problem 2.3:7
a: Heat is produced inside a sphere of radius R at the rate H (H = energy per unit time and volume). The surface is kept at the temperature $T_0 = 0$, the thermal conductivity is κ. Find the steady state temperature distribution.
b: Answer the same question in the case of a cylinder of radius R length L. The ends of the cylinder are thermally insulated from the surroundings.

Problem 2.3:8
When solving a heat conduction problem we may encounter the differential equation

$$\frac{d^2u(x)}{dx} - xu(x) = x$$

with boundary conditions $u = 0$ for $x = 0$ and $x = 1$.If you try to solve this equation e.g. using Maple you will find a very complicated solution involving the so-called "Airy" functions.
a: Instead, solve the problem numerically using the finite difference method

of lecture 6 using $\Delta = 0.005$ for the distance between mesh points and plot your result.
b: Estimate the likely error by comparing with the exact result or a numerical calculation using $\Delta = 0.01$.

Problem 2.3:9
The thermal conductivity may depend on the temperature. This gives rise to a nonlinear differential equation for the steady state temperature. Consider the following steady state differential equation

$$\frac{d}{dx}((\kappa_0 + \alpha T)\frac{dT}{dx}) = Q$$

Assume that for $x = 0$ and $x = L$ the temperature is kept at $T = 0$
a:
Integrating of the differential equation gives rise to a quadratic equation for T with two roots. Solve this equation Which root is appropriate if $\alpha > 0$? $\alpha < 0$? What is the solution when $\alpha = 0$?
b:
What goes wrong if α is too large and negative?

Problem 2.3:10
Solve the boundary value problem

$$\frac{d^2t}{dz^2} - t = -1 \tag{3}$$

with boundary conditions $t = 0, z = 0$ and $t = 1, z = 1$

Problem 2.3:11
Solve the following boundary value problem using the finite difference method

$$\frac{d^2u(x)}{dx^2} + x(1-x)u(x) = 1, \quad 0 < x < 1$$

$$u(0) = 0; \quad \frac{du(x)}{dx}|_{x=1} = u(1)$$

Choose enough points so that you feel reasonably sure that your result is accurate to 3 significant figures throughout the region $0 < x < 1$.

Problem 2.3:12
Heat is produced uniformly at rate H per unit volume inside a spherical "sun" of radius R. Heat escapes from the surface at the rate σT^4 per unit area at the surface, where T is the surface temperature. Find the steady state temperature distribution as a function or the radius r inside the sun. The thermal conductivity of the sun is κ.

3 Partial differential equations

3.1 Wave equation in one dimension. Sound waves.

In the last lecture we analyzed a heat flow boundary value problem. In this problem we were only interested in the steady state. If we wish to study time-dependent phenomena there would be more than one independent variable (time and one or more spatial coordinates) and we would be dealing with partial differential equations. Rather than continuing with the heat equation we will start our discussion of partial differential equations by considering waves.
TODAY
We wish to derive the *wave equation*

- We will use sound propagation in a gas such as air as an example.
- Sound waves are elastic waves that propagate in a gas liquid or solid.
- In fluids long wavelength sound waves consist of an alternating pattern of rarefaction and compression.
- In a solid *transverse waves* can also propagate.
- In ordinary sound the changes in pressure tend to be very small.

The *intensity* of a sound wave is often measured in decibels

$$I_{dB} = 20\log_{10}(\frac{P_e}{P_{ref}})$$

The reference pressure is defined as

$$P_{ref} = 2\times 10^{-10} bar$$

and P_e is the root mean square pressure amplitude ($1/\sqrt{2}$ of the peak excess pressure). Even at the *pain level* of $120dB$ the peak excess pressure will be

$$2\sqrt{2} \times 10^{-4} bar$$

which is small compared to the ambient pressure $\simeq 1bar$.

The mechanism for a sound wave is that gas motion generates a change in the density, which causes a change in pressure. There will then be an unbalanced force which accelerates the gas and causes the cycle to repeat.

We write for the pressure and density

$$\begin{aligned} P &= P_a + P_e \\ \rho &= \rho_a + \rho_e \end{aligned}$$

where the subscript a stands for average while e stands for excess.

The relationship between changes in density and pressure depends on the properties of the medium in which the sound waves propagate. We will assume that the processes are fast enough that they can be considered to be *adiabatic*, i.e. without any heat transport. Since the amplitudes P_e and ρ_e are small compared to the ambient conditions we assume that they are proportional to each other

$$P_e = \kappa \rho_e$$

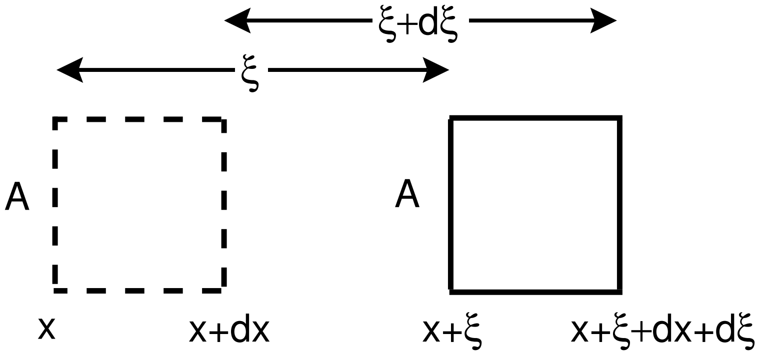
with

$$\kappa = \left.\frac{\partial P}{\partial \rho}\right|_{adiabatic}$$

Let us consider a column of cross-section A.
When the air is at rest in equilibrium this column extends from x to $x + dx$. We assume that a sound wave is traveling in the x-direction and that at some instant the left end of the column is displaced an amount $\xi(x, t)$ which is small compared to the wavelength of the sound wave.

Conservation of mass yields

$$A\rho_a dx = A(\rho_a + \rho_e)(d\xi + dx)$$



or

$$\rho_e dx = -(\rho_a + \rho_e) d\xi \simeq -\rho_a d\xi$$

since $\rho_e << \rho_a$, giving

$$\rho_e \simeq -\rho_a \frac{\partial \xi}{\partial x}$$

The column is subject to a net force

$$A[P(x+\xi,t) - P(x+\xi+dx+d\xi,t)] \approx -A\frac{\partial P_e(x,t)}{\partial x}dx$$

causing an acceleration

$$Adx\rho_a \frac{\partial^2 \xi}{\partial t^2} = -A\kappa dx \frac{\partial \rho_e}{\partial x} = A\kappa dx \frac{\partial^2 \xi}{\partial x^2}\rho_a$$

We thus obtain the *wave equation*

$$\boxed{\frac{\partial^2 \xi}{\partial t^2} = \kappa \frac{\partial^2 \xi}{\partial x^2}}$$

For an ideal gas assuming an adiabatic process

$$PV^\gamma = constant$$

with $\gamma = C_P/C_V$ or

$$P = Const.\rho^\gamma$$

Differentiating we find

$$\left.\frac{\partial P}{\partial \rho}\right|_{adiabatic} = \frac{\gamma P}{\rho} = \frac{\gamma k_B T}{m} = \frac{\gamma RT}{\mu}$$

where m is the mass of a molecule and μ the molecular weight. We finally get

$$c = \sqrt{\frac{\gamma k_B T}{m}} = \sqrt{\frac{\gamma RT}{\mu}}$$

It is interesting to compare the speed of sound with typical molecular speeds. From the thermodynamics of ideal gases we have for the root mean square (rms) speed

$$v_{rms} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3}{\gamma}}c$$

Since $\gamma \simeq 1.4$ for air we see that the rms speed and the sound speed are quite comparable.

By substituting into the wave equation we see that it admits solutions in the form of traveling waves $\psi(x \pm ct)$, where $c = \sqrt{\kappa}$ We will next show that the general solution to the wave equation can be written

$$\boxed{\xi(x,t) = f_1(x+ct) + f_2(x-ct)}$$

where f_1 and f_2 are arbitrary functions of the argument.
Before we proceed let us make a few general points:

- The same equation that we derived for sound waves also holds for a vibrating string.
- The speed of the wave is then

  $$c = \sqrt{T/\rho}$$

  where T is the *tension* and ρ is the *mass/unit length* of the string.

- We will later generalize the wave equation to more than one dimension. The two dimensional analog to the vibrating string is then the vibrating membrane (section 5.4)
- Often one is not interested in the details of the solution but only in knowing the frequencies which can be excited. We will need to develop methods which provides this kind of information.

FINDING THE GENERAL SOLUTION

The general solution to ordinary differential equations involves *arbitrary constants*, while for partial differential equations we will need *arbitrary functions.* Similarly, to determine a particular solution to an ordinary differential equation we need specify the solution and/or it derivative on one or more *points.* In the case of partial differential equations we need to specify the solution along one or more *curves.* In order to see how this works let us begin with the simple first order equation:

$$c\frac{\partial\xi}{\partial x} - \frac{\partial\xi}{\partial t} = 0$$

Consider the families of curves

$$u = x - ct = const.$$

$$v = x + ct = condt.$$

we have

$$\frac{\partial\xi}{\partial x} = \frac{\partial u}{\partial x}\frac{\partial\xi}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial\xi}{\partial v} = \frac{\partial\xi}{\partial u} + \frac{\partial\xi}{\partial v}$$
$$\frac{\partial\xi}{\partial t} = \frac{\partial u}{\partial t}\frac{\partial\xi}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial\xi}{\partial v} = -c\frac{\partial\xi}{\partial u} + c\frac{\partial\xi}{\partial v}$$

Substituting into the differential equation we find

$$\frac{\partial\xi}{\partial v} = 0$$

The significance of this result is that the solution cannot depend on v, but any dependence on u is allowed. The general solution is thus

$$\xi(x,t) = f(u) = f(x - ct)$$

where f is an *arbitrary function.* If we know the solution $\xi(x,0) = f(x)$ at $t = 0$ the solution for different times can then be obtained trivially.

Let us generalize the above result to the wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$$

We again introduce the new coordinates

$$u = x + ct$$

$$v = x - ct$$

The curves $u = const$ and $v = const.$ are commonly referred to as *characteristics.* We have

$$\begin{aligned}
\frac{\partial \xi}{\partial x} &= \frac{\partial u}{\partial x}\frac{\partial \xi}{\partial u} + \frac{\partial v}{\partial x}\frac{\partial \xi}{\partial v} = \frac{\partial \xi}{\partial u} + \frac{\partial \xi}{\partial v} \\
\frac{\partial \xi}{\partial t} &= \frac{\partial u}{\partial t}\frac{\partial \xi}{\partial u} + \frac{\partial v}{\partial t}\frac{\partial \xi}{\partial v} = c\frac{\partial \xi}{\partial u} - c\frac{\partial \xi}{\partial v} \\
\frac{\partial^2 \xi}{\partial x^2} &= \frac{\partial u}{\partial x}\frac{\partial^2 \xi}{\partial u^2} + \frac{\partial v}{\partial x}\frac{\partial^2 \xi}{\partial u \partial v} + \frac{\partial v}{\partial x}\frac{\partial^2 \xi}{\partial v^2} + \frac{\partial u}{\partial x}\frac{\partial^2 \xi}{\partial u \partial v} \\
&= \frac{\partial \xi}{\partial^2 u^2} + \frac{\partial^2 \xi}{\partial v^2} + 2\frac{\partial^2 \xi}{\partial u \partial v} \\
\frac{\partial^2 \xi}{\partial t^2} &= c\frac{\partial u}{\partial t}\frac{\partial^2 \xi}{\partial u^2} + c\frac{\partial v}{\partial t}\frac{\partial^2 \xi}{\partial u \partial v} - c\frac{\partial v}{\partial t}\frac{\partial^2 \xi}{\partial v^2} - c\frac{\partial u}{\partial t}\frac{\partial^2 \xi}{\partial u \partial v} \\
&= c^2\frac{\partial^2 \xi}{\partial u^2} + c^2\frac{\partial^2 \xi}{\partial v^2} - 2c^2\frac{\partial^2 \xi}{\partial u \partial v}
\end{aligned}$$

Collecting terms we find that in the new coordinate system the wave equation takes on the form

$$4c^2 \frac{\partial^2 \xi}{\partial u \partial v} = 0$$

Let us define

$$\phi(u,v) \equiv \frac{\partial \xi}{\partial u}$$

and rewrite the wave equation

$$\frac{\partial \phi}{\partial v} = 0$$

We can integrate this equation to give

$$\phi = F(u) \equiv \frac{df_1(u)}{du}$$

where $f_1(u)$ is an *arbitrary function* and $F(u)$ its derivative. Integrating once more we find the desired result

$$\xi = \int \phi du = f_1(u) + f_2(v)$$

where f_2 is an other arbitrary function. Substituting for u and v gives the desired result

$$\xi(x,t) = f_1(x+ct) + f_2(x-ct)$$

Since two arbitrary functions are involved, to find a particular solution of the wave equation we must specify two conditions. Typically, we will be dealing with an initial value problem. Then both the initial displacement $\xi(x,0)$ and the velocity $\dot{\xi}(x,0)$ are required to find the solution for later times.

SUMMARY
We have

- derived the wave equation using propagation of sound as an example.
- showed that the equation admitted solutions in the form of traveling waves.
- used the method of characteristics to find the general solution to the wave equation.

PROBLEMS

Problem 3.1:1
a:
Find the solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \; -\infty < x < \infty; \; t > 0$$

satisfying

$$u(x,0) = a\ for\ |x| < a; u(x,0) = 0\ otherwise$$
$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = 0$$

Hint: try solutions on the form

$$f(x+ct) + g(x-ct)$$

b: Sketch the solutions for times

$$t = \frac{a}{2c}, \frac{a}{c}, \frac{2a}{c}$$

or use the maple command "animate" to visualize the time evolution of the solution.

Problem 3.1:2
a: Find the solution to the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \ -\infty < x < \infty;\ t > 0$$

satisfying

$$u(x,0) = e^{-x^2};\ \ \frac{\partial u(x,t)}{\partial t}|_{t=0} = -2xce^{-x^2}$$

b: Describe qualitatively the solutions found in **a:**.

Problem 3.1:3
a: Solve the wave equation

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}, \ -\infty < x < \infty;\ t > 0$$

satisfying

$$u(x,0) = \sin(x)$$
$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = 0$$

using D'Alembert's solution.
b: Solve the same problem if

$$u(x,0) = \sin(x)$$
$$\frac{\partial u(x,t)}{\partial t}|_{t=0} = \cos(x)$$

3.2 Traveling and standing waves. Characteristic frequencies. Reflection at boundaries

LAST TIME

- Derived the wave equation

$$\frac{\partial^2 \xi}{\partial t^2} = c^2 \frac{\partial^2 \xi}{\partial x^2}$$

  using propagation of sound as an example.

- Argued that one could express the general solution of the wave equation

$$\xi(x,t) = f_1(x - ct) + f_2(x + ct)$$

  in terms of traveling waves.

TODAY
We will present solutions to the wave equation in terms of standing waves and indicate some important generalizations.

TRAVELING vs. STANDING WAVES
Standing waves can be expressed as linear combinations of traveling waves and vice versa. Consider for example the case where the functions f_1 and f_2 are sine and cosine waves respectively: We use the trigonometric identities

$$\cos A \cos B = \frac{1}{2}[\cos(A + B) + \cos(A - B)]$$

$$\cos A \sin B = \frac{1}{2}[\sin(A + B) - \sin(A - B)]$$

A standing wave

$$\alpha \cos(\lambda x) \cos(c\lambda t)$$

can thus be written as a superposition of traveling waves

$$\frac{1}{2}\alpha[\cos(\lambda(x + ct)) + \cos(\lambda(x - ct))]$$

Similarly since

$$\beta \cos(\lambda x) \sin(c\lambda t) = \frac{1}{2}\beta[\sin(\lambda(x+ct)) - \sin(\lambda(x-ct))]$$

we can write a standing wave as a superposition of traveling waves.

EXAMPLE:
An organ pipe is open in one end ($x = 0$) and closed at the other end ($x = a$) At the closed end the amplitude ξ has to be zero while at the open end the excess pressure is zero. Since the excess pressure is proportional to the derivative of the amplitude we have

$$\left.\frac{\partial \xi}{\partial x}\right|_{x=0} = 0$$

Let us try to find solutions to the wave equation as standing waves

$$\xi(x,t) = \phi(x)\tau(t)$$

Substitution gives

$$c^2\tau(t)\frac{d^2\phi(x)}{dx^2} = \phi(x)\frac{d^2\tau(t)}{dt^2}$$

We employ the symbol $\prime$ (prime) to indicate differentiation

$$\frac{\phi''(x)}{\phi(x)} = \frac{\tau''(t)}{c^2\tau(t)}$$

Since the left hand side is a function of x only, and the right hand side a function of t only, neither side can depend on x nor t. They must thus be equal to a constant!

$$\frac{\phi''(x)}{\phi(x)} = \frac{\tau''(t)}{c^2\tau(t)} = const = c_1$$

The boundary conditions on $\phi(x)$ are

$$\phi'(0) = 0; \quad \phi(a) = 0$$

If $c_1 > 0$, the solution to

$$\phi''(x) - c_1\phi(x) = 0$$

will be on the form

$$\phi(x) = A \sinh(x\sqrt{c_1}) + B \cosh(x\sqrt{c_1})$$

The boundary condition at $x = 0$ requires

$$A = 0$$

The condition at $x = a$ is then impossible to satisfy. In order to remind ourselves that $c_1 < 0$ we write

$$c_1 = -\lambda^2$$

The solution to the differential equation for $\phi(x)$ is now

$$\phi(x) = A \sin(\lambda x) + B \cos(\lambda x)$$

The boundary condition at $x = 0$ gives $A = 0$. The boundary condition at $x = a$ tells us that the allowed values of λ are

$$\lambda_n = \frac{(2n+1)\pi}{2a}, \quad n = integer$$

The differential equation for the time dependent part is

$$\tau''(t) + c^2\lambda_n^2\tau(t) = 0$$

with solution

$$\tau(t) = \alpha_n \cos(c\lambda_n t) + \beta_n \sin(c\lambda_n t)]$$

Any solution of this form with arbitrary integer values of n thus solves the boundary value problem and the differential equation. To proceed further we need to know *initial values* of the ξ and its derivatives. We will later learn how to find the solution as a linear combination of the allowed solutions

$$\xi(x,t) = \sum_{n=1}^{\infty} \cos(\lambda_n x)[\alpha_n \cos(c\lambda_n t) + \beta_n \sin(c\lambda_n t)]$$

The method we have employed to find the standing wave solutions is called *separation of variables.* We will employ this technique often in what follows. In practice, one is often only interested in knowing the frequencies

$$\omega_n = \frac{c(2n+1)\pi}{2a}$$

If we want a detailed solution we must determine α_n and β_n from initial conditions.

REFLECTION AT A BOUNDARY

It is instructive to interprete the boundary conditions at closed or open ends in terms of traveling waves. To do this we write the solution to the wave equation approaching a boundary at

$$x = 0$$

as

$$f_1(ct + x) + f_2(ct - x)$$

to indicate a wave f_1 which moves to the left towards $x = 0$ and then is reflected into a wave f_2 which moves towards positive x.

- If the boundary condition at $x = 0$ is $\xi(0, t) = 0$ we find

  $$f_1(ct) = -f_2(ct)$$

  that is the *reflected wave has the opposite phase of the incoming wave.*

- If we have open end boundary conditions

  $$\frac{\partial \xi(x,t)}{\partial x}|_{x=0} = 0$$

  we find

  $$f_1'(ct) = f_2'(ct)$$

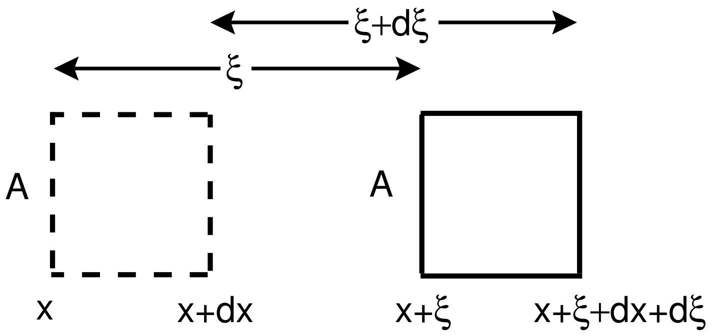
  which we interprete by saying that *the two waves are in phase.*

DAMPING

There will always be some damping of the sound wave due to dissipative effects. (Some of the kinetic energy associated with the organized wave motion will be lost as heat).

To take this into account we go back to the derivation of the wave equation (see section 3.1). In the expression

$$A dx \rho_a \frac{\partial^2 \xi}{\partial t^2} = A c^2 dx \frac{\partial^2 \xi}{\partial x^2} \rho_a \tag{4}$$



the left hand side represented mass times acceleration, while the right hand side represented a restoring force.

It is natural to include the effect of damping by adding a term

$$-Adx\rho_a k\frac{\partial\xi}{\partial t}$$

proportional to the velocity of the volume element opposing the motion, i.e. the constant k should be positive.

The modified wave equation thus becomes:

$$\frac{\partial^2\xi}{\partial t^2}+k\frac{\partial\xi}{\partial t}=c^2\frac{\partial^2\xi}{\partial x^2}$$

To get a slightly different situation from what we had before let us assume that both ends are closed so that the boundary conditions are

$$\xi(0,t)=\xi(a,t)=0$$

Again we can attempt solve this problem by looking for standing wave solutions. Put

$$\xi(x,t)=\phi(x)\xi(t)$$

We find

$$\frac{1}{\phi(x)}\frac{d^2\phi}{dx^2}=\frac{1}{c^2\tau(t)}(\frac{d^2\tau}{dt^2}+k\frac{d\tau}{dt})=c_1=const$$

As before the constant c_1 must be negative and we put $c_1 = -\lambda^2$ and have

$$\phi'' + \lambda^2 \phi = 0$$

with general solution

$$\phi = A\sin(\lambda x) + B\cos(\lambda x)$$

This time the boundary conditions require that $B = 0$ and the eigenvalues are

$$\lambda_n = \frac{n\pi}{a}$$

The differential equation for τ is now

$$\frac{d^2\tau}{dt^2} + k\frac{d\tau}{dt} + c^2\lambda^2\tau = 0$$

This is a differential equation with constant coefficients and it can be solved by trying solutions on the form

$$\tau(t) \propto e^{\gamma t}$$

Substituting into the differential equation gives the *characteristic equation*

$$\gamma^2 + k\gamma + \lambda_n^2 c^2 = 0$$

If the damping is not too large this equation has two complex conjugate solutions

$$\gamma = -\frac{k}{2} \pm i\sqrt{\lambda_n^2 c^2 - \frac{k^2}{4}}$$

The general solution to the differential equation for τ is then

$$\tau(t) = e^{-kt/2}[\alpha\sin(t\sqrt{\lambda^2 c^2 - \frac{k^2}{4}}) + \beta\cos(t\sqrt{\lambda^2 c^2 - \frac{k^2}{4}})]$$

We will later show how to express the solution to the partial differential equations as a *Fourier series*

$$\xi(x,t) = \sum_{n=1}^{\infty}\sin(\lambda_n x)e^{-kt/2}[\alpha_n\sin(t\sqrt{\lambda_n^2 c^2 - \frac{k^2}{4}}) + \beta_n\cos(t\sqrt{\lambda_n^2 c^2 - \frac{k^2}{4}})]$$

where the coefficients α_n and β_n must be determined from the initial conditions.

SOURCE TERM
As our final example let us consider the inhomogeneous wave equation

$$\frac{\partial^2 \xi}{\partial t^2} - c^2 \frac{\partial^2 \xi}{\partial x^2} + k\frac{\partial \xi}{\partial t} = f(x)\cos(\omega t)$$

i.e there is a source of sound with spatial extent $f(x)$ generating waves at the frequency ω. Again, we will find that this problem can be solved using the method of Fourier series expansion.

SUMMARY
We have

- solved a typical boundary value problem for the one-dimensional wave equation in terms of standing waves
- showed how the standing wave solution could be re-expressed in terms of traveling waves
- analyzed the open and closed end boundary condition for traveling waves in terms of reflected waves
    - at a closed end the reflected wave is phase shifted by π
    - at an open end the reflected wave is in phase with the incoming wave
- generalized the wave equation to include the effect of damping
- also generalized the equation to include the effect of a source term.

PROBLEMS

Problem 3.2:1
a: A triangular sound pulse

$$\xi(x+ct) = \begin{cases} 0 & for \quad x+ct<0 \\ x+ct & \quad 0<x+ct<b \\ 2b-x-ct & \quad b<x+ct<2b \\ 0 & 2b<x+ct \end{cases}$$

is traveling down a tube. As the pulse reaches an open end it is reflected and returns in the direction of positive x. Sketch the shape of the pulse for times $t = b/c,\ 2b/c; 4b/c$ after the front of the pulse has reached the open end.
b: Solve the same problem if the end is closed.

Problem 3.2:2
Solve **Problem 3.1:3** using the method of separation of variables.

Problem 3.2:3
a: Find the general solution to the fourth order differential equation

$$a^4 \frac{d^4 u}{d\,x^4} - \omega^2 u = 0$$

b: The transverse vibrations of a thick rod can be shown to satisfy the fourth order partial differential equation

$$a^4 \frac{\partial^4 u}{\partial x^4} + \frac{\partial^2 u}{\partial t^2} = 0$$

Find solutions on the separated variable form for the vibrations.
c: If the ends are clamped down *both* u and $\partial u/\partial x$ vanish. Find an equation that the frequencies of vibration must satisfy for a rod of length L with both ends clamped down.

Problem 3.2:4
a:
For which values of λ will

$$\frac{d^2\phi}{dx^2} + 2\frac{d\phi}{dx} + \lambda^2\phi = 0$$

have solutions satisfying

$$\phi(0) = \phi(\pi) = 0$$

Sketch the eigenfunctions corresponding to the three lowest eigenvalues.
b:
Find the three lowest eigenvalues and sketch the eigenfunctions of the problem

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0;\ 0 < x < a$$

3.3 The potential equation

LAST LECTURES
Concluded discussion of *wave equation.*

TODAY
Wish to move on to the *potential equation* (also called the Laplace equation). In two dimension this equation can be written

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

while in three dimensions we write

$$\frac{\partial^2 u(x,y,z)}{\partial x^2} + \frac{\partial^2 u(x,y,z)}{\partial y^2} + \frac{\partial^2 u(x,y,z)}{\partial z^2} = 0$$

Either equation may be written

$$\nabla^2 u = 0$$

and we will have to learn how to express the *Laplacian*

$$\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$$

in different ways such as in polar, cylindrical or spherical coordinates.

ORIGIN OF LAPLACE EQUATION
We will frequently need to be able to solve the Laplace equation as a stepping-stone to the solution to a more complicated problem:
E.g. in *electrostatics* one finds that in a region of space in which there are no charges the *electrostatic potential* satisfies

$$\nabla \cdot \vec{E} = 0$$

where

$$\vec{E} = -\nabla V$$

giving

$$\nabla \cdot \nabla V = \nabla^2 V = 0$$

Of course, electrostatics would be uninteresting if there weren't any charges around. However, to solve Poisson's equation

$$\nabla^2 V = -\frac{\rho}{\epsilon_0}$$

one needs to be able to solve the homogenous equation.

FUNCTIONS OF A COMPLEX VARIABLE

The 2-D Laplace equation plays a special role in the theory of complex variables. Suppose $V(z)$ is a differentiable function of a complex variable

$$z = x + iy$$

then

$$\frac{\partial V}{\partial x} = \frac{dV}{dz}$$
$$\frac{\partial V}{\partial y} = i\frac{dV}{dz}$$

and

$$\frac{\partial^2 V}{\partial x^2} = \frac{d^2 V}{dz^2}$$
$$\frac{\partial^2 V}{\partial y^2} = -\frac{d^2 V}{dz^2}$$

Hence

$$\frac{\partial^2 V}{\partial x^2} + \frac{\partial^2 V}{\partial y^2} = 0$$

In the case of the wave equation in one dimension we argued that the general solution could be written $f_1(x - ct) + f_2(x + ct)$, as a of linear combinations of traveling waves. Similarly the general solution to Laplace equation in 2 dimensions can be written

$$f_1(x + iy) + f_2(x - iy) = f_1(z) + f_2(z^*)$$

where z^* is the complex conjugate (see section (2.1) of $z = x + iy$ and f_1 and f_2 can be differentiated twice.

FINITE DIFFERENCES

To get a feeling for what the Laplace equation "does" let us attempt to solve

it approximately by the *method of finite differences.*
We showed in 2.3 that we can use the values of a function on three neighboring points

$$x - \Delta, x, x + \Delta$$

and the method of Taylor expansion

$$f(x - \Delta) \approx f(x) - \Delta f' + \frac{\Delta^2}{2} f''$$

$$f(x + \Delta) \approx f(x) + \Delta f' + \frac{\Delta^2}{2} f''$$

to get approximate expressions for the derivatives

$$f' = \frac{1}{2\Delta}(f(x + \Delta) - f(x - \Delta))$$

$$f'' = \frac{1}{\Delta^2}(f(x + \Delta) + f(x - \Delta) - 2f(x))$$

Similarly in two dimensions

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} \approx$$

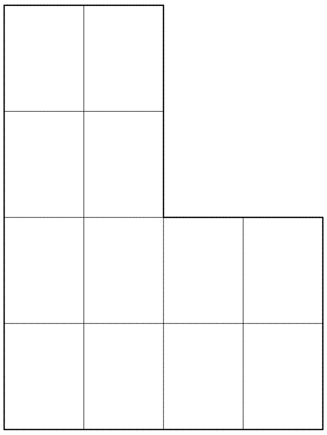
$$\frac{1}{\Delta_x^2}(f(x+\Delta_x, y)+f(x-\Delta_x, y)-2f(x))+\frac{1}{\Delta_y^2}(f(x, y+\Delta_y)+f(x, y-\Delta_y)-2f(x))$$

Hence,*we can find an approximate solution to the Laplace equation by averaging over the surrounding points.*
This result can be used to solve Laplace equation numerically just as we did in the boundary value problem in (2.3).

EXAMPLE
Suppose we want to find the solution to the Laplace equation

$$\nabla^2 f = 0$$

in an L-shaped region.



To specify the problem we assume the function is known at the boundary Let us crudely discretize the problem

$$\begin{array}{ccccc} & 1 & & & \\ 2 & 3 & 4 & & \\ 5 & 6 & 7 & 8 & \\ 9 & 10 & 11 & 12 & 13 \\ & 14 & 15 & 16 & \end{array}$$

Here

$$1, 2, 4, 5, 7, 8, 9, 13, 14, 15, 16$$

are *exterior points* where the function is known.

$$3, 6, 10, 11, 12$$

are *interior points* where we wish to find f.

The approximate solution is then obtained by solving the set of equations

$$f_3 = \frac{1}{4}(f_1 + f_2 + f_4 + f_6)$$

$$f_6 = \frac{1}{4}(f_3 + f_5 + f_7 + f_{10})$$

$$f_{10} = \frac{1}{4}(f_6 + f_9 + f_{11} + f_{14})$$

$$f_{11} = \frac{1}{4}(f_7 + f_{15} + f_{10} + f_{12})$$

$$f_{12} = \frac{1}{4}(f_8 + f_{11} + f_{13} + f_{16})$$

Of course, if one wishes to find an accurate solution it is necessary to use a finer mesh. The generalization to three dimensions is

$$\frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$$

$$= \frac{1}{\Delta^2}(f(x+\Delta, y, z) + f(x, y+\Delta, z) + f(x, y, z+\Delta)$$
$$+f(x-\Delta, y, z) + f(x, y-\Delta, z) + f(x, y, z-\Delta) - 6f(x))$$

An important consequence of the averaging property of solutions to the Laplace equations is that *maxima and minima only occurs at boundaries, never in interior regions.*

LAPLACE EQUATION IN DIFFERENT COORDINATE SYSTEMS

When solving boundary value problems in more than one dimension it is often necessary to use other coordinate systems than the Cartesian. It is then important to be able to express the Laplacian operator in these coordinate systems. We first consider

POLAR COORDINATES

$$x = r\cos\theta$$

$$y = r\sin\theta$$

Laplace's equation in this coordinate system can be shown to be:

$$\nabla^2 u(r,\theta) = \frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}$$

PROOF: Let us for now apply the convention that subscript implies partial differentiation e.g.

$$u_r \equiv \frac{\partial u}{\partial r}$$

Applying the chain rule we find

$$u_x = u_r r_x + u_\theta \theta_x$$

$$\begin{aligned} u_{xx} &= (u_r)_x r_x + u_r r_{xx} + (u_\theta)_x \theta_x + u_\theta \theta_{xx} \\ (u_r)_x r_x &= u_{rr}(r_x)^2 + u_{r\theta}\theta_x r_x \\ (u_\theta)_x \theta_x &= u_{\theta r} r_x \theta_x + u_{\theta\theta}(\theta_x)^2 \end{aligned}$$

We have

$$r = \sqrt{x^2 + y^2}$$

$$\theta = \tan^{-1}\frac{y}{x}$$

and

$$r_x = \frac{x}{\sqrt{x^2+y^2}} = \frac{x}{r}$$

$$r_{xx} = \frac{1}{r} - \frac{x^2}{r^2} = \frac{y^2}{r^3}$$

$$\theta_x = \frac{1}{1+(\frac{y}{x})^2}(\frac{-y}{x^2}) = \frac{-y}{r^2}$$

$$\theta_{xx} = -y(-\frac{2}{r^3})r_x = \frac{2xy}{r^4}$$

Collecting terms

$$u_{xx} = \frac{x^2}{r^2}u_{rr} + \frac{y^2}{r^4}u_{\theta\theta} - \frac{2xy}{r^3}u_{r\theta} + \frac{y^2}{r^3}u_r + \frac{2xy}{r^4}u_\theta$$

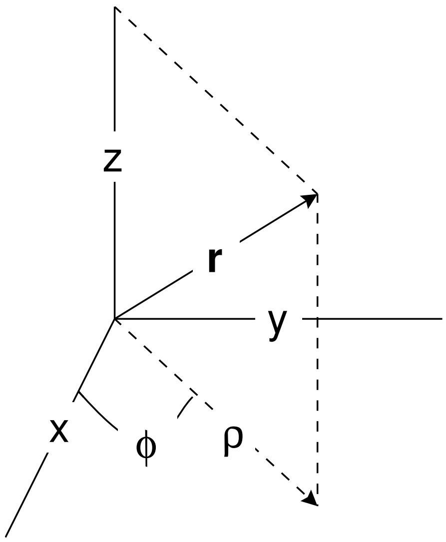
Similarly we can show that

$$u_{yy} = \frac{y^2}{r^2}u_{rr} + \frac{x^2}{r^4}u_{\theta\theta} + \frac{2xy}{r^3}u_{r\theta} + \frac{x^2}{r^3}u_r - \frac{2xy}{r^4}u_\theta$$

Again, collecting terms

$$\nabla^2 u = u_{xx} + u_{yy} = \frac{\partial^2 u(r,\theta)}{\partial r^2} + \frac{1}{r}\frac{\partial u(r,\theta)}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u(r,\theta)}{\partial \theta^2}$$

which is the desired result!



CYLINDRICAL COORDINATES

It is easy to generalize the result for polar coordinates to cylindrical coordinates

$$x = \rho\cos\phi$$

$$y = \rho\sin\phi$$

$$z = z$$

$$\nabla^2 u(\rho,\phi,z) = \frac{\partial^2 u(\rho,\phi,z)}{\partial\rho^2} + \frac{1}{\rho}\frac{\partial u(\rho,\phi,z)}{\partial\rho}$$

$$+\frac{1}{\rho^2}\frac{\partial^2 u(\rho,\phi,z)}{\partial\phi^2} + \frac{\partial^2 u(\rho,\phi,z)}{\partial z^2}$$

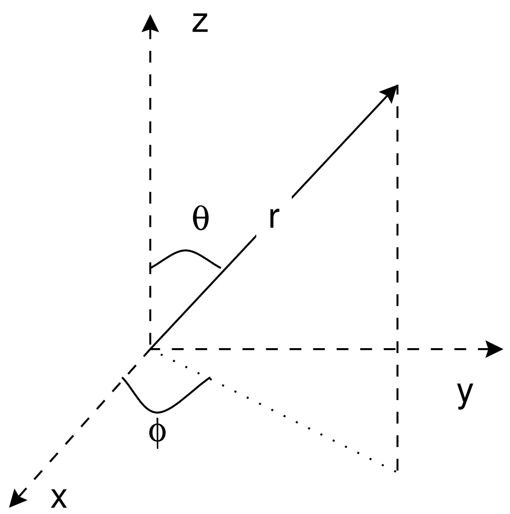
SPHERICAL COORDINATES

Finally we give without proof the result for Laplace's equation in spherical coordinates:

$$\nabla^2 u(r,\theta,z) = \frac{1}{r^2}[\frac{\partial}{\partial r}(r^2\frac{\partial u(r\theta,\phi)}{\partial r})$$

$$+\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u(r,\theta,\phi)}{\partial\theta}) + \frac{1}{\sin^2\theta}\frac{\partial^2 u(r,\theta,\phi)}{\partial\phi^2}]$$

(You will find proofs in Riley et al. or Arfken and Weber).



SUMMARY
We have

- started our discussion of the Laplace equation
- discussed some typical situation where it arises
- demonstrated a connection with the theory of complex variables
- discussed numerical solutions using finite differences
- written down expressions for the Laplacian in different coordinate systems.

PROBLEMS

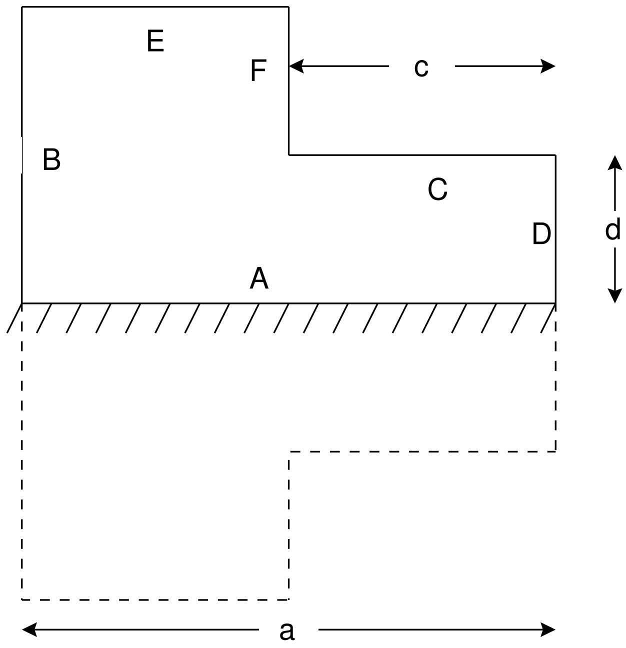
Problem 3.3:1
Solve Laplace equation

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$$

in an $L-$shaped region (see figure)

Use the following constants:

$$a = 16; \quad b = 8; \quad c = 8; \quad d = 4$$



Use a coordinate system where the origin is the lower left hand corner where sides A and B meet. The $x-$axis is along A and the $y-$axis along B. Along the side A the boundary condition is

$$\frac{\partial u}{\partial y} = 0$$

along the side B we have

$$u(0, y) = y$$

along E we have

$$u(x, b) = b + x$$

while along F

$$u(a - c, y) = a - c + b$$

and along C

$$u(x, d) = x + b$$

and finally along D

$$u(a, y) = a + b$$

Use the finite difference method to find u inside the $L-$shaped region using a mesh of size 1×1. and plot the result.

Problem 3.3:2
Solve the same problem as the one above except

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 1$$

The differential equation corresponds to the Poisson equation for the electrostatic potential with a uniform negative charge distribution. Plot your result.

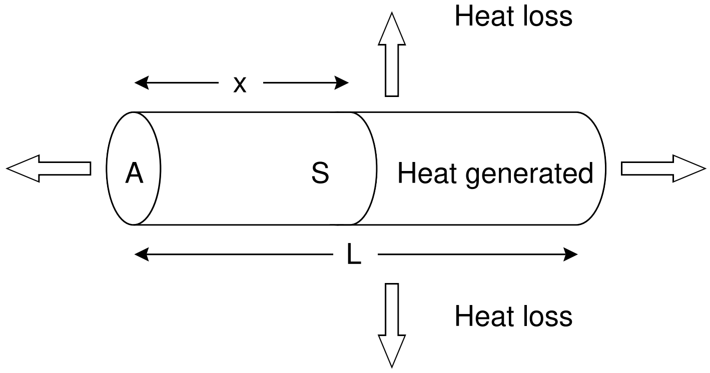
3.4 Heat equation in one dimension. Separation of variables.

LAST TIME
Discussed the Laplace equation.

Today we wish to derive time dependent heat conduction equation and discuss some properties of solutions to that equation.

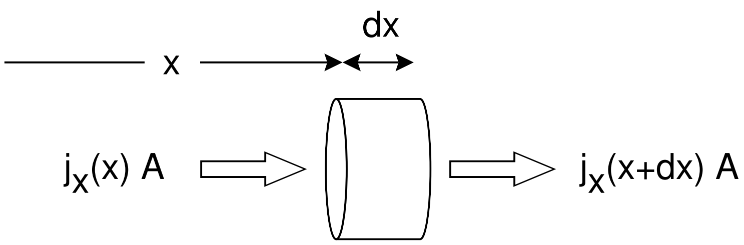
We have earlier (section 2.3) discussed the steady state problem.

DERIVATION OF TIME DEPENDENT EQUATION



The temperature now depends on both x and t, but is still assumed constant over cross-sectional area A!

$L =$ length of rod; $S =$ area of cylinder
$T_0 =$ ambient temperature; $T(x) =$ temperature along rod
$\dot{Q}$ =heat generated per unit volume and time
$j_x =$ axial heat flow per unit time and area
$j_S =$ radial heat flow per unit time and area
$C_V = \frac{dQ}{dT}$ bulk heat capacity of rod
$\lambda = \frac{j_S}{T-T_0} =$ coefficient of heat loss
$\kappa = -\frac{j_x}{dT/dx} =$ thermal conductivity



The axial flow of heat into the element dx is

$$-Aj_x(x+dx) + Aj_x(x) = -Adx\frac{\partial j_x}{\partial x}$$

$$= Adx\kappa\frac{\partial^2 T}{\partial x^2} \equiv Adx\dot{Q}_{in}$$

The radial heat flow out of the element is

$$\frac{dx\lambda(T-T_0)S}{L} \equiv Adx\dot{Q}_{out}$$

The difference causes a rise in temperature per unit time

$$\kappa\frac{\partial^2 T}{\partial x^2} + \dot{Q} - \dot{Q}_{out} = C_V\frac{\partial T}{\partial t}$$

If we neglect radial losses and heat generation we get the *heat equation.*

$$\kappa\frac{\partial^2 T}{\partial x^2} = C_V\frac{\partial T}{\partial t}$$

BOUNDARY AND INITIAL CONDITIONS

Let us define $k = \kappa/C_V$. The heat equation then becomes

$$\boxed{k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}}$$

On *physical grounds* we expect an unique solution if we specify

- Initial temperature distribution.
- What happens at the boundaries.

Some common boundary conditions:

- Temperature held fixed at ends $\to$ *Dirichlet boundaries.*
- Heat flow constant at ends $\to$ *Neumann boundaries.*
- Mixed boundary conditions (Dirichlet at one end Neumann at the other).
- Heat flow at ends governed by temperature difference with surroundings $\to$ *convective boundaries.*

If there is heat generated, or radial heat transfer, add an inhomogeneous term to differential equation.

There are a number of important situations which are governed by equations of simular structure (linear equations containing second spatial derivatives together first order time derivatives) for example

- The *Wave function* of a particle in quantum mechanics is governed by the Schrödinger equation.
- The concentration of particles subject to random collisions with other particles is governed by *diffusion equation.*
- The probability distribution for the position of a particle subject to a random force can be described by the *Fokker-Planck equation.*

We can approach the problem of solving the heat equation (or related problems) by finite element methods. The approach is somewhat similar to the ones used for boundary value ordinary differential equations (section 2.3) and in potential problems (section 3.3). However, because of problems with numerical stability, the heat equation is more difficult to handle this way, and we will not pursue this method here. For a discussion see *Numerical Recipes*[9]. Today we will approach the problem by *separating the variables* as in (section 3.2).

PLAN OF ATTACK

Step 1: Solve steady state problem $T_S(x)$.

Step 2: Redefine problems in terms of

$$u(x,t) = T - T_S(x)$$

Step 3: Try to find solutions to the differential equations in which the variables are separated (as we did for the standing wave solutions to the wave equation):

$$u(x,t) = \tau(t)\phi(x)$$

We will find that this leads to ordinary differential equations with a separation of variable constant.

Step 4: Determine the separation of variable constant by considering the boundary value conditions at the ends. This will lead to an expression for the solution, typically in the form of a *Fourier series.*

Step 5: Determine the coefficients of the series from the initial conditions.

EXAMPLE: FIXED END TEMPERATURES

We wish to solve the heat conduction problem

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

with the temperature held fixed at

$$T = T_0 \; for \; x = 0$$

$$T = T_1 \; for \; x = a$$

Initially ($t = 0$) the temperature distribution is

$$T(x, 0) = f(x)$$

where $f(x)$ is some known function.

Wanted:

$$T(x, t) \; for \; t > 0$$

THE STEADY STATE

The steady state temperature T_S is a function of x alone and hence satisfies

$$k \frac{d^2 T_s}{dx^2} = 0$$

The general solution to this equation is

$$T_S(x) = C_1 + C_2 x$$

If

$$T_S(0) = T_0; \;\; T_S(a) = T_1$$

we can solve for C_0 and C_1 to get

$$T_S(x) = T_0 + (T_1 - T_0)\frac{x}{a}$$

Let us define the new temperature variable

$$u(x, t) = T(x, t) - T_S(x)$$

with initial condition

$$u(x, 0) = h(x) = f(x) - T_S(x)$$

The function $u(x,t)$ satisfies the same partial differential equations as $T(x,t)$

$$k\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{\partial u}{\partial t}$$

but the boundary conditions are now *homogeneous*:

$$u(0,t) = u(a,t) = 0 \ \ for\ all\ t$$

As we shall see homogeneous boundary conditions makes the problem much easier to solve!

SEPARATION OF VARIABLES
The next step is to try to find solutions of the on the form

$$u(x,t) = \tau(t)\phi(x)$$

where $\tau(t)$ depends on t *only*
$\phi(x)$ depends on x *only*
Substitute into the partial differential equation:

$$\tau(t)\frac{d^2\phi}{dx^2} = \frac{1}{k}\frac{d\tau(t)}{dt}\phi(x)$$

We divide both sides by $\phi\tau$ to find

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = \frac{1}{k\tau}\frac{d\tau}{dt} = c_1 = const$$

The first term can only depend on x, while the second term can only depend on t. Hence, their value must be *independent of both x and t.*
We solve the temporal equation to obtain

$$\tau(t) = c_2 \exp(c_1 k t)$$

Solutions that grow exponentially in time are not acceptable on physical grounds. The constant c_1 must therefore be *negative.* To remind us of this put

$$c_2 = -\lambda^2$$

The differential equation for ϕ thus becomes

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0$$

with general solution

$$\phi_\lambda(x) = \alpha \sin \lambda x + \beta \cos \lambda x \tag{5}$$

In the case of fixed temperature boundary conditions we require that $\phi = 0$ at both ends, and put $\beta = 0$. We are interested in solutions

$$\phi_n = \sin \lambda_n x; \quad \lambda_n = \frac{n\pi}{a}, n = \textit{positive integer}$$

and we see that the *eigenfunctions* $\phi_n(x)$ with *eigenvalues* λ_n can be used to build up solutions of the form

$$u(x,t) \sum_{n=1}^{\infty} \alpha_n \phi_n(x) \exp(-\lambda_n^2 kt)$$

using the initial condition, where the coefficients α_n remains to be determined. This is an example of a Fourier series expansion, which is our next major topic.

$$T(x,t) = T_s + \sum_{n=0}^{\infty} \alpha_n \exp\left(k\frac{n^2\pi^2 t}{a^2}\right) \sin\frac{n\pi x}{a}$$

- We note that the time dependent solution decays exponentially towards the steady state solution.
- The higher order terms in Fourier expansion decay the fastest.
- Non-steady temperature distributions smoothen out before they decay!

We still have to solve the problem of finding the coefficients α_n!

PROBLEMS

Problem 3.4:1
Consider the conduction problem for $0 < x < a$, $t > 0$, $S > 0$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$\left.\frac{\partial u(x,t)}{\partial x}\right|_{x=0} = 0,\ \left.\frac{\partial u(x,t)}{\partial x}\right|_{x=a} = S,\ u(x,0) = 0$$

a: Give a physical interpretation of the problem. Will it approach a steady state?
b:
Show that

$$\nu(x,t) = \frac{S}{2a}(x^2 + 2kt)$$

satisfies the heat equations and boundary conditions (but not the initial condition).
c:
In order t satisfy the initial condition try a solution

$$u(x,t) = \nu(x,t) + w(x,t)$$

Find the differential equation, boundary and initial conditions that $w(x,t)$ must satisfy. Will $w(x,t)$ approach a steady state. What is it?

Problem 3.4:2
Show that the four "heat polynomials"

$$u_0 = 1,\ u_1 = x,\ u_2 = x^2 + 2kt,\ u_3 = x^3 + 6kxt$$

are solutions to the heat equation. Find a linear combination of them that satisfies the boundary conditions $u(0,t) = 0,\ u(a,t) = t$.

Problem 3.4:3
In which of the following cases can solutions be found by the method of separation of variables?

$$\frac{\partial^2 u}{\partial x^2} + x\frac{\partial^2 u}{\partial t^2} = 0 \tag{6}$$

$$\frac{\partial^2 u}{\partial x^2} + t^2\frac{\partial^2 u}{\partial t^2} = 0 \tag{7}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial t^2} = e^{-x} \tag{8}$$

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = e^{-x} + e^{-y} \tag{9}$$

In cases where the variables don't separate directly, transform the equations so that product solutions can be found. You don't need to solve the resulting ordinary differential equations.

3.5 Sturm-Liouville Problem

LAST TIME

- We formally solved a time dependent heat conduction problem with *fixed end temperatures* (Dirichlet boundary conditions).
- The first step in the method was to solve the steady state problem.
- The non-steady solution was expanded as a series.

Before we get into the details of Fourier series. we today consider more general boundary conditions!

EXAMPLE INSULATED BAR
The boundary conditions are now

$$\frac{\partial T(x,t)}{\partial x} = 0; \quad for\ x = 0; \quad x = a$$

(no heat flow through the ends.)
The initial condition is

$$T(x,0) = f(x)$$

where $f(x)$ is some known function. The boundary condition at the ends is now homogeneous. We shall see this means that there is no need to subtract a steady state solution. (It is intuitively obvious that the steady state solution is a uniform temperature equal to the average initial temperature throughout the bar.) As before we look for solutions

$$T(x,t) = \tau(t)\phi(x)$$

where $\tau(t)$ depends on t *only*
$\phi(x)$ depends on x *only*
Substitute into the partial differential equation: and divide both sides by $\phi\tau$ to find

$$\frac{1}{\phi}\frac{d^2\phi}{dx^2} = \frac{1}{k\tau}\frac{d\tau}{dt} = c_1 = const$$

As before the first expression can only depend on x, while the second only depends on t, hence both must be constant. Solutions to the temporal equation that grow exponentially in time are not acceptable on physical grounds. The constant c_1 must therefore be *negative* and again we put

$$c_2 = -\lambda^2$$

We find for $\tau(t)$

$$\tau(t) = const.e^{-\lambda^2 kt}$$

The differential equation for ϕ is

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0$$

with general solution

$$\phi_\lambda(x) = \alpha \sin \lambda x + \beta \cos \lambda x \tag{10}$$

If we require that

$$\frac{d\phi}{dx} = 0,\ for\ x = 0;\ \ x = a$$

the boundary conditions are automatically satisfied. We pick from the general solution (10) those with

$$\alpha = 0,\ \lambda_n = \frac{n\pi}{a}, n = 0\ or\ a\ positive\ integer$$

The solution $T(x,t)$ is now

$$T(x,t) = \sum_{n=0}^{\infty} \beta_n \exp(-\lambda_n^2 kt)\cos(n\pi x/a)$$

where the coefficients β_n must be determined by making a cosine expansion of the initial condition (something I haven't yet told you how to do). Note that since $\cos(0) = 1 \neq 0$ the sum now extends from $0 \to \infty$. Negative values are uninteresting since $\cos(x) = \cos(-x)$ The $n = 0$ term in the sum represents the steady state solution, while the $n \neq 0$ terms decay with time.

EIGENFUNCTIONS AND EIGENVALUES

Let us next consider the more general problem

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0, \;\; l < x < r$$

where the *eigenfunctions* are required to satisfy the boundary conditions

$$a_l\phi(l) - b_l\frac{d\phi(x)}{dx}|_{x=l} = 0$$

$$a_r\phi(l) - b_r\frac{d\phi(x)}{dx}|_{x=r} = 0$$

and we wish to find the *eigenvalues* λ_n for which solutions exist.

ORTHOGONALITY OF EIGENFUNCTIONS

An extremely important general result is that the eigenfunctions that correspond to different eigenvalues are *orthogonal*. By this we mean that

$$\boxed{\int_l^r \phi_n(x)\phi_m(x)dx = 0, \; if \lambda_n \neq \lambda_m}$$

To prove this, first note that the left boundary conditions require that

$$a_l\phi_n(l) - b_l\phi_n'(l) = 0$$

$$a_l\phi_m(l) - b_l\phi_m'(l) = 0 \qquad (11)$$

where we use the notation

$$\phi\prime \equiv \frac{d\phi}{dx}$$

From (11) we see that the determinant

$$\begin{vmatrix} \phi_n(l) & -\phi_n'(l) \\ \phi_m(l) & -\phi_m'(l) \end{vmatrix} = 0$$

Hence

$$-\phi_n(l)\phi'_m(l) + \phi_m(l)\phi'_n(l) = 0$$

Similarly at the other end

$$-\phi_n(r)\phi'_m(r) + \phi_m(r)\phi'_n(r) = 0$$

From the differential equation we have

$$\phi''_m = -\lambda_m^2 \phi_m$$

$$\phi''_n = -\lambda_n^2 \phi_n$$

Multiplying the first equation above by ϕ_n and the second by ϕ_m and integrating over x from l to r we find

$$(\lambda_m^2 - \lambda_n^2) \int_l^r dx \phi_n(x)\phi_m(x) = \int_l^r dx (\phi_n(x)''\phi_m(x) - \phi_m(x)''\phi_n(x))$$

Integrating the last integral by parts and assuming that

$$\lambda_m^2 - \lambda_n^2 \neq 0$$

we find

$$(\lambda_m^2 - \lambda_n^2) \int_l^r dx \phi_n(x)\phi_m(x) = [-\phi_n(x)\phi'_m(x) + \phi_m(x)\phi'_n(x)]|_l^r = 0$$

q.e.d.

THE STURM LIOUVILLE PROBLEM

We shall later encounter boundary value problems of the same type as before

$$a_l\phi(l) - b_l \frac{d\phi(x)}{dx}|_{x=l} = 0$$

$$a_r\phi(r) - b_r \frac{d\phi(x)}{dx}|_{x=r} = 0 \tag{12}$$

but where the differential equation is the more complicated

$$\frac{d}{dx}(s(x)\frac{d\phi}{dx}) - q(x)\phi(x) + \lambda^2 p(x)\phi$$

where $s(x), q(x), p(x)$ are known functions. The problem of finding the eigenfunctions and eigenvalues in this case is called the *Sturm-Liouville problem.* We will show in (section 4.5): that in the general case the orthogonality relations are a bit more complicated

$$\int_l^r dxp(x)\phi_n(x)\phi_m(x) = 0, \quad \lambda_n^2 \neq \lambda_m^2$$

We will encounter many examples later on, especially when we move to more than one dimension.
SUMMARY

- Illustrated the use of method of separation of variables by considering heat equation. with fixed temperature and with insulated end boundary conditions.
- Formulated the Sturm Liouville problem.
- Discussed eigenfunctions and eigenvalues.
- Demonstrated orthogonality relations for the eigen-functions.

PROBLEMS

Problem 3.5:1
Find the eigenvalues and sketch the first three eigenfunctions of the problem

$$\phi'' + \lambda^2\phi = 0; \; 0 < x < a$$

a:

$$\phi'(0) = 0; \phi(a) = 0$$

b:

$$\phi(0) - \phi'(0) = 0; \phi(a) + \phi'(a) = 0$$

In case **b:** you will end up with a transcendental equation which can only be solved numerically. To do this choose a=1. If you wish to solve the equation with Maple the command

$$fsolve(equation, x = lowerlimit..upperlimit)$$

searches for a numerical solution of the equation for x in the specified interval. It only finds one solution. If there are several solutions, you must narrow the search interval and repeat the procedure to find them all.

Problem 3.5:2
The concentration c of a certain chemical in a column of water as a function of height z and time is given by the diffusion equation

$$D\frac{\partial^2 c}{\partial z^2} = \frac{\partial c}{\partial t}$$

where D is the diffusivity. The boundary conditions are

$$\frac{\partial c}{\partial z} = 0, \quad for\ z = 0,\ z = h$$

where h is the height of the water column. The initial concentration is

$$c(z) = c_0 \sin^2(\frac{\pi z}{2h})$$

a: Find the steady state concentration. **b:** Find the concentration as a function of height and time. **c:** Find the time taken for the concentration at any point to reach the mean of the initial and steady state values.
The trigonometric identity

$$\sin^2(x) = \frac{1}{2}(1 - \cos(2x))$$

may prove handy.

Problem 3.5:3
a: Show that the function

$$\phi(x, y) = \sin(\pi x)\sin(2\pi y) - \sin(2\pi x)\sin(\pi y)$$

are solutions to the differential equation

$$\frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} = -\lambda\phi$$

with the boundary condition, for a certain value of λ

$$\phi = 0$$

on the boundary of the triangle T bounded by the lines

$$y = 0,\ y = x,\ x = 1$$

What is the value of λ(eigenvalue) associated with this solution?
b: Try to guess some other solutions with other eigenvalues λ.

4 Fourier methods

4.1 Fourier sine and cosine series

LAST TIMES

- Attacked a one dimensional heat conduction problem by applying the method of separation of variables.
- Found ordinary differential equation whose solutions were trigonometric functions. The general solution could then be expressed in terms of trigonometric series or *Fourier series.*
- Formulated the Sturm-Liouville problem and proved some orthogonality relationships.

TODAY

- Start discussing Fourier methods
- Technique named after Jean Baptiste Fourier (1768-1830) who used them to solve heat conduction problems.
- Many other applications in
  - optics

- image analysis
- electrical networks
- spectral analysis
- crystallography
- probability theory.

- Three different approaches
  - Fourier series ⇒ used for periodic functions and functions with finite range.
  - Discrete Fourier series ⇒ used for functions defined on a set of points.
  - Fourier integral ⇒ used for functions defined for $-\infty < x < \infty$, or $0 < x < \infty$.

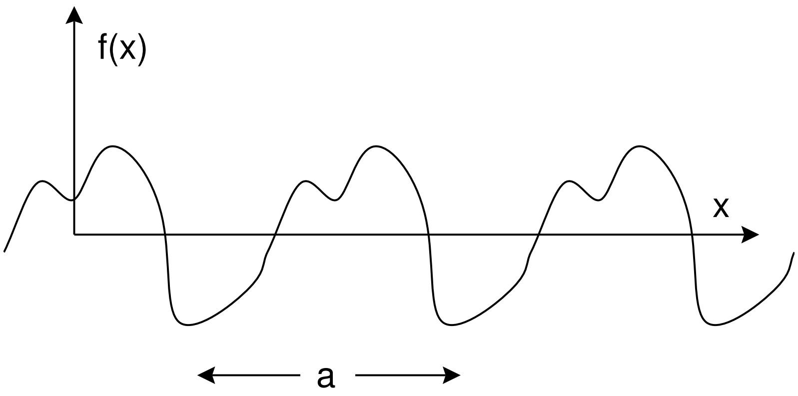
PERIODIC FUNCTIONS

A function $f(x)$ is *periodic* with period a if

$$f(x) = f(x+a)$$

Such a function has infinitely many periods

$$f(x) = f(x+a) = f(x+2a) = f(x+3a) = \cdots$$



The trigonometric functions $\sin(\frac{2\pi x}{a}),\ \cos(\frac{2\pi x}{a})$ are periodic with period a.
If n is integer $\sin(\frac{2n\pi x}{a}),\ \cos(\frac{2n\pi x}{a})$ are also periodic with period a
a doesn't have to be the shortest period!

ORTHOGONALITY
Let us define the *Kronecker-delta* as

$$\delta_{nm} = \begin{cases} 0 & for \quad n \neq m \\ 1 & for \quad n = m \end{cases}$$

where n and m are *integers.*
We next show that

$$\boxed{\begin{gathered} \int_0^a dx \sin\frac{2n\pi x}{a} = 0 \\ \int_0^a dx \cos\frac{2n\pi x}{a} = a\delta_{n0} \\ \int_0^a dx \sin\frac{2n\pi x}{a}\sin\frac{2m\pi x}{a} = \frac{a}{2}\delta_{nm} \\ \int_0^a dx \sin\frac{2n\pi x}{a}\cos\frac{2m\pi x}{a} = 0 \\ \int_0^a dx \cos\frac{2n\pi x}{a}\cos\frac{2m\pi x}{a} = \frac{a}{2}\delta_{nm} \end{gathered}}$$

The above set of equations are commonly referred to as *orthogonality relations.* Since these functions occur as solutions to Sturm-Liouville problems (section 3.5) we expect such relationships. However, since they will prove so important in what follows we will prove them by explicit integration.

To prove these relation first note that for any integer n

$$\cos(2\pi n) = 1;\ \sin(2\pi n) = 0$$

For $n = 0$ we have trivially

$$\int_0^a dx \sin \frac{2n\pi x}{a} = 0$$

$$\int_0^a dx \cos \frac{2n\pi x}{a} = a$$

For $n \neq 0$

$$\int_0^a dx \sin \frac{2n\pi x}{a} = \frac{a}{2\pi n}(1 - \cos(2\pi n)) = 0$$

$$\int_0^a dx \cos \frac{2n\pi x}{a} = \frac{a}{2\pi n}(\sin(2\pi n) - 0) = 0$$

The last three integrals can easily be proved using the trigonometric identities

$$\sin A \sin B = \frac{1}{2}[\cos(A - B) - \cos(A + B)]$$

$$\cos A \cos B = \frac{1}{2}[\cos(A - B) + \cos(A + B)]$$

$$\sin A \cos B = \frac{1}{2}[\sin(A - B) + \sin(A + B)]$$

FOURIER SERIES

An arbitrary function $f(x)$ defined for $0 < x < a$ can be expressed as:

$$\boxed{f(x) = \alpha_0 + \sum_{n=1}^{\infty}[\alpha_n \cos \frac{2n\pi x}{a} + \beta_n \sin \frac{2n\pi x}{a}]}$$

where

$$\boxed{\begin{aligned} \alpha_0 &= \frac{1}{a}\int_0^a f(x)dx \\ \alpha_n &= \frac{2}{a}\int_0^a f(x)\cos\frac{2n\pi x}{a}dx \\ \beta_n &= \frac{2}{a}\int_0^a f(x)\sin\frac{2n\pi x}{a}dx \end{aligned}}$$

The equations for the coefficients $\alpha_n,\ \beta_n$ follow from the orthogonality conditions. The restrictions on the function $f(x)$ for the series to converge to the function are not severe and will be discussed more in (section 4.3). The main restriction is that the function should be bounded and piecewise continuous i.e. have only a finite number of discontinuities. At discontinuities the series converges conditionally to the average value

$$\frac{1}{2}(f(x+0)+f(x-0))$$

we will come back later to the question of completeness of the expansion when discussing the Dirac $\delta-$function (section 6.1).

For examples of evaluation of Fourier coefficients see the maple worksheet at http://www.physics.ubc.ca/˜birger/p312l6.html.

PERIODIC EXTENSION

Suppose we define a function $f(x)$ for the range

$$0 < x < a$$

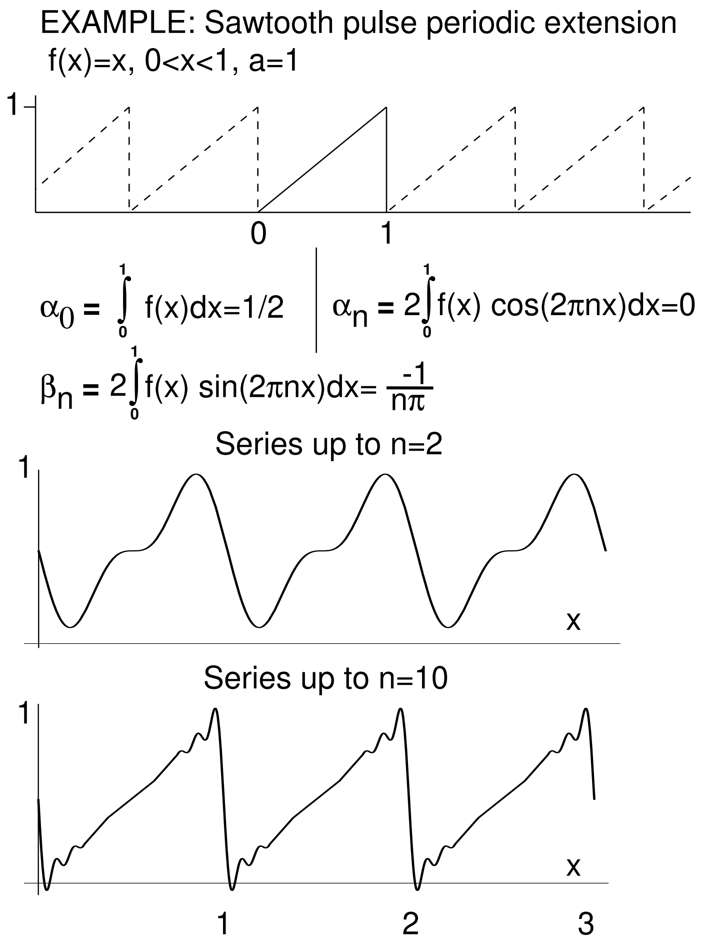
and compute the coefficients of the Fourier series

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty}[\alpha_n \cos\frac{2n\pi x}{a} + \beta_n \sin\frac{2n\pi x}{a}]$$

What happens if we substitute values for x outside the range?

Since $\sin(2n\pi/a)$ and $\cos(2n\pi/a)$ are periodic function with period a the Fourier series will evaluate to the *periodic extension*

$$f(x+na) = f(x) \quad for \;\; n = \pm 1, \pm 2 \cdots$$



EVEN EXTENSION: **Cosine series**

The periodic extension is not the only way to extend the function f(x) beyond the range

$$0 < x < a$$

An alternative is

$$f(x) = f(-x); \quad f(x \pm 2a) = f(x)$$

The period is now $2a$ instead of a. Since

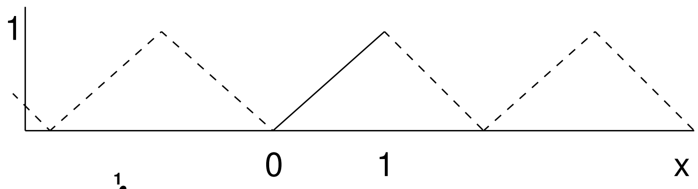
$$\sin(-x) = -\sin(x); \quad \cos(-x) = \cos(x)$$

all the sine terms in the Fourier series expansion are zero. We find

$$\boxed{f(x) = \alpha_0 + \sum_{n=1}^{\infty} \alpha_n \cos \frac{n\pi x}{a}}$$

$$\boxed{\begin{aligned} \alpha_0 &= \frac{1}{a}\int_0^a f(x)dx \\ \alpha_n &= \frac{2}{a}\int_0^a f(x)\cos\frac{n\pi x}{a}dx \end{aligned}}$$

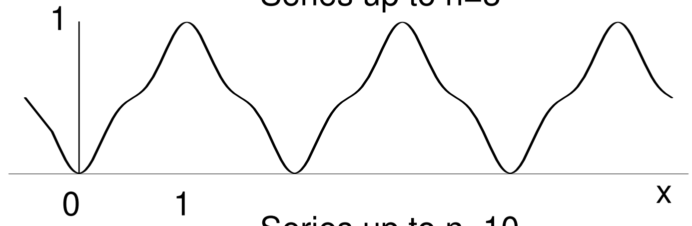
EXAMPLE: Sawtooth pulse even extension
$f(x)=x,\ 0<x<1,\ a=1$



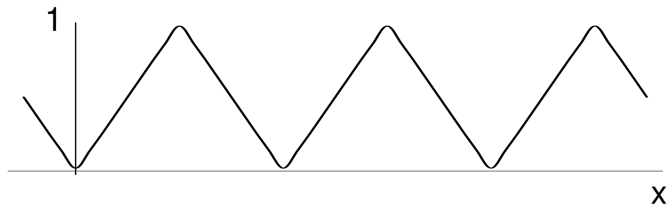
$$\alpha_0 = \int_0^1 f(x)dx = 1/2$$

$$\alpha_n = \int_{-1}^{1} f(x)\cos(\pi n x)dx = \begin{cases} 0, & n=\text{even} \\ \dfrac{4}{n^2\pi^2}, & n=\text{odd} \end{cases}$$

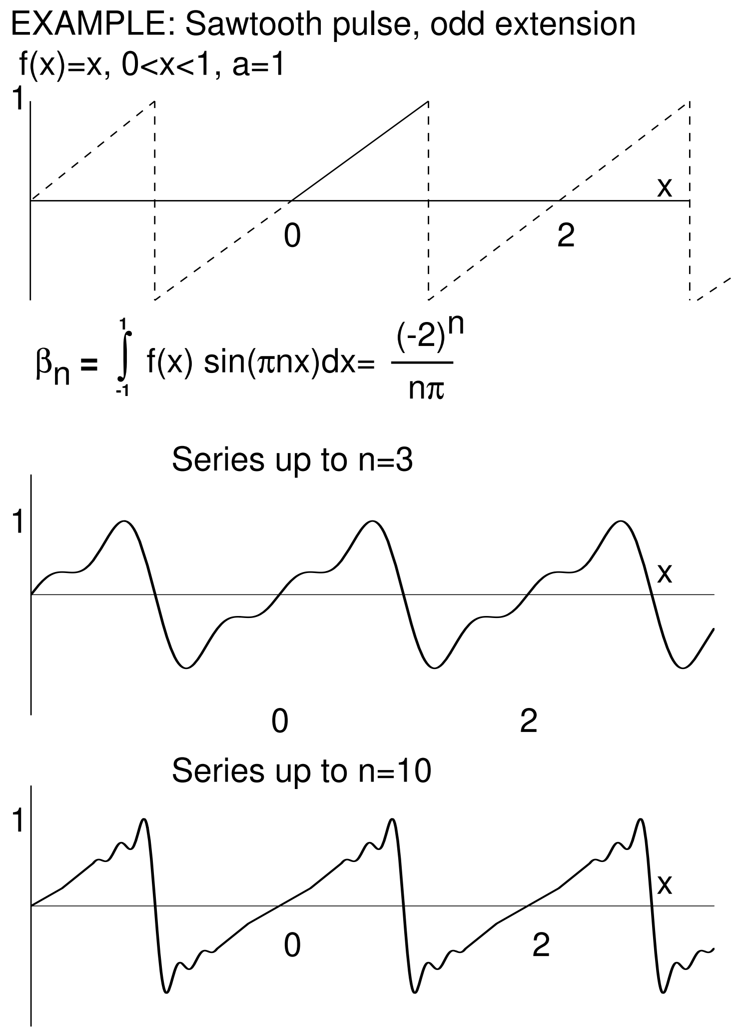
Series up to n=3



Series up to n=10



ODD EXTENSION: **Sine series**



We now extend the definition of $f(x)$ defined on $0 < x < 1$ as follows

$$f(x) = -f(-x); \quad f(x \pm 2a) = f(x)$$

Since the function is odd all the cosine terms in the Fourier series will vanish and we have

$$\boxed{f(x) = \sum_{n=1}^{\infty} \beta_n \sin \frac{n\pi x}{a}}$$

$$\boxed{\beta_n = \frac{2}{a} \int_0^a f(x) \sin \frac{n\pi x}{a} dx}$$

SUMMARY
We have

- defined the Fourier series of a function $f(x)$ defined in a finite interval, and illustrated by some examples.
- defined the *periodic, even* and *odd* periodic extensions.
- derived formulas for the coefficients of the series in terms of integrals over trigonometric functions.

PROBLEMS

Problem 4.1:1
The function $f(x)$ is defined for $0 < x, \pi$ as

$$f(x) = \begin{cases} \frac{1}{2}, & 0 < x < \frac{\pi}{2} \\ 1, & \frac{\pi}{2} < x < \pi \end{cases}$$

Find the Fourier series and plot sum of terms to order $n = 10$ for the

a: periodic extension (sin/cos series).

b: even periodic extension.

c: odd periodic extension.

Problem 4.1:2 Consider the function

$$f(x) = |\sin(x)|, \ -\infty < x < \infty$$

a: What is the period?
b: Expand the function in a Fourier series and plot a comparison between $f(x)$ and the series up to $n = 5$.

Problem 4.1:3 Consider the function

$$f(x) = x(1-x), 0 < x \leq 1$$

a: Find the Fourier series for the *periodic extension.*
b: Find the Fourier series for the *even periodic extension.*
c: Find the Fourier series for the *odd periodic extension.*
d: Plot the results after summing to terms to order n=5 in the expansion for the three cases above.

Problem 4.1:4 The ends of a thin bar is kept at $T = 0$. The temperature inside satisfies the differential equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

The boundary conditions are

$$T(0,t) = T(a,t) = 0$$

It was shown in class that the solution can be expressed as a Fourier series

$$T(x,t) = \sum_{n=0}^{\infty} \beta_n \exp(-n^2\pi^2 kt/a^2)\sin(n\pi x/a)$$

Assume that the initial temperature distribution is

$$T(x,0) = \frac{T_0 x(a-x)}{a^2}$$

Plot T/T_0 in the middle of the bar as a function of kt/a^2. Alternatively use the Maple "animation" command to visualize the time evolution of the system.

Problem 4.1:5 Given the function

$$f(x) = \sin(x) + \cos(x);\ 0 < x < \pi$$

Find the function $g(x)$ in the range $-\pi < x < 0$ that constitutes the
a: periodic extension of $f(x)$.
b: the odd periodic extension of $f(x)$.
c: the even periodic extension of $f(x)$. **c:** Is the derivative of a periodic function periodic? Is the integral of a periodic function periodic?

Problem 4.1:6

a:
Two identical metal bars are each of length a. Initially one is at temperature 0^oC while the other is at temperature 100^oC. They are joined together end to end and the assembly thermally isolated. Assume the temperature satisfies the heat equation

$$k\frac{\partial^2 T}{\partial x^2} = \frac{\partial T}{\partial t}$$

with boundary conditions

$$\frac{\partial T(x,t)}{\partial x} = 0, \quad x = -a \;\; and \; x = a$$

Find in the form of a Fourier series the temperature $T(x,t)$ at later times t.
b:
Plot the solution for $x = -a$ as a function of time (in units of a^2/k).

Problem 4.1:7

Consider the functions

$$f(x) = \sin(x) + \cos(x)$$

defined *only* in the restricted range $0 < x < \pi$ and

$$g(x) = \left\{ \begin{array}{lll} \sin(x) + \cos(x) & for & 2n\pi < x < (2n+1)\pi \\ -\sin(x) + \cos(x) & for & (2n+1)\pi < x < (2n+2)\pi \\ & n = -\infty, \cdots -2, -1, 0, 1, 2, \cdots \infty & \end{array} \right\}$$

defined for all real x.
a: Which of the following statements are true

1. $g(x)$ is the periodic extension of $f(x)$.
2. $g(x)$ is the odd periodic extension of $f(x)$.

3. $g(x)$ is the even periodic extension of $f(x)$.

4. none of the above.

Justify your answer. We make a Fourier expansion of the function $g(x)$

$$g(x) = \alpha_0 + \sum_{n=1}^{\infty} (\alpha_n \cos(nx) + \beta_n \sin(nx)))$$

$$\alpha_0 = \frac{1}{2\pi} \int_0^{2\pi} dx g(x)$$

$$\alpha_n = \frac{1}{\pi} \int_0^{2\pi} dx g(x) \cos(nx)$$

$$\beta_n = \frac{1}{\pi} \int_0^{2\pi} dx g(x) \sin(nx)$$

b: Which of the following are true

1. all the coefficients $\beta_n = 0$.

2. the coefficients $\alpha_n = 0$ for $n > 1$.

3. none of the above.

c: Calculate the coefficient α_0.

4.2 Complex Fourier series.

LAST TIME

- derived the Fourier series of a function $f(x)$ defined in a finite interval.
- defined the *periodic, even* and *odd* periodic extensions.
- derived formulas for the coefficients.

TODAY we will

- introduce the complex Fourier series

- make some comments on convergence. This being a physics course we will avoid getting too involved in such issues. The texts by Arfken and Weber and Riley et. al. have more details than given here.

COMPLEX FOURIER SERIES
The trigonometric functions are intimately related to the exponential function with imaginary arguments (see section 1.3).

$$\cos\theta = \frac{1}{2}(e^{i\theta} + e^{-i\theta})$$

$$\sin\theta = \frac{1}{2i}(e^{i\theta} - e^{-i\theta})$$

or alternatively

$$e^{i\theta} = \cos\theta + i\sin\theta$$

Consider the Fourier series of a function $f(x)$ defined for $0 < x < a$

$$f(x) = \alpha_0 + \sum_{n=1}^{\infty}[\alpha_n \cos\frac{2n\pi x}{a} + \beta_n \sin\frac{2n\pi x}{a}]$$

Substitution of the exponential forms into the Fourier series expression yields

$$f(x) = \alpha_0 + \frac{1}{2}\sum_{n=1}^{\infty}\{(\alpha_n - i\beta_n)e^{2\pi inx/a} + (\alpha_n + i\beta_n e^{-2\pi inx/a})\}$$

Combining terms we find that the series can be written

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx/a} \tag{13}$$

where

$$c_n = \begin{cases} \alpha_0; & n = 0 \\ \frac{1}{2}(\alpha_n - i\beta_n); & n > 0 \\ \frac{1}{2}(\alpha_n + i\beta_n); & n < 0 \end{cases}$$

By explicit integration we can show that

$$\int_0^a e^{i2\pi nx/a}e^{-i2\pi mx/a}dx = a\delta_{nm}$$

from which we can find the coefficients. To see this multiply 13 by

$$e^{-2\pi imx/a}$$

and integrate over a from 0 to $+a$. This gives

$$\int_0^a f(x)e^{-2\pi imx/a}dx = a\sum_{n=-\infty}^{\infty}\delta_{nm}c_n = ac_m$$

Since this holds for any m we find for the Fourier coefficients in (13)

$$c_n = \frac{1}{a}\int_0^a f(x)e^{-i2\pi nx/a}dx$$

The complex Fourier series is completely equivalent to the other types of series, but gives often rise to simpler looking formalism.

CONVERGENCE

The restrictions on the function $f(x)$ for it to have a Fourier series are mild:

- the series will exist if $f(x)$ is continuous except for a finite number of jump discontinuities.
- if there are jump discontinuities the series will fall off as $1/n$ for large n and only be conditionally convergent.
- if the function jumps from the value f_- to f_+ at some value of x the Fourier series will converge to

$$\frac{1}{2}(f_- + f_+)$$

- if the function is continuous the series will be uniformly convergent.
- if the function and the first m derivatives are continuous the series the series will fall off no slower than n^{-m-2} for large n.

MANIPULATION OF FOURIER SERIES:

Suppose a function $f(x)$ has a Fourier series

$$f(x) = \sum_{n=-\infty}^{+\infty} c_n \exp(2\pi inx/a)$$

Integration of the series term by term is generally harmless. Since integration

$$\int dx \exp(2\pi inx/a) = \frac{1}{2\pi inx/a} \exp(2\pi inx/a)$$

brings down a factor of $1/n$ a series which is convergent for large n will generally be more convergent after it is integrated.

Differentiation term by term requires uniform convergence i.e. the function $f(x)$ must be continuous. If the series is only conditionally convergent (see 1.4) differentiation will make the series divergent. We can see this by noting that differentiation brings down an extra factor of n which becomes large for large n.

$$\frac{d}{dx} \exp(2\pi inx/a) = \frac{2\pi inx}{a} \exp(2\pi inx/a)$$

Conditional convergence typically occurs when the function $f(x)$ is discontinuous. The derivative at a discontinuity is infinite. Divergence of the series must thus be expected.

Many useful tricks in summing up series involve integrating or differentiating a series term by term.

SOME GENERAL PROPERTIES OF THE COMPLEX FOURIER SERIES

The function $f(x)$ which is expanded in a Fourier series can be defined in intervals other than $0 \leq x \leq a$. Suppose c_n is the Fourier coefficient associated with the function $f(x)$, $-a \leq x \leq a$. Then

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/a}$$

$$\frac{1}{2a} \int_{-a}^{a} e^{i\pi nx/a} e^{-i\pi mx/a} dx = \delta_{nm}$$

$$c_n = \frac{1}{2a} \int_{-a}^{a} f(x) e^{-i\pi nx/a} dx \tag{14}$$

we find

- if $f(x)$ is *real* then $c_n = c^*_{-n}$.
- if $f(x) = f(-x)$ (even) then $c_n = real$
- if $f(x) = -f(-x)$ (odd) then $c_n = purely\ imaginary$

POWER SPECTRA

From (14) we find

$$\frac{1}{2a}\int_{-a}^{a} |f(x)|^2 dx = \sum_{n=-\infty}^{\infty} |c_n|^2$$

which is known as *Parseval's* theorem. When x represents *time*, $f(x)$ is typically an *amplitude*, $|f(x)|^2$ represents an *intensity* and $|c_n|^2$ will then be a measure of the contribution of a particular *frequency* to the total power of the signal, or its *power spectrum.* Since $c_n = c^*_{-n}$ for a real signal, the sum over negative n is sometimes neglected and the definition of the power spectrum may differ by a factor of 2.

MULTIPLE FOURIER SERIES

The generalization to the case of several variables is straight forward. Suppose $f(x, y)$ is defined on the domain $0 < x < a,\ 0 < y < b$. Then

$$f(x, y) = \sum_{n,m=-\infty}^{\infty} c_{n,m} \exp(2\pi i(\frac{nx}{a} + \frac{my}{b}))$$

$$c_{n,m} = \frac{1}{a}\int_0^a dx \exp(\frac{-2\pi inx}{a})\frac{1}{b}\int_0^b dy f(x, y) \exp(-\frac{2\pi iny}{b})$$

The Maple worksheet at
http://www.physics.ubc.ca/~birger/p312l6.mws (or.html)
illustrates how to evaluate Fourier coefficients and sum the different types of Fourier series.

SUMMARY

We have:

- introduced the complex Fourier series.
- made comments about convergence.
- discussed rules for manipulating the series.

PROBLEMS

Problem 4.2:1 The function $f(x)$ is periodic with period 4 and is defined by

$$f(x) = \begin{cases} 0 & for \quad -2 < x \leq -1 \\ 1+x & \quad -1 < x \leq 0 \\ 1-x & \quad 0 < x \leq 1 \\ 0 & \quad 1 < x \leq 2 \end{cases}$$

a: Expand the function in a complex Fourier series.
b: Plot the partial sum from $n = -5$ to $n = 5$. and compare with the original function.
c: Verify that since $f(x)$ is *even* the Fourier coefficients are *real*.
d: Verify that since $f(x)$ is continuous with discontinuous derivative the coefficients fall off as n^{-2} for large n.
e: Compute the power spectrum of $f(x)$ and verify that Parceval's theorem is satisfied.

4.3 Potential inside a rectangle.

LAST TIMES
Developed the theory of Fourier series including

- general Fourier series (sine and cosine) $\Rightarrow$ periodic extension.
- complex Fourier series $\Rightarrow$ periodic extension.
- sine series $\Rightarrow$ odd extension.
- cosine series $\Rightarrow$ periodic extension.

TODAY
Want to work through a complete example of how the Fourier series method

works when combined with the method of separating the variables. As we shall see the methods do get somewhat involved and again we shall resort to Maple to complete the job.

POTENTIAL INSIDE A RECTANGLE

Consider the problem of solving the Laplace equation

$$\frac{\partial^2 u(x,y)}{\partial x^2} + \frac{\partial^2 u(x,y)}{\partial y^2} = 0$$

inside the rectangle

$$0 < x < a, \ 0 < y < b$$

with the boundary conditions

$$u(x,0) = s(x); \ u(x,b) = n(x)$$

$$u(0,y) = w(y); \ u(a,y) = e(y)$$

We first note that the boundary conditions are not homogeneous. Hence, we cannot apply the method of separation of variables method directly. We get around this by splitting up the problem into two sub-problems

$$u(x,y) = u_1(x,y) + u_2(x,y) \tag{15}$$

where u_1 and u_2 both satisfy the Laplace equation, but with different boundary conditions

$$\begin{aligned} u_1(x,0) &= s(x), \ u_2(x,0) = 0 \\ u_1(x,b) &= n(x), \ u_2(x,b) = 0 \\ u_1(0,y) &= 0, \ u_2(0,y) = w(x) \\ u_1(a,y) &= 0, \ u_2(a,y) = e(y) \end{aligned} \tag{16}$$

Equation (15) together with the boundary conditions (16) represent the solution to the full problem. Let us first try to find an eigenfunction expansion of $u_1(x,y)$ by the method of separation of variables:

$$u_1(x,y) = \xi(x)\eta(y)$$

Substituting into the Laplace equation and dividing by $\xi\eta$ we find

$$\frac{d^2\xi}{\xi dx^2} = -\frac{d^2\eta}{\eta dy^2} = c_1 = const.$$

The trick is to try to find eigenfunctions $\xi(x)$ so that

$$\xi(0) = \xi(a) = 0$$

We must first determine if the constant c_1 is positive or negative. Assume first $c_1 = \lambda^2 > 0$. The solution to the differential equation for ξ is

$$\xi(x) = d_1 \exp(\lambda x) + d_2 \exp(-\lambda x)$$

where d_1 and d_2 are constants. The boundary condition at $x = 0$ gives $d_1 = -d_2$. However it is then impossible to satisfy the boundary condition at $x = a$. Hence

$$0 > c_1 = -\lambda^2$$

The solution to the differential equation for ξ is now

$$\xi(x) = \gamma \sin(\lambda x) + \delta \cos(\lambda x)$$

The boundary condition at $x = 0$ now gives $\delta = 0$ while the condition at $x = a$ gives

$$\lambda = \lambda_n = \frac{n\pi}{a};\ n = 1, 2...$$

The solution to the differential equation for η is then

$$\eta_n = \alpha_n \exp(\frac{n\pi y}{a}) + \beta_n \exp(-\frac{n\pi y}{a})$$

We express the function $u_1(x, y)$ as a series

$$u_1(x, y) = \sum_{n=1}^{\infty}(\alpha_n \exp(\frac{n\pi y}{a}) + \beta_n \exp(-\frac{n\pi y}{a})) \sin\frac{n\pi x}{a}$$

The coefficients α_n and β_n can now be determined by the boundary conditions at the south and north edge

$$s(x) = \sum_{n=1}^{\infty}(\alpha_n + \beta_n) \sin\frac{n\pi x}{a}$$

$$n(x) = \sum_{n=1}^{\infty} (\alpha_n \exp(\frac{n\pi b}{a}) + \beta_n \exp(-\frac{n\pi b}{a})) \sin \frac{n\pi x}{a}$$

We expand $s(x), n(x)$ in a sine series (odd periodic extension)

$$s(x) = \sum_{n=0}^{\infty} s_n \sin \frac{n\pi x}{a}$$

$$n(x) = \sum_{n=0}^{\infty} n_n \sin \frac{n\pi x}{a}$$

$$s_n = \frac{2}{a} \int_0^a s(x) \sin(\frac{n\pi x}{a}) dx$$

$$n_n = \frac{2}{a} \int_0^a n(x) \sin(\frac{n\pi x}{a}) dx$$

The coefficients α_n and β_n can then be obtained by solving the system of equations

$$s_n = \alpha_n + \beta_n$$

$$n_n = \alpha_n \exp(\frac{nb\pi}{a}) + \beta_n \exp(-\frac{nb\pi}{a})$$

We find

$$\alpha_n = \frac{n_n \exp(\frac{n\pi b}{a}) - s_n}{\exp(\frac{2n\pi b}{a}) - 1}$$

$$\beta_n = \frac{s_n \exp(\frac{n\pi b}{a}) - n_n}{\exp(\frac{n\pi b}{a}) - \exp(-\frac{n\pi b}{a})}$$

The procedure to find u_2 is analogous. We write

$$u_2(x, y) = \sum_{n=1}^{\infty} (\gamma_n \exp(\frac{n\pi x}{b}) + \delta_n \exp(-\frac{n\pi x}{b})) \sin \frac{n\pi y}{b}$$

we again expand boundary conditions into a sine series

$$w(y) = \sum_{n=0}^{\infty} w_n \sin \frac{n\pi y}{b}$$

$$e(y) = \sum_{n=0}^{\infty} e_n \sin \frac{n\pi y}{b}$$

$$w_n = \frac{2}{b}\int_0^b w(y)\sin(\frac{n\pi y}{b})dy$$
$$e_n = \frac{2}{b}\int_0^b e(y)\sin(\frac{n\pi y}{b})dy$$

The coefficients γ_n and δ_n can then be determined by solving the coupled equations

$$w_n = \gamma_n + \delta_n$$
$$e_n = \gamma_n \exp(\frac{na\pi}{b}) + \delta_n \exp(-\frac{na\pi}{b})$$

We find

$$\gamma_n = \frac{e_n \exp(\frac{n\pi a}{b}) - w_n}{\exp(\frac{2n\pi a}{b}) - 1}$$
$$\delta_n = \frac{w_n \exp(\frac{n\pi a}{b}) - e_n}{\exp(\frac{n\pi a}{b}) - \exp(-\frac{n\pi a}{b})}$$

As we have seen, although each step is straightforward, there are rather many steps and the procedure gets rather involved unless one has access to software such as Maple. You will find a worked example on the worksheet
http://www.physics.ubc.ca/˜birger/n312l14a.mws (or .html)
where we also adress the problem that unless the functions $s(x), n(x), w(y), e(y)$ vanish at the corners of the rectangle, the boundary conditions for u_1 and u_2 will be discontinuous there. This will result in slow convergence of the Fourier series for u_1 and u_2 unless steps are taken to bypass the problem. You may also wish to compare with the finite difference method of (section 3.3). I am not convinced that the Fourier method is necessarily better in this case unless the problem somehow can be simplified, particularly since the finite difference is more flexible. For instance, in the finite difference method the room doesn't have to be perfectly square.

PROBLEMS

Problem 4.3:1
Here is an example of a problem illustrating the points made in this section but which does not have the full complexity of the general case.
a: Solve Laplace's equation

$$\nabla^2 u(x,y) = 0$$

inside a square of side a with boundary conditions

$$u(0,y) = u(a,y) = \sin(\frac{\pi y}{a})$$

$$u(x,0) = u(x,a) = \sin(\frac{\pi x}{a})$$

b: Solve the problem numerically using the method of finite differences, using a grid of 400 internal points, and plot the solution.
c: Compare the exact and finite difference solution by plotting them along the diagonal of the square.

Problem 4.3:2
Solve

$$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = \cos x$$

subject to the boundary conditions

$$u(-\frac{\pi}{2}, y) = u(\frac{\pi}{2}, y) = u(x, -\frac{\pi}{2}) = u(x, \frac{\pi}{2}) = 0$$

4.4 Convective boundary conditions.

So far we have only considered cases where the period of the functions entering the Fourier series was either equal to or twice the interval on which the function was defined. Let us next consider a more complicated boundary value problem. A conducting rod has one end kept at a fixed temperature, while the other end is subject to convective heat transfer. The result will still be expressible in the form of sine and cosine series, but the eigenvalue problem now forces us to go back to the more general Sturm Liouville problem of (section 3.5) The heat equation is still

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t}$$

The initial condition is

$$T(x,0) = f(x)$$

At one end

$$T(0,t) = T_0$$

The other end at $x = a$ is in contact with a fluid at temperature T_1. The heat flow at this end is governed by the boundary condition

$$-\kappa \frac{\partial T}{\partial x}\bigg|_{x=a} = h(T(a,t) - T_1);\ t > 0$$

(In order to avoid confusion with the eigenvalues λ_n we now call the convection constant h)

THE STEADY STATE

The steady state temperature satisfies

$$\frac{d^2 T_S}{dx^2} = 0$$

with boundary conditions

$$T_s(0) = T_0, \quad \frac{-\kappa dT_S}{dx}\bigg|_{x=a} = h[T_S(a,t) - T_1]$$

with solution

$$T_S = T_0 + \frac{xh(T_1 - T_0)}{\kappa + ha}$$

The transient thus must obey

$$u(x,t) = T(x,t) - T_S(x)$$

$$\frac{\partial^2 u}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

$$u(0,t) = 0; \ \kappa \frac{\partial u(x,t)}{\partial x} + hu(a,t) = 0$$

$$u(x,0) = f(x) - T_S \equiv g(x)$$

EIGENVALUE PROBLEM

The next step is to find solutions on the form

$$u(x,t) = \tau(t)\phi(x)$$

to the heat equation. As before

$$\frac{d^2\phi}{dx^2} + \lambda^2\phi = 0$$

$$\frac{d\tau}{dt} = -\lambda^2 k\tau = 0$$

giving

$$\phi_\lambda = \alpha\sin(\lambda x) + \beta\cos(\lambda x)$$

$$\tau = \exp(-\lambda^2 kt)$$

Since $\phi(0) = 0$ we must have $\beta = 0$. The boundary condition at $x = a$ is thus

$$\kappa\lambda\cos(\lambda a) + h\sin(\lambda a) = 0$$

This is a *transcendental* equation that needs to be solved *numerically.* As we shall see there are infinitely many solutions

$$\lambda_n, \quad n = 1, 2...$$

to this equation.

We know from our discussion of the Sturm Liouville problem that if $\lambda_n \neq \lambda_m$ we have

$$\int_0^a dx\sin(\lambda_n x)\sin(\lambda_m x) = 0$$

We must next compute the normalization constants

$$b_n = \int_0^a dx[\sin(\lambda_n x)]^2$$

and the coefficients ($g(x)$ is the initial transient)

$$\alpha_n = \frac{1}{b_n}\int_0^a dx\sin(\lambda_n x)g(x)$$

The temperature distribution is then given by

$$T(x,t) = T_S(x) + \sum_{n=1}^{\infty}\alpha_n\sin(\lambda_n x)\exp(-k\lambda_n^2 t)$$

We illustrate this next in the form of a Maple worksheet at
http://www.physics.ubc.ca/~birger/p312l12.mws (or .html).

SUMMARY

- We have constructed a solution to a one dimensional heat conduction problem.
- In this example one end was kept at fixed temperature while the other had convective contact with a surrounding fluid.
- The eigenvalues λ_n had to be evaluated numerically.
- In practice, problems involving convective boundary conditions involve use of computer programs such as Maple.

PROBLEMS

Problem 4.4:1 Consider the following eigenvalue problem (assume $\lambda > 0$)

$$\frac{d^2\phi}{dx^2} + \lambda\phi = 0$$

$$\phi(0) + \frac{d\phi}{dx}|_{x=0} = 0$$

$$\phi(a) + \frac{d\phi}{dx}|_{x=a} = 0$$

Normally boundary value conditions that involve combinations of the function and its derivatives lead to transcendental equations, but in this case the equations can be solved explicitly.
Find the eigenvalues λ and eigenfunctions ϕ of the problem!

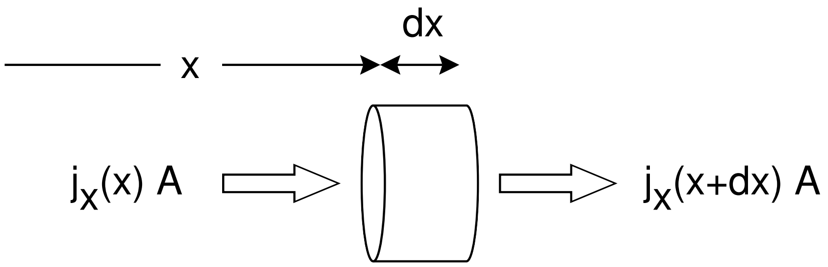
4.5 Generalization of the heat conduction problem

LAST TIME

- We constructed a solution to a one dimensional heat conduction problem in which one end was kept at fixed temperature while the other had convective contact with a surrounding fluid.

- The eigenvalues λ_n had to be evaluated numerically and we found an approximate solution to the problem using Maple.

TODAY
we will consider some further generalizations of the heat equation.



As before let us consider heat conduction in a long rod... Again we neglect radial heat losses. However, we now wish to be able to take into account the possibility that the thermal conductivity κ and heat capacity C_V varies along the length of the rod. To do this let us go back to the derivation of the heat equation. The axial influx at x is

$$Aj(x) = -A\kappa(x)\frac{\partial T(x,t)}{\partial x}$$

The heat leaving at the other end is

$$Aj(x+dx) = -A\kappa(x+dx)\frac{\partial T(x+dx,t)}{\partial x}$$

The difference will be the heat build-up in the element

$$C_V(x)\frac{\partial T(x,t)}{\partial t}$$

We are thus left with the modified heat equation:

$$\frac{\partial}{\partial x}(\kappa(x)\frac{\partial T}{\partial x}) = C_V(x)\frac{\partial T}{\partial t}$$

THE STEADY STATE

The steady state problem can still be solved. We have

$$\frac{d}{dx}(\kappa(x)\frac{dT_S(x)}{dx}) = 0$$

or

$$\kappa(x)\frac{dT_s(x)}{dx} = c_1$$

This equation can be integrated to yield

$$T_S = c_2 + c_1 \int_l^x \frac{dx'}{\kappa(x')}$$

where the two constants c_1 and c_2 must be determined by the boundary conditions at $x = l$ and $x = r$ (the ends of the rod).

BOUNDARY VALUE PROBLEM

We now wish to express the problem as an example of the regular Sturm-Liouville problem that we briefly alluded to in section 3.5. To do this we let

$$u(x,t) = T(x,t) - T_S(x)$$

$$\kappa(x) = s(x)\bar{\kappa}$$

$$C_V(x) = p(x)\bar{C}_V$$

where $\bar{\kappa}$ and $\bar{C}_V$, represent some "average" values of the thermal conductivity and heat capacity, and let $k = \bar{\kappa}/\bar{C}_V$. With these substitutions the partial differential equation for u becomes

$$\frac{\partial}{\partial x}(s(x)\frac{\partial u}{\partial x}) = \frac{1}{k}p(x)\frac{\partial u}{\partial t}$$

We use the method of separation of variables to solve this equation.

$$u(x,t) = \phi(x)\tau(t)$$

$$\tau(t)\frac{d}{dx}s(x)\frac{d\phi}{dx} = \phi(x)\frac{p(x)}{k}\frac{d\tau}{dt}$$

Dividing through gives

$$\frac{1}{\phi(x)p(x)}\frac{d}{dx}s(x)\frac{d\phi(x)}{dx} = \frac{1}{k\tau(t)}\frac{d\tau(t)}{dt} = const = -\lambda^2$$

As before we have

$$\tau = c_1 e^{-k\lambda^2\tau}$$

where we put $c_1 = 1$ absorbing the constant in the function ϕ. The differential equation for ϕ is then

$$\frac{d}{dx}s(x)\frac{d\phi}{dx} + \lambda^2 p(x)\phi = 0 \tag{17}$$

As we did in section 3.5 we assume that the boundary conditions are

$$a_l\phi(l) + b_l\frac{d\phi(l)}{dx} = 0$$

$$a_r\phi(r) + b_r\frac{d\phi(r)}{dx} = 0$$

ORTHOGONALITY

In lecture 12 we showed that when ϕ satisfied

$$\phi'' + \lambda_n\phi = 0$$

the eigenfunctions satisfied the orthogonality relationship

$$\int_l^r dx\phi_n(x)\phi_m(x) = 0, \quad \lambda_n^2 \neq \lambda_m^2$$

We also claimed that in the case of the more complicated differential equation (17) the orthogonality relationship must be modified to read

$$\int_l^r dxp(x)\phi_n(x)\phi_m(x) = 0, \quad \lambda_n^2 \neq \lambda_m^2$$

As in the previous case the boundary conditions require that

$$a_l\phi_n(l) - b_l\phi_n'(l) = 0$$

$$a_l\phi_m(l) - b_l\phi'_m(l) = 0$$

and the determinant

$$\begin{vmatrix} \phi_n(l) & -\phi'_n(l) \\ \phi_m(l) & -\phi'_n(l) \end{vmatrix} = 0$$

Hence

$$-\phi_n(l)\phi'_m(l) + \phi_m(l)\phi'_n(l) = 0$$

Similarly at the other end

$$-\phi_n(r)\phi'_m(r) + \phi_m(r)\phi'_n(r) = 0$$

From the differential equation we have

$$(s\phi'_m)' = -\lambda_m^2 p\phi_m$$

$$(s\phi'_n)' = -\lambda_n^2 p\phi_n$$

We thus find

$$(\lambda_m^2 - \lambda_n^2)\int_l^r dx\; p(x)\phi_n(x)\phi_m(x) = \int_l^r dx(s(x)\phi_n(x)')'\phi_m(x) - (s(x)\phi_m(x)')'\phi_n(x))$$

Integrating the last integral by parts and assuming that

$$\lambda_m^2 - \lambda_n^2 \neq 0$$

we find

$$\int_l^r dx\; p(x)\phi_n(x)\phi_m(x) = \frac{1}{\lambda_m^2 - \lambda_n^2}[-\phi_n(x)s(x)\phi'_m(x) + \phi_m(x)s(x)\phi'_n(x)]|_l^r = 0$$

q.e.d.

REGULAR STURM LIOUVILLE PROBLEM:
If $p(x)$ and $s(x)$ are both non-zero and positive the eigenvalue problem is called *regular* and one can prove some rather strong results. In the present case the restriction to positive values will always be satisfied because of the physical interpretation of these functions as thermal conductivity and specific heat, later on we will, however encounter *singular* Sturm-Liouville problems.

- The eigenfunctions are unique except for a multiplicative constant.

- Since the eigenvalue λ enters as a square in the differential equation defining the eigenvalue problem we can choose λ to be positive.
- If we order the eigenvalues $\lambda_1 < \lambda_2 \cdots < \lambda_n$ the nth eigenfunction will have exactly $n-1$ zeroes in the range $l < x < r$.
- The eigenfunctions are *complete*, that is an arbitrary function

$$f(x), \quad l < x < r$$

can be expanded in terms of the eigenfunctions

$$f(x) = \sum_{n=1}^{\infty} \alpha_n \phi_n(x)$$

where

$$\alpha_n = \frac{1}{b_n} \int_l^r dx p(x) f(x) \phi_n(x)$$
$$b_n = \int_l^r dx p(x) \phi_n^2(x)$$

We will revisit the question of completeness later when discussing the Dirac $\delta-$function (section 6.2).

The generalizations of the heat equation above leaves the equation *linear*. As we have seen the problem becomes more difficult when κ and C_V no longer are constant but depend on the coordinate x. Still, because of the linearity, the superposition principle still applies (section 2.1) we can still employ the methods of separation of variables and eigenfunction expansions.

Material properties such as κ and C_V may also depend on the temperature. When this happens the differential equation becomes non-linear and the superposition principle will not be valid, making the problem much more difficult. If the temperature dependence is *weak* one can sometimes make progress using successive approximations. To do this let us first solve the problem assuming values of κ and C_V corresponding to some *average temperature* T_0. We then substitute the recalculated value of the temperature into κ and C_V and repeat the procedure. For the steady state temperature the method is in principle straightforward, but for the transient we encounter the additional property that the resulting differential equation may not be separable.

SUMMARY

- We have generalized the heat conduction equation to one in which the thermal conductivity and the heat capacity could depend on position.
- The boundary value problem then became an example of the *regular Sturm-Liouville problem.*
- We proved the orthogonality relationship for the eigenfunctions in the regular Sturm-Liouville problem.
- We also stated without proof some important general properties of the eigenfunctions and eigenvalues.
- When the thermal conductivity and the heat capacity in addition depend on temperature the problem becomes non-linear and much more difficult.

PROBLEMS

Problem 4.5:1
A rod of length a is initially $(t = 0)$ at temperature $T = 0$. It is put into convective contact with a fluid with temperature T_0. The temperatures at the ends $(x = 0)$ and $(x = a)$ are kept at $T = 0$, but towards the middle the temperature rises towards T_0. Assume that the temperature distribution satisfies the differential equation

$$\frac{\partial^2 T}{\partial x^2} - \gamma^2(T - T_0) = \frac{1}{k}\frac{\partial T}{\partial t}$$

for $t > 0$, with k and γ constants.

a:
Find the steady state temperature distribution $T_S(x)$.

b:
Find the differential equation satisfied by the transient.

$$u(x,t) = T(x,t) - T_S(x)$$

c:
What is the initial temperature distribution for the transient?
d:
Solve the transient problem *formally*, by assuming that the initial temperature $u(x,0)$ has a Fourier sine series:

$$u(x,0) = \sum_{n=0}^{\infty} c_n \sin(\frac{n\pi x}{a})$$

(for this part of the problem you don't need to calculate the Fourier coefficients).
e:
Could you find the time dependent temperature distribution directly without subtracting the steady state?
f:
Plot the temperature distribution at time $t = 1$ for $a = k = \gamma = T_0 = 1$. (For this part of the problem you need to compute the first few Fourier coefficients either with or without the steady state subtracted.)

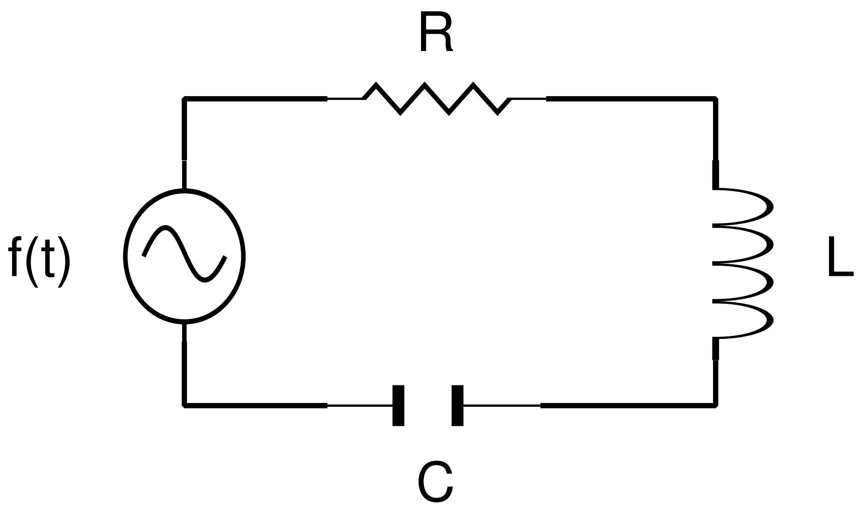
4.6 Non-homogeneous equations. LRC circuit.

LAST TIME

- Discussed generalizations of the 1 dimensional heat equation
- Showed how more generalized Sturm-Liouville problems could arise than we had considered so far

TODAY: Consider another generalization, we wish to show how Fourier methods can be used to solve inhomogeneous equations. As an example we consider the simple LRC-circuit.

LRC-CIRCUIT



The charge q on the capacitor satisfies the differential equation

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

where
$L =$ inductance
$R =$ resistance
$C =$ capacitance
$V =$ applied voltage

As we did in section 2.3 our first step is to reduce the number of parameters. Let us define the new *time* variable

$$\tau = \frac{t}{\sqrt{LC}}$$

and replace q by

$$y = \frac{q}{C}$$

and let

$$f(\tau) = V(t); 2\alpha = R\sqrt{\frac{C}{L}}$$

The differential equation in the new variables is

$$\frac{d^2y}{d\tau^2} + 2\alpha\frac{dy}{d\tau} + y = f(\tau)$$

CHARACTERISTIC EQUATION
Let us first look at the solutions to the homogeneous equation

$$\frac{d^2y_h}{d\tau^2} + 2\alpha\frac{dy_h}{d\tau} + y_h = 0$$

You may recall from MATH 215 that for equations with constant coefficients it pays to try solutions on form

$$y = e^{\lambda\tau}$$

This gives the *characteristic equation*

$$\lambda^2 + 2\alpha\lambda + 1 = 0$$

with roots

$$\lambda = -\alpha \pm \sqrt{\alpha^2 - 1}$$

There are three main cases depending on whether the roots are distinct, real or complex:

OVER DAMPED CASE $\alpha > 1$
The characteristic equation has two real roots. The solution to the homogeneous equation is

$$y_h(\tau) = C_1 e^{-(\alpha+\sqrt{\alpha^2-1})\tau} + C_2 e^{-(\alpha-\sqrt{\alpha^2-1})\tau}$$

CRITICAL DAMPING $\alpha = 1$
The two roots coincide and the general solution is

$$y_h = (C_1 + C_2\tau)e^{-\tau}$$

as can easily be verified by substituting into the differential equation.

UNDER DAMPED CASE $\alpha < 1$
The characteristic equation has two complex roots. The solution to the homogeneous equation is now

$$\begin{aligned} y_h &= c_1 \exp(\tau[-\alpha + i\sqrt{1-\alpha^2}]) \\ &+ c_2 \exp(\tau[-\alpha - i\sqrt{1-\alpha^2}]) \end{aligned}$$

which can be rewritten in terms of real functions

$$y_h = [C_1 \sin(\tau\sqrt{1-\alpha^2}) + C_2 \cos(\tau\sqrt{1-\alpha^2})]e^{-\alpha\tau}$$

TRANSIENTS
The general solution to the homogeneous equation approaches zero for long times

$$\lim_{\tau\to\infty} y_h = 0$$

The physical solution only depends on initial conditions for a *transient period*, which often is quite short. *After that the memory of initial conditions is lost and the solution is independent of initial conditions.*

If we mainly care about the transient we can solve the inhomogeneous differential equation by adding the particular solution

$$y_p(\tau) = \int_0^\tau G(\tau, \tau')f(\tau')d\tau'$$

to the homogeneous solution with the Green's function computed as shown in (section 2.2. However, the Green's function method is unnecessarily cumbersome if we don't care about the initial conditions and the transient!

FOURIER TRANSFORM OF DERIVATIVES
One of the chief advantages of the Fourier method is that taking derivatives is easy:

Suppose a function $y(t)$ has the Fourier series

$$y(\tau) = \sum_{n=-\infty}^{\infty} \eta_n e^{i2\pi n\tau/a}$$

The derivative of y is

$$\frac{dy}{d\tau} = \sum_{n=-\infty}^{\infty} \frac{i2\pi n}{a} \eta_n e^{i2\pi n\tau/a}$$

Hence the derivative has Fourier coefficients

$$\frac{i2\pi n}{a} \eta_n$$

The process can be repeated. The Fourier coefficients of the *second derivative* are

$$-(\frac{2\pi n}{a})^2 \eta_n$$

PERIODIC FORCING

If the forcing term $f(\tau)$ is periodic we expand it in a *Fourier series.* We may use either the sine&cosine series or the complex form. We prefer the latter since it leads to simpler formulas.

$$f(\tau) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi n\tau/a}$$

where we assume that the period is a and

$$c_n = \frac{1}{a} \int_0^a f(\tau) d\tau e^{-i2\pi n\tau/a}$$

We assume the particular solution of interest also has the Fourier series

$$y_p(\tau) = \sum_{n=-\infty}^{\infty} \eta_n e^{i2\pi n\tau/a}$$

We require that the left and right hand side of the differential equation

$$\frac{d^2y}{d\tau^2} + 2\alpha \frac{dy}{d\tau} + y = f(\tau)$$

have the same series. Using our formulas for derivatives we find

$$(-(\frac{2\pi n}{a})^2 + 2\alpha i \frac{2\pi n}{a} + 1)\eta_n = c_n$$

we find

$$\eta_n = \frac{c_n}{-(\frac{2\pi n}{a})^2 + 2i\alpha\frac{2\pi n}{a} + 1}$$

The solution of interest is thus

$$y_p(\tau) = \sum_{n=-\infty}^{\infty} \frac{e^{i2\pi n\tau/a} c_n}{-(\frac{2\pi n}{a})^2 + 2i\alpha\frac{2\pi n}{a} + 1}$$

The Fourier coefficients can be rewritten in terms of an *amplitude* and *phase*

$$\eta_n = \frac{c_n}{\sqrt{[1-(\frac{2\pi n}{a})^2]^2 + [2\alpha\frac{2\pi n}{a}]^2}} e^{i\phi_n}$$

where the *shift in phase* is

$$\phi_n = \tan^{-1} \frac{\frac{-4\alpha\pi n}{a}}{1-(\frac{2\pi n}{a})^2}$$

Let us illustrate this by an example:
http://www.physics.ubc.ca/˜birger/p312l8.mws (or.html)

ORGAN PIPE WITH PERIODIC FORCING

The method outlined above can be extended to problems in both space and time. As and example consider the inhomogeneous wave equation of (section 3.2)

$$\frac{\partial^2\xi}{\partial t^2} - c^2\frac{\partial^2\xi}{\partial x^2} + k\frac{\partial\xi}{\partial t} = f(x)e^{i\omega t}$$

where we use the complex form $e^{i\omega t}$ rather than $\cos\omega t$ with the understanding that the physical solution is the real part of ξ. We showed in section 3.2 that the solutions to the homogeneous equation will decay exponentially in time. If we are not interested in the transient the solution to our problem will be proportional to the driving term. To be specific let us assume that the pipe is closed at both ends. We expand $f(x)$ in a sine series

$$f(x) = \sum_{n=1}^{\infty} f_n \sin(\frac{n\pi x}{a})$$

where a is the length of the pipe. We try

$$\xi(x,t) = \sum_{n=1}^{\infty} \xi_n \sin\frac{n\pi x}{a} e^{i\omega t}$$

Substitution into the differential equation gives

$$\sum_{n=1}^{\infty}(-\omega^2 + \frac{c^2n^2\pi^2}{a^2} + i\omega k)\xi_n e^{i\omega t}\sin\frac{n\pi x}{a} = \sum_{n=1}^{\infty} f_n e^{i\omega t}\sin\frac{n\pi x}{a}$$

Because of the orthogonality relationship of the sine functions we can equate term by term and find

$$\xi_n = \frac{f_n}{-\omega^2 + \frac{c^2n^2\pi^2}{a^2} + i\omega k}$$

and with the understanding that we should take the real part

$$\xi(x,t) = \mathcal{R}\sum_{n=1}^{\infty}\frac{f_n \sin\frac{n\pi x}{a}e^{i\omega t}}{-\omega^2 + \frac{c^2n^2\pi^2}{a^2} + i\omega k}$$

To proceed further we need to specify $f(x)$. Since we already have discussed how to make and sum Fourier series we stop here.

SUMMARY

- We have used the complex Fourier series to analyze a LRC circuit.
- The Fourier method is easier to apply than the Green's function method, particularly when transients can be neglected.
- When the forcing term is periodic, the Fourier series is appropriate.
- We used a Maple worksheet to work out an example.
- Finally we generalized the method to a spatial problem.

PROBLEMS

Problem 4.6:1

a: For what values of p is

$$T(x,t) = const.e^{-px}e^{i(\omega t - px)}$$

solution to the one-dimensional heat equation.

$$\frac{\partial^2 T}{\partial x^2} = \frac{1}{k}\frac{\partial T}{\partial t}$$

b: Use the solutions found in **a:** to find a solution to the heat equation with boundary conditions

$$T(0,t) = T(a,t) = Ae^{i\omega t}$$

c: Show that the real part of the solution solves the heat equation with boundary conditions

$$T(0,t) = T(a,t) = A\cos(\omega t)$$

d: Show that for long times solutions with arbitrary initial conditions will approach the solutions found in **c:** at long times.

Problem 4.6:2
Consider the LCR circuit of section 4.6 in reduced units

$$\frac{d^2 y}{d\tau^2} + 2\alpha\frac{dy}{d\tau} + y = V(\tau)$$

Here the terms on the left hand side represents the voltage over, respectively, the inductor, the resistor and the capacitor. Let

$$V(\tau) = f(\omega\tau)$$

where $f(x)$ is the rectified sine wave considered above.
Find the complex Fourier coefficients c_n of $V(\tau)$ in the interval $0 < \tau < \pi/\omega$.
Plot in the same graph the voltage over the capacitor with $\omega = 1/2, 1$ *and* 2.
Plot in the same graph the voltage over the inductor with $\omega = 1/2, 1$ *and* 2.

4.7 Discrete Fourier series. Time series. Fast Fourier transform.

Up until now we have, in our treatment of Fourier methods, expanded in *functions* such as

$$\sin\frac{n\pi x}{a}, \cos\frac{n\pi x}{a}, \exp\frac{2n\pi i x}{a}$$

The sum of the resulting series can then be evaluated for a continuum of $x-$values. There are a number of situations where it is more convenient to evaluate a function on a *discrete lattice of points.* Some examples:

- The position of atoms in a crystal can best be described as a discrete set of points, when describing e.g. lattice vibrations.
- X-ray diffraction is an important technique in determining crystal structures. Again the description involves positions of atoms on a discrete set of points. The discrete Fourier transform of the lattice is called the reciprocal lattice and it plays a crucial rôle in solid state physics.
- There are many practical situations where one samples a variable at regular time intervals. Fourier methods provide important tools when analyzing such time series. Examples are data for environmental variables such as temperatures, rainfall, pollution levels, water levels of rivers, or financial data such values of currencies, stock market prices etc.
- Modern technology places an increasing importance on digital rather than analog representation of information.
- The discrete Fourier transform plays an important rôle in image processing.
- As we already have found in our treatment of finite difference methods, some problems are more efficiently approximated by determining variables on a discrete grid. We may then use interpolation methods to approximate the results on intermediate points.
- The availability of the *fast Fourier transform (FFT)* algorithm allows one to use Fourier method efficiently for discrete systems.

DISCRETE FOURIER TRANSFORM
Suppose we have available the values of a function f_k on a discrete set of points labeled by the index k, $k = 0, 1, 2 \ldots N-1$. In the case of a *time series* the point k corresponds to the *sampling times*

$$t_k = k\Delta$$

if the first time is $t = 0$. We define the *discrete Fourier transform* of the N numbers f_k as F_n, $n = 0, 1, 2, \ldots N-1$ where

$$F_n = \sum_{k=0}^{N-1} f_k \exp\left(-\frac{2\pi i k n}{N}\right)$$

(Be warned that this is not the only convention. I am trying to adhere to the notation used by Maple. Some people don't put the minus sign in the exponential, or they place a factor $1/N$ or $1/\sqrt{(N)}$ in front of the sum).

If we allow n, k to be outside the range $0, 1, 2..N-1$ the extension is

$$f_{k\pm N} = f_k, \quad F_{n\pm N} = F_n$$

i.e. the *periodic extension* is understood. If f_k is a time series the numbers n are associated with the *frequencies*

$$\nu_n = \frac{n}{N\Delta}$$

if measured in Hertz (periods/second) or

$$\omega_n = \frac{2\pi n}{N\Delta}$$

if measured in radians/second. Consider the *geometric series* (section 1.4)

$$S_{n,m} = \sum_{k=0}^{N-1} \exp(\frac{2\pi ik(m-n)}{N}) = \frac{1-\exp(2\pi i(m-n))}{1-\exp(\frac{2\pi i(m-n)}{N})}$$

If $n - m \neq 0, \pm N, \pm 2N...$ we see that $S_{n,m} = 0$. If $n = m$ or $n - m$ is some multiple of N the formula for S is undetermined $(0/0)$, but by inspecting the sum we see that each term is unity so

$$S_{n,n} = N$$

The orthogonality relation

$$S_{n,m} = \begin{cases} N & n = m \\ 0 & n \neq m \end{cases}$$

allows us to find the *inverse* of the discrete Fourier transform

$$f_k = \frac{1}{N} \sum_{n=0}^{N-1} F_n \exp(\frac{2\pi ikn}{N})$$

Some properties of the discrete Fourier transform:
If

- $f_k = f_{k+N}; \quad F_n = F_{n+N}$
- f_k is real $\Rightarrow F_n = F^*_{-n} = F_{N-n}*$
- f_k is imaginary $\Rightarrow F_n = -F^*_{-n}$
- f_k is even $(f_k \equiv f_{-k}) \Rightarrow F_n = F_{-n}$
- f_k is odd $(f_k \equiv -f_{-k}) \Rightarrow F_n = -F_{-n}$

where * indicates complex conjugate. Most signals which we wish to analyze are *real.* F_n and F^*_{-n} and by extension F^*_{N-n} will then be the same, and we see that there are only $N/2$ distinct frequencies. The frequency

$$\nu_c = \frac{1}{2\Delta}$$

is commonly referred to as the *Nyquist frequency.*

POWER SPECTRUM

The magnitude square $|f_k|^2$ of a signal is commonly referred to as its *intensity* and we refer to the sum

$$\sum_{k=0}^{N-1} |f_k|^2$$

as the *total power.* It is easy to verify using the orthogonality relations that the discrete Fourier series satisfies *Parseval's theorem*

$$\sum_{k=0}^{N-1} |f_k|^2 = \frac{1}{N} \sum_{n=0}^{N-1} |F_n|^2$$

The magnitude square of the Fourier coefficients is called the *power spectral density*

$$P(\nu_n) = |F_n|^2 + |F_{-n}|^2$$

(It is conventional to restrict the frequencies to $|\nu_n| < \nu_c$ and not distinguish between positive and negative frequencies. Sometimes one doesn't add the two terms and the power spectral density is given as 1/2 of the above)

SAMPLING THEOREM

Suppose $f(t)$ is a function defined for $0 \leq t < T$ and we wish to reconstruct it by sampling the function at the N regularly spaced points (excluding the end point $t = T$). We say that the function $f(t)$ is *bandwidth limited* to the frequency ν (in periods per second) if its Fourier series

$$f(t) = \sum_{k=-\infty}^{\infty} c_n \exp(\frac{2\pi int}{T})$$

does not contain any Fourier coefficients with n larger in magnitude than

$$n = \nu\Delta$$

The *Nyquist sampling theorem* states that, if the function $f(t)$ is bandwidth limited to the Nyquist frequency, it can be completely reconstructed by discrete sampling. (We will find a explicit form for this reconstructions in connection with our discussion of completeness and the Dirac delta-function in section 6.2. If the sampled function contains Fourier coefficient higher than the Nyquist frequency to a significant degree, we run into the following problem: signal components that vary as $\exp 2\pi\nu_1 t$ and $\exp 2\pi\nu_2 t$ sample the same way, if $\nu_1 - \nu_2$ differ by a multiple of the sampling frequency $1/\Delta$. Then higher order Fourier coefficients gets folded back into F_n. This phenomenon is called *aliasing*, and is something one wishes to avoid in signal processing. A number of filtering methods have been developed for this purpose, but a discussion of this point is beyond the scope of this course.

MULTIPLE DISCRETE FOURIER SERIES

The generalization to the case of more than one dimension is in principle straightforward. Consider e.g. a two dimensional $N_x \times N_y$ grid on which we sample the function $f(x, y)$

$$x_k = k\Delta_x, \quad k = 0, 1...N_x - 1, \quad y_l = l\Delta_y, \quad l = 0, 1...N_y - 1$$

$$f_{k,l} = f(x_k, y_l)$$

The discrete Fourier transform of $f(x, y)$ is

$$F_{n,m} = \sum_{k=0}^{N_x-1} \sum_{l=0}^{N_y-1} f(k, l) \exp\{-2\pi i(nk/N_x + ml/N_y)\}$$

with inverse

$$f_{k,l} = \frac{1}{N_x N_y} \sum_{k=0}^{N_x-1} \sum_{l=0}^{N_y-1} F(n,m) \exp\{2\pi i(nk/N_x + ml/N_y)\}$$

FAST FOURIER TRANSFORM

A naive implementation of the discrete Fourier transform would involve at least N^2 multiplications. This mean the computing time should increase with the square of the number of terms in the series. For long series with, say, a millions of data points the computational effort would be prohibitive. In the mid 60's the so-called Fast Fourier algorithm became available and reduced the computation time by a potentially enormous factor. It is not necessary to understand the details of this algorithm in order to use it, but one quirk of it must be pointed out. In order to take full advantage of the algorithm the number of points to be transformed should be exactly 2^m where m is an integer. In the Maple worksheet
http://www.physics.ubc.ca/~birger/n312l20a.mws (or .html)
we give examples of the discrete Fourier transform and the use of the FFT method. In particular we explore the connection between the discrete Fourier transform and *stochastic processes* such as the *random walk*.

Further reading: "Numerical recipes" [9] contains a lot of details on the discrete Fourier transform. This book is also available on the web
http://www.nr.com/
For a readable introduction to time series see the book by Feder [4].

PROBLEMS

Problem 4.7:1

Throughout this problem set we will consider the rectified sine wave

$$f(x) = |\sin(x)|$$

It is periodic with period π, and can be considered as the periodic extension of

$$g(x) = \sin(x), \quad 0 < x < \pi$$

a:
Expand $f(x)$ in a *complex* Fourier series in the interval $0 < x < \pi$

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(2inx)$$

and show that the Fourier coefficients are

$$c_n = \frac{2}{\pi(1-4n^2)}$$

b: Suppose we hadn't noticed that the period is π, and expand in a complex Fourier series in the interval $0 < x < 2\pi$. The Fourier series is now

$$f(x) = \sum_{n=-\infty}^{\infty} d_n \exp(inx)$$

Find the Fourier coefficients d_n and compare to the result found in **a:**
c: Let us next **sample** the function $f(x)$ at $N = 2^m$ values of x

$$x_k = \frac{k\pi}{N}$$

We wish to find the discrete Fourier transform F_n of $f_k = f(x_k)$ for some values of m. Is F_n real? Plot Re F_n/N and c_n in the same graph between $n = 0$ and $n = N/2 - 1$ for $m = 3, N = 8$. Repeat for $m = 6, N = 64$. **d:**

Problem 4.7:2
Consider the function

$$f(x) = 1 - x^2, \quad -1 \leq x < 1$$

a:
Find the coefficient c_n in the Fourier expansion

$$f(x) = \sum_{n=-\infty}^{\infty} c_n \exp(in\pi x)$$

Are the Fourier coefficients c_n real?
b:
Let us next sample the function $f(x)$ at the $N = 2^m$ values of x

$$x = \frac{2k}{N}, \quad k = \frac{-N}{2}, \frac{-N}{2} + 1, \cdots \frac{N}{2} - 1$$

Calculate the discrete Fourier transform F_n of $f_k = f(x_k)$ for $m = 6$.
c:
Approximate the sum over k in the definition of F_n by an integral. By what factor α must F_n be multiplied to be approximately equal to the coefficient c_n in part **a:**?
d: Check your previous result by plotting the first few Fourier coefficients c_n and αF_n in the same graph for $m = 6$.

5 Bessel and Legendre functions

5.1 Laplace equation in polar coordinates

Up until now we have concentrated on problems which either involved only one spatial dimension, or where Cartesian coordinates were used.

TODAY
We wish to start on problems where polar, spherical or cylindrical coordinates are employed. We begin by solving the 2-dimensional Laplace equation in polar coordinates (section 3.3)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0$$

We assume, as boundary condition, that the potential $u(r, \theta)$ is known along a circle $r = a$, where a is a constant:

$$u(a, \theta) = f(\theta) = given$$

we also assume that the solution is bounded in the region of interest. We now encounter a new type of boundary condition. If we increase (or decrease) the angle θ by 2π, we return to our starting point. Hence, we require that

$$u(r, \theta) = u(r, \theta + 2\pi)$$

this type of boundary condition is called *periodic*.

We attempt to solve the problem using the method of separation of variables and put

$$u(r, \theta) = \rho(r) T(\theta)$$

Substituting into the partial differential equation and multiplying by r^2 and dividing by $\rho\theta$ we obtain, letting the prime indicate differentiation:

$$\frac{r^2\rho'' + r\rho'}{\rho} = -\frac{T''}{T} = c = const$$

If c is negative we put $c = -\gamma^2$ and obtain

$$T'' = \gamma^2 T$$

with general solution

$$T = A\sinh(\gamma\theta) + B\cosh(\gamma\theta)$$

There is no way this solution can be used to produce periodic functions in θ, and we reject the possibility that $c < 0$. Writing $c = \lambda^2$ we find

$$T'' + \lambda^2 T = 0$$

with

$$T = A\sin(\lambda\theta) + B\cos(\lambda\theta)$$

The *periodicity condition* now requires that

$$\lambda = n = integer\ (or\ zero)$$

The differential equation for ρ is now

$$r^2\rho'' + r\rho' - n^2\rho \tag{18}$$

If $n \neq 0$ the general solution to (18) is

$$\rho(r) = cr^n + dr^{-n} \tag{19}$$

If $n = 0$ the general solution is

$$\rho = c + d\ln r \tag{20}$$

We now distinguish three cases

CASE I.
We require the solution *inside* the circle $r = a$ for which

$$u(a, \theta) = f(\theta)$$

The solution should be bounded at $r = 0$. This requires that we put $d = 0$ in (19). Similarly the logarithmic term in (20) must be zero. The solution is then be on the form

$$u(r, \theta) = \sum_{n=0}^{\infty} \alpha_n r^n \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^n \sin(n\theta) \tag{21}$$

The coefficients α_n and β_n can be determined by expanding $f(\theta)$ in a Fourier series

$$\alpha_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$\alpha_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) d\theta \cos(n\theta)$$

$$\beta_n = \frac{1}{a^n \pi} \int_0^{2\pi} f(\theta) d\theta \sin(n\theta)$$

Consider the special case $r = 0$ (center of circle). When $r = 0$ all the terms except the $n = 0$ term in the cosine series vanish and we find

$$u(0) = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

The potential at the center is the average of the potential over a circle surrounding it. The average cannot be larger (or smaller) than the largest (smallest) value over which we average. Hence: *maxima and minima of solutions to the Laplace equation only occur on the boundary of the solution region.*

CASE II.
We require the solution *outside* the circle $r = a$ for which

$$u(a, \theta) = f(\theta)$$

We also require that the solution is *bounded* in the limit $r \to \infty$. This demands that we put $c = 0$ in (19). Again, the logarithmic term in (20) must be zero. The solution will then be on the form

$$u(r, \theta) = \sum_{n=0}^{\infty} \alpha_n r^{-n} \cos(n\theta) + \sum_{n=1}^{\infty} \beta_n r^{-n} \sin(n\theta) \tag{22}$$

As before, the coefficients α_n and β_n can be determined by expanding $f(\theta)$ in a Fourier series

$$\alpha_0 = \frac{1}{2\pi} \int_0^{2\pi} f(\theta) d\theta$$

$$\alpha_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) d\theta \cos(n\theta)$$

$$\beta_n = \frac{a^n}{\pi} \int_0^{2\pi} f(\theta) d\theta \sin(n\theta)$$

We note that the *external solution* (case II) will not be bounded as $r \to 0$, while the *internal solution* (case I) is not bounded as $r \to \infty$. Hence: *the only solution to the Laplace equation which is bounded in the entire $x-y$-plane is $u = const$.*

CASE III.
We require the solution in an *annulus* (region between two circles).
We now need to combine the external and internal solution. For simplicity let us assume that the potential is $u(r = a) = u_a = const$ and $u(r = b) = u_b = const$, i.e. there is no angular dependence. and the $n \neq 0$ terms in the Fourier series vanish.
We seek a solution to the potential equation on the form

$$u(r) = A \ln r + B$$

The boundary conditions require that

$$A \ln a + B = u_a$$

$$A \ln b + B = u_b$$

The solution is

$$A = \frac{u_b - u_a}{\ln \frac{b}{a}}; \; B = \frac{u_a \ln b - u_b \ln a}{\ln \frac{b}{a}}$$

SUMMARY
We have found solutions to the Laplace equation in polar coordinates by the method of separation of variables for a variety of situations.

PROBLEMS

Problem 5.1:1
A thin metal ring is thermally insulated from its surroundings. The radius of the ring is L meter and its thermal diffusivity is k meter^{2}sec^{-1}. Assume that the temperature of the ring is $T(\theta, t)$.

a: Assume the initial temperature is $T(\theta, 0) = f(\theta)$. Find the subsequent temperature distribution.

b: Assume that

$$f(\theta) = T_0 \cos\theta$$

How long will it take for the temperature difference between the hottest and coldest spot on the ring to halve?

Problem 5.12:
Solve the 2-dimensional Laplace equation

$$\nabla^2 u(r, \theta) = 0$$

in polar coordinates in the region $a < r < 2a$ with boundary conditions

$$u(a, \theta) = \cos\theta; \; u(2a, \theta) = \cos\theta$$

Problem 5.13:
Solve

$$\nabla^2 u = 1$$

inside a sphere of radius 1. The boundary condition is

$$u(1, \theta, \phi) = 0$$

Hint: Assume that u is independent of θ and ϕ. It is possible to find a particular solution proportional to r^2. The solution should be finite at the origin.

Problem 5.1:1
In class we wrote for the Laplace equation in polar coordinates

$$\nabla^2 u = \frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = 0 \tag{23}$$

a: Show that the first two terms on the left hand side of (23) can be rewritten as

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} = \frac{1}{r}\frac{\partial}{\partial r}(r\frac{\partial u}{\partial r})$$

b: If the boundary conditions are such that the solution is independent of the angle θ the Laplace equation becomes an ordinary differential equation in the variable r only. Show that

$$\nabla^2 u = f(r)$$

has a particular solution

$$u(r) = \int_a^r \frac{dr_1}{r_1}\int_b^{r_1} dr_2\ r_2 f(r_2)$$

where the lower limits of integration a and b are arbitrary.
c: Use the above result to find the solution to

$$\nabla^2 u = 1;\ u(0) = 0, u\ \mathit{independent\ of}\ \theta$$

5.2 Derivation of wave and heat equation in higher dimension.

LAST TIME
We solved the Laplace equation in polar coordinates.

TODAY
We wish to proceed to the wave and heat equations.

THE VIBRATING STRING

In section 3.1 we derived the one-dimensional wave equation for sound waves. For the two dimensional wave equation I have chosen a different example: the vibrating membrane. First consider the one-dimensional version of this problem the flexible vibrating string. Let us assume that the string is stretched along the $x-$direction and executes transverse oscillations in the $z-$direction. We assume that the tension T of the string is uniform, as is the mass density ρ (mass per unit length). We will throughout assume that the amplitude z is small. Define θ as the angle of a segment of dx with respect to the x-axis

$$\frac{\partial z(x,t)}{\partial x} = \tan\theta \approx \theta$$

The projection of the tension in the $z-$ direction is then

$$T\sin\theta \approx T\theta \approx T\frac{\partial z(x,t)}{\partial x}$$

The net $z-$component of the force on the segment dx centered at x is

$$f_z = T(\frac{\partial z(x+dx/2,t)}{\partial x} - \frac{\partial z(x-dx/2,t)}{\partial x}) \approx Tdx\frac{\partial^2 z(x,t)}{\partial x^2}$$

The mass of this segment is ρdx. From Newton's second law (force = mass $\times$ acceleration) we find

$$Tdx\frac{\partial^2 z(x,t)}{\partial x^2} = \rho dx\frac{\partial^2 z(x,t)}{\partial t^2}$$

We define

$$c = \frac{T}{\rho}$$

and obtain

$$c^2\frac{\partial^2 z(x,t)}{\partial x^2} = \frac{\partial^2 z(x,t)}{\partial t^2}$$

If there is an external force $f(x,t)$ per unit length of the string, acting in the transverse direction (e.g. the bow of a violin), we obtain instead

$$T\frac{\partial^2 z(x,t)}{\partial x^2} + f(x,t) = \rho\frac{\partial^2 z(x,t)}{\partial t^2}$$

THE VIBRATING MEMBRANE

The generalization to the two dimensional case is fairly straightforward. We now consider an element of area of mass $\rho dxdy$ vibrating in the $z-$direction. We assume that there is a uniform tension T (force per unit length), acting on the area element. We again make the small amplitude approximation and assume that the segment is centered at a point with coordinates x, y. The net force in the $z-$direction is now

$$f_z = Tdy(\frac{\partial z(x+dx/2,y,t)}{\partial x} - \frac{\partial z(x-dx/2,y,t)}{\partial x})$$
$$+Tdx(\frac{\partial z(x,y+dy/2,t)}{\partial x} - \frac{\partial z(x,y-dy/2,t)}{\partial x})$$
$$\approx Tdxdy(\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2})$$

This force must be equal to the mass $\times$ acceleration of the volume element. We find

$$Tdxdy(\frac{\partial^2 z(x,y,t)}{\partial x^2} + \frac{\partial^2 z(x,y,t)}{\partial y^2}) = \rho dxdy\frac{\partial^2 z(x,y,t)}{\partial t^2}$$

or

$$c^2\nabla^2 z = \frac{\partial^2 z}{\partial t^2}$$

THREE DIMENSIONAL HEAT EQUATION

We derived the one-dimensional heat equation in section 3.4. Let us consider an element of volume

$$dV = dxdydz$$

centered at a point with coordinates

$$\vec{r} = [x, y, z]$$

Consider a face of the volume element perpendicular to the $z-$axis located at $z-dz/2$. The area of the face is $dxdy$. Heat flows *into* the volume element through the face at the rate

$$-dxdy\kappa\frac{\partial T(x,y,z-dz/2,t)}{\partial z}$$

while the heat flowing *out* of the opposing face at $z + dz/2$ is

$$-dxdy\kappa\frac{\partial T(x, y, z + dz/2, t)}{\partial z}$$

We have

$$heat\ in - heat\ out = CdV \times temperature\ change$$

where C is the heat capacity per unit volume. We also have

$$dxdy\kappa(\frac{\partial T(x, y, z + dz/2, t)}{\partial z} - \frac{\partial T(x, y, z - dz/2, t)}{\partial z})$$

$$\approx dxdydz\kappa\frac{\partial^2 T(x, y, z, t)}{\partial z^2}$$

Collecting terms we find

$$dxdydz\kappa(\frac{\partial^2 T(x, y, z, t)}{\partial x^2} + \frac{\partial^2 T(x, y, z, t)}{\partial y^2} + \frac{\partial^2 T(x, y, z, t)}{\partial z^2}) = Cdxdydz\frac{\partial T}{\partial t}$$

Defining

$$k = \frac{\kappa}{C}$$

we obtain the *heat equation*

$$k\nabla^2 T = \frac{\partial T}{\partial t}$$

Sometimes it is convenient to use vector calculus to describe the situation. The heat *current density*(heat flow per unit area) $\vec{j}$ is given by the temperature *gradient*

$$\vec{j_Q} = -\kappa\nabla T$$

The rate at which heat is accumulating per unit volume is given by the *divergence* of the current density

$$-\nabla \cdot \vec{j_Q} = \kappa\nabla^2 T$$

equating this to (rate of change of the temperature)$\times$(heat capacity) yields the heat equation.

Almost identical arguments can be used to describe *diffusion*. We have

$$D\nabla^2 c = \frac{\partial c}{\partial t}$$

where D is the *diffusion constant* and c is the concentration of some substance.

SUMMARY
We have derived the wave equation for a vibrating membrane and also the three dimensional heat equation.

5.3 Bessel's equation

LAST TIME
We derived the two dimensional wave equation for the case of a vibrating membrane and also derived the three-dimensional heat equation.

TODAY
We wish to show that, although the two equations describe very different physics, when applying the method of separation of variables, the mathematics is often very similar for the two cases. In particular we wish to demonstrate how the *Bessel differential equation* arises naturally when using polar or cylindrical coordinates. We will first consider the simplest case: polar coordinates.

The wave equation in polar coordinates is (see section 3.3for the formulas for the Laplacian in different coordinate systems)

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

while the heat equation in the same coordinate system is

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \theta^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

In both cases we attempt to solve the equations by separating the variables writing

$$u = R(\vec{r})\tau(t)$$

As we have done before we let prime ($\prime$) indicate differentiation. We find for the wave equation

$$\frac{\nabla^2 R(\vec{r})}{R} = \frac{1}{c^2}\frac{\tau\prime\prime}{\tau} = const. = -\lambda^2$$

(as before the constant has to be negative in order that τ remains bounded for large positive and negative times). To mark this we write (as we have done before) $-\lambda^2$ for the separation constant.

$$\tau'' + \lambda^2\tau = 0$$

Giving

$$\tau(t) = A\cos(\lambda t) + B\sin(\lambda t)$$

In the case of the heat equation we have

$$\frac{\nabla^2 R(\vec{r})}{R} = \frac{1}{k}\frac{\tau'}{\tau} = const. = -\lambda^2$$

The solution for $\tau(t)$ is now

$$\tau = const\,\exp(-k\lambda^2 t)$$

so that *in both cases* we have the equation for the spatial part

$$\nabla^2 R(r,\theta) + \lambda^2 R = \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \frac{1}{r^2}\frac{\partial^2 R}{\partial \theta^2} + \lambda^2 R = 0$$

We again attempt to separate the variables writing

$$R(r,\theta) = \rho(r)T(\theta)$$

We find

$$\frac{r^2\rho'' + r\rho' + \lambda^2 r^2\rho}{\rho} = -\frac{T''}{T} = n^2$$

The equation for T

$$T'' + n^2 T = 0$$

has solution

$$T = A\cos(n\theta) + B\sin(n\theta)$$

and the condition that T is periodic with period 2π requires that n is an integer or zero. This leaves us with the differential equation for ρ

$$\boxed{r^2\rho'' + r\rho' + (\lambda^2 r^2 - n^2)\rho = 0}$$

The above equation is called *Bessel's equation.* We first note that if we introduce the new dependent variable

$$x = \lambda r$$

the differential equation simplifies to

$$x^2\frac{d^2\rho}{dx^2} + x\frac{d\rho}{dx} + (x^2 - n^2)\rho = 0$$

and becomes independent of λ. This means that we can write the solutions of Bessel's equation as

$$\rho(\lambda r) = \rho(x)$$

We also note that the differential equation is singular at $r = 0$. It is instructive to study the behavior of the solutions for small r. If $\lambda r << 1$ we neglect the term proportional to $\lambda^2 r^2$ and approximate

$$r^2\rho'' + r\rho' - n^2\rho \approx 0$$

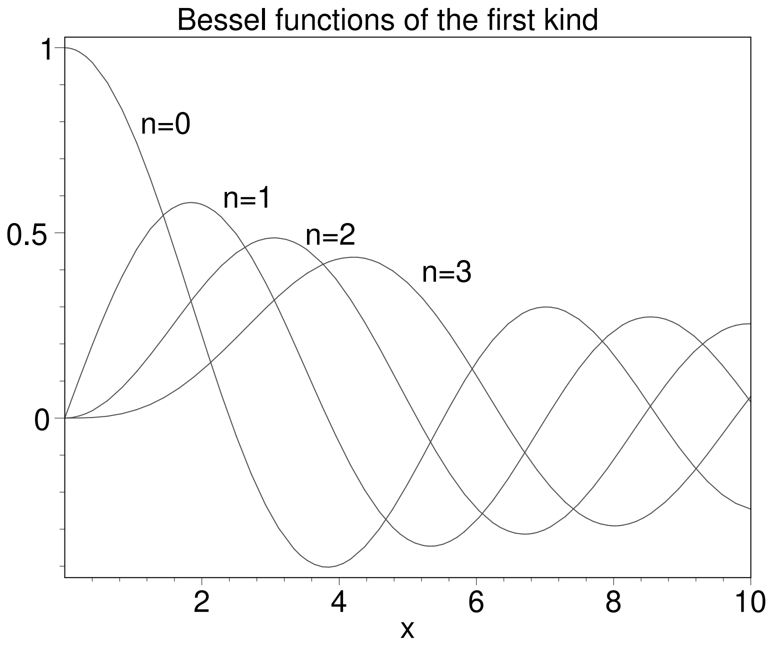
We have encountered this equation before when solving the Laplace equation in polar coordinates the general solution is

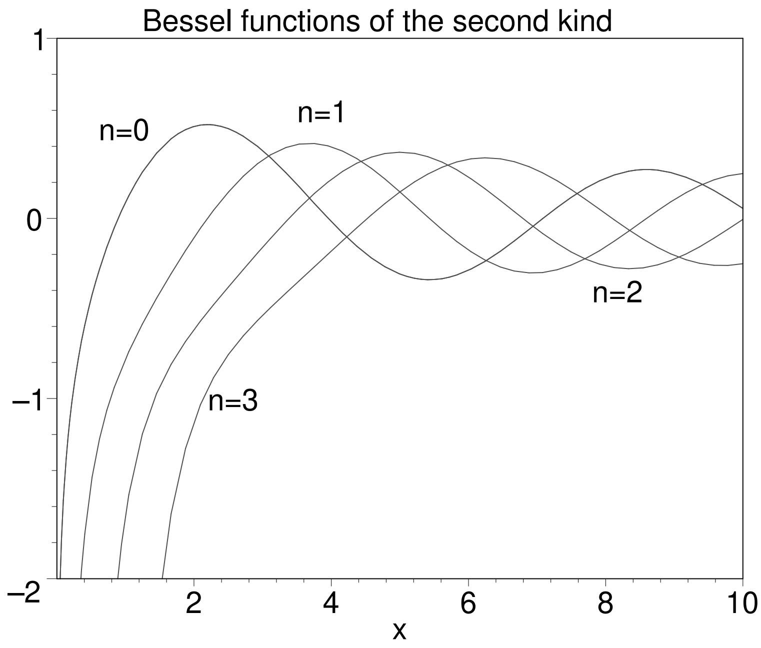
$$\rho = Ar^n + Br^{-n};\ \ if\ n \neq 0$$

$$\rho = A + B\ln r;\ \ n = 0$$

The terms proportional to A are bounded as $r \to 0$, while the terms proportional to B are not. Indeed, it turns out to be possible by direct substitution into the Bessel differential equations to find a power series solution (we prove this result in section 5.6)

$$\rho = J_n(\lambda r) = (\frac{\lambda r}{2})^n \sum_{m=0}^{\infty} \frac{(-1)^m}{m!(n+m)!}(\frac{\lambda r}{2})^{2m} \qquad (24)$$


Bessel functions of the first kind
n=0
n=1
n=2
n=3
1
0.5
0
2
4
6
8
10
x


Bessel functions of the second kind
n=0
n=1
n=2
n=3
1
0
–1
–2
2
4
6
8
10
x

One can also use the method of variation of parameters (see section 2.2) to find a second independent solution. It is conventional to write the general solution to Bessel's equation as

$$\rho = AJ_n(\lambda r) + BY_n(\lambda r)$$

where $J_n(\lambda r)$ is called the *Bessel function of the first kind*, and is given by the power series expansion (24). This expansion converges for all bounded values of r. The solution $Y_n(\lambda r)$ is called *Bessel function of the second kind.* It is singular as $r \to 0$:

$$Y_n(x) \propto x^{-n} \; as \; x \to 0 \; if \; n \neq 0$$

$$Y_0(x) \propto \ln x \; as \; x \to 0$$

Because of its singular behavior, as the argument goes to zero, the Bessel functions of the second kind are used much less than those of the first kind. In the next two lectures we will through examples explore the behavior of Bessel functions in more detail. The figures on the previous page show the first few Bessel functions of the first and second kind.

5.4 Vibrating membrane

LAST TIME
when separating variables in polar coordinates we encountered Bessel's equation on the form

$$\frac{d^2\rho(r)}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} + (\lambda^2 - \frac{n^2}{r^2})\rho(r) = 0$$

The general solutions to this equation can be written

$$\rho(r) = AJ_n(\lambda r) + BY_n(\lambda r)$$

where A and B are arbitrary constants and $J_n(\lambda r)$ and $Y_n(\lambda r)$ are the Bessel functions of the first and second kind, respectively. The Bessel functions have the properties that

$$J_0(\lambda r) \to 1, \; r \to 0$$

$$J_n(\lambda r) \propto r^n, \; r \to 0 \; if \; n \neq 0$$

$$Y_0(\lambda r) \propto \ln(r),\ r \to 0$$
$$Y_n(\lambda r) \propto r^{-n},\ r \to 0\ if\ n \neq 0$$

TODAY
we wish to familiarize ourselves more with the Bessel functions, using a vibrating membrane as an example. We wish to find solutions to the two dimensional wave equation

$$\nabla^2 u = \frac{1}{c^2}\frac{\partial^2 u}{\partial t^2}$$

The amplitude u has physical interpretation as the vertical displacement of a membrane, which we assume to be circular in shape and clamped down at radius a. We use polar coordinates so

$$u = u(r, \theta, t)$$

The boundary condition at the rim is

$$u(a, \theta, t) = 0$$

We also require that $u(0, \theta, t)$ is finite (bounded). A further boundary condition is that the amplitude is a periodic function of θ

$$u(r, \theta, t) = u(r, \theta + 2\pi, t)$$

We are now ready to find particular solutions to the wave equation using the method of separation of variables

$$u(r, \theta, t) = R(r, \theta)\tau(t) = \rho(r)T(\theta)\tau(t)$$

The first step is to find $\tau(t)$. We have

$$\frac{\nabla^2 R}{R} = \frac{\tau''}{c^2\tau} = -\lambda^2$$

giving

$$\tau(t) = A\sin(\lambda ct) + B\cos(\lambda ct)$$

$$\frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \frac{1}{r^2}\frac{\partial^2 R}{\partial \theta^2} + \lambda^2 R = 0$$

We now substitute $R = \rho T$ and rearrange

$$\frac{r^2\rho'' + r\rho' + \lambda^2 r^2 \rho}{\rho} = -\frac{T''}{T} = n^2 = const$$

We have

$$T'' + n^2 T = 0$$

$$T(\theta) = \beta\cos(n\theta) + \gamma(\sin(n\theta)$$

and the periodicity condition requires that n is an integer or zero. We are left with the equation for ρ

$$r^2\rho'' + r\rho' + (\lambda^2 r^2 - n^2)\rho = 0$$

with general solution

$$\rho = CJ_n(\lambda r) + DY_n(\lambda r)$$

Since ρ is required to be bounded at $r = 0$ we must have $D = 0$. The eigenvalues λ can then be determined by the requirement that

$$J_n(\lambda a) = 0$$

where a is the radius of the membrane. It can be shown that the Bessel function $J_n(x)$ has infinitely many zeroes for each value of n. We define α_{ni} as the i'th positive zero of the Bessel function of the first kind, of order n. The allowed values of λ are then

$$\lambda_{ni} = \frac{\alpha_{ni}}{a}$$

We conclude that the vibrating string admits solutions on the form

$$\boxed{u(r,\theta,\phi) = J_n(\frac{\alpha_{ni} r}{a})[\beta\cos(n\theta) + \gamma\sin(n\theta)](A\cos(\frac{\alpha_{ni} ct}{a}) + B\sin(\frac{\alpha_{ni} ct}{a}))}$$

Each value of n and i gives rise to a *different frequency*

$$\omega = \frac{\alpha_{ni} ct}{a}$$

We show in the Maple worksheet at
http://www.physics.ubc.ca/˜birger/n312l24.mws (or html) how one can compute the different frequencies and visualize the different modes.

The coefficients A, B, β, γ can in principle be determined by initial conditions. In the case of the vibrating membranes one is typically more interested in determining the different modes and associated frequencies. We will next time show how one can find a complete solution in the case of a heat conduction problem.

5.5 Heat equation in cylindrical coordinates

LAST TIME

- Discussed the vibrating membrane.
- Identified, using the method of separation of variables, the different eigenmodes of the membrane and found the associated frequencies.
- Showed, using Maple, how the frequencies could be computed and the modes visualized.

In the case of the vibrating membrane, we were most interested in the properties of the individual modes of vibration, and not so much in constructing a complete solution from initial value conditions.

TODAY
We wish to work out in detail the complete solution to a boundary/initial value problem of a three dimensional heat equation problem.

Before we proceed we need to establish some more properties of Bessel functions. Some time ago (sections 3.5and 4.5) we discussed the regular Sturm-Liouville problem

$$\frac{d}{dr}(s(r)\frac{d\phi}{dr}) - q(r)\phi(r) + \lambda^2 p(r)\phi = 0$$

where $s(r), q(r), p(r)$ are known functions, and the eigenfunctions satisfied homogeneous boundary conditions

$$a_{left}\phi(n) - b_{left}\frac{d\phi(r)}{dr}|_{r=left} = 0$$

$$a_{right}\phi(n) - b_{right}\frac{d\phi(r)}{dr}|_{r=right} = 0$$

and showed that the eigenfunctions satisfied orthogonality relations

$$\int_{left}^{right} drp(r)\phi_n(r)\phi_m(r) = 0, \;\; \lambda_n^2 \neq \lambda_m^2$$

We note that Bessel's equation

$$\frac{d^2\rho(r)}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} + (\lambda^2 - \frac{n^2}{r^2})\rho(r) = 0$$

Can be rewritten on the "Sturm-Liouville form"

$$\frac{d}{dr}(r\frac{d\rho}{dr}) + (\lambda^2 r - \frac{n^2}{r})\rho(r) = 0$$

with the weight function $p(r) = r$. Because of the singularity in Bessel's equation at $r = 0$, eigen-value problems with this equation are not of the regular Sturm-Liouville type. Nevertheless, it can be shown that the Bessel functions satisfy the orthogonality relation

$$\int_0^a rdrJ_n(\frac{\alpha_{ni}r}{a})J_n(\frac{\alpha_{nj}r}{a}) = 0$$

if $i \neq j$ and where (as in the previous lecture) α_{ni} is the i-th zero of $J_n(x)$. Note that the orthogonality relations only hold for Bessel functions with the same n!

There are a number of relationships that can be proven using the power series expansion for the Bessel functions and the differential equations (A good source for the properties of Bessel functions is Abramowitz and Stegun [1965]). In what follows we will need some definite integrals over Bessel functions of order $n = 0$

$$\int_0^a rdrJ_0(\frac{\alpha_{0i}r}{a}) = \frac{a^2}{\alpha_{0i}}J_1(\alpha_{0i}) \tag{25}$$

$$\int_0^a rdr J_0(\frac{\alpha_{0i}r}{a})^2 = \frac{a^2}{2}J_1^2(\alpha_{0i}) \tag{26}$$

EXAMPLE PROBLEM

Heating of a roast.

We idealize the roast as a cylinder of height b radius a. Initially the roast is at the temperature T_0. It is put in an oven with temperature T_1 and we assume this gives rise to the boundary condition that the surface of the cylinder is kept at the fixed temperature T_1. This is not very realistic -it would be better to assume the convective boundary conditions of section 4.4. In order to avoid complications we adopt the simpler boundary conditions. The steady state temperature is $T = T_1$ and we put

$$u = T - T_1$$

for the difference between the actual and steady state temperature. Mathematically, we wish to solve the heat equation

$$\nabla^2 u(z,r,t) = \frac{1}{k}\frac{\partial u}{\partial t}$$

where for reasons of symmetry we do not expect the temperature to depend on the angle θ. The solution is subject to the boundary condition

$$u(0,r,t) = 0;\ u(b,r,t);\ u(z,a,t) = 0;$$

and the initial condition

$$u(z,r,0) = \Delta \equiv T_0 - T_1$$

We next attempt to solve the problem by the method of separation of variables and look for particular solutions on the form

$$u(z,r,t) = R(z,r)\tau(t)$$

We find

$$\tau = const. \exp(-\alpha^2 kt)$$

$$\nabla^2 R = \frac{\partial^2 R}{\partial r^2} + \frac{1}{r}\frac{\partial R}{\partial r} + \frac{\partial^2 R}{\partial z^2} + \alpha^2 R = 0$$

We next separate the variables again putting

$$R(z,r) = Z(z)\rho(r)$$

We find

$$\frac{\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} + \alpha^2\rho}{\rho} = -\frac{d^2 Z}{Z dz^2} = \gamma^2 = const$$

and

$$\frac{d^2 Z}{dz^2} + \gamma^2 Z = 0$$

with general solution

$$Z = A\sin(\gamma z) + B\cos(\gamma z)$$

The boundary condition at $z = 0$ gives $B = 0$ while the boundary condition at $z = b$ gives for the eigenvalue γ

$$\gamma = \frac{m\pi}{b}; \quad m = 1, 2...$$

and the eigenfunctions are

$$Z = const. \sin\frac{m\pi z}{b}$$

We recognize the equation for ρ

$$\frac{d^2\rho}{dr^2} + \frac{1}{r}\frac{d\rho}{dr} + (\alpha^2 - \frac{m^2\pi^2}{b^2})\rho = 0$$

as Bessel's equation with $n = 0$. Writing

$$\lambda^2 = \alpha^2 - \frac{m^2\pi^2}{b^2}$$

And find (since ρ must be bounded for $r = 0$)

$$\rho = const. J_0(\lambda r)$$

The boundary condition ar $r = a$ requires that

$$\lambda = \frac{\alpha_{0i}}{a}$$

where as before α_{0i} is the location of the i'th zero of $J_0(x)$ and $i = 1, 2...$

Collecting terms we write

$$u(z,r,t) = \sum_{mi} \beta_{mi} J_0(\frac{\alpha_{0i} r}{a}) \sin(\frac{m\pi z}{b}) \exp(-k[(\frac{\alpha_0 i}{a})^2 + (\frac{m\pi}{b})^2]t)$$

The next step is to find the coefficients β_{mi} from the initial condition

$$\Delta = \sum_{mi} \beta_{mi} J_0(\frac{\alpha_{0i} r}{a}) \sin(\frac{m\pi z}{b}) \tag{27}$$

We multiply both sides of (27) by

$$\sin(\frac{n\pi z}{b})$$

and integrate over z from 0 to b using

$$\int_0^b dz \sin(\frac{m\pi z}{b}) \sin(\frac{n\pi z}{b}) = \delta_{n,m} \frac{b}{2}$$

$$\int_0^b dz \sin(\frac{n\pi z}{b}) = \frac{-b}{n\pi} \cos(\frac{n\pi z}{b})\Big|_0^b = \begin{cases} 0 & n = even \\ \frac{2b}{n\pi} & n = odd \end{cases}$$

To mark that n has to be odd we put

$$n = 2j - 1$$

to get

$$\frac{2\Delta}{(2j-1)\pi} = \frac{1}{2} \sum_i J_0(\frac{\alpha_{0i} r}{a}) \beta_{2j-1,i} \tag{28}$$

The final step in calculating $\beta_{2j-1,i}$ is to multiply both sides of (28) by

$$J_0(\frac{\alpha_{0l}}{a})$$

and integrate from 0 to a using (25)and (26). We get with

$$B_{jl} = \beta_{2j-1,l}$$

The final result is

$$\boxed{\begin{array}{c} B_{jl} = \dfrac{8\Delta}{\pi(2j-1)J_1(\alpha_{0l})\alpha_{0l}} \\[2ex] u(z,r,t) = \displaystyle\sum_{j,l=1}^{\infty} B_{jl} J_0(\frac{\alpha_{0l} r}{a}) \sin(\frac{(2j-1)\pi z}{b}) e^{-k[(\frac{\alpha_{0l}}{a})^2 + (\frac{(2j-1)\pi}{b})^2]t} \end{array}}$$

Numerical results follow on the Maple worksheet at
http://www.physics.ubc.ca/˜birger/p312l20.mws (or .html)

5.6 Properties of Bessel functions.

LAST TIME
we worked our way through a boundary value heat conduction problem using cylindrical coordinates. The solution made use of some of the known properties of Bessel functions.

TODAY
We wish to prove formally some properties of Bessel functions.
In sections 5.3 and 5.4 we made use of the fact that solutions of the differential equation

$$r^2 \frac{d^2\rho}{dr^2} + r\frac{d\rho}{dr} + (\lambda^2 r^2 - n^2)\rho = 0$$

could be written as functions of the product λr. To show that this is allowed we introduce the new variable

$$x = \lambda r$$

and define the function

$$J(x) = \rho(r)$$

(same value different functional dependence)
We note that

$$r^2 \frac{d^2\rho}{dr^2} = \lambda^2 r^2 \frac{d^2\rho}{d(\lambda r)^2} = x^2 \frac{d^2 J}{dx^2}$$

$$r\frac{d\rho}{dr} = \lambda r \frac{d\rho}{d\lambda r} = x\frac{dJ}{dx}$$

The differential equation for J is thus

$$x^2\frac{d^2J}{dx^2}+x\frac{dJ}{dx}+(x^2-n^2)J=0$$

which shows that we can write J as a function of the single argument x.

POWER SERIES EXPANSION OF $J_n(x)$

Many of the formal properties of Bessel functions can be obtained from their power series. In lecture 23 we argued that for small values of the argument

$$J_n(x)\propto x^n$$

It is therefore tempting to look for solutions of the form

$$J_n(x)=x^n(c_0+c_1x+c_2x^2\cdots+c_mx^m\cdots)=x^n\sum_{m=0}^{\infty}c_mx^m$$

to Bessel's differential equation. We find

$$x^2\frac{d^2J_n}{dx^2}+x\frac{dJ_n}{dx}-n^2J_n=\sum_{m=0}^{\infty}c_mx^{m+n}[(n+m)(n+m-1)+n+m-n^2]$$

$$x^2J_n(x)=\sum_{i=0}^{\infty}c_ix^{i+2+n}=\sum_{m=2}^{\infty}c_{m-2}x^{m+n}$$

We thus find

$$x^2\frac{d^2J_n}{dx^2}+x\frac{dJ_n}{dx}+(x^2-n^2)J_n$$

$$=x^n\left(c_0\cdot 0+c_1x((n+1)^2-n^2)+\sum_{m=2}^{\infty}x^m(((n+m)^2-n^2)c_m+c_{m-2})\right)$$

We demand that the terms multiplying each power of x should separately be zero. We conclude that c_0 can be chosen arbitrarily

$$c_1=0$$

and that for $m\geq 2$

$$c_m=\frac{c_{m-2}}{m(2n+m)}\tag{29}$$

We first note that since $c_1 = 0$ Equation (29) implies that $c_m = 0$ for *all odd values of* m. For even values of we can use the *recursion relation (29)* to express c_m in terms of c_0. For example

$$c_2 = \frac{-c_0}{2(2n+2)}$$

$$c_4 = \frac{-c_2}{4(2n+4)} = \frac{c_0}{2 \cdot 4(2n+2)(2n+4)}$$

and in general

$$c_{2m} = \frac{(-1)^m (\frac{1}{2})^{2m} c_0}{m!(n+1)(n+2)\cdots(n+m)}$$

It is conventional to choose

$$c_0 = \frac{(\frac{1}{2})^n}{n!}$$

This gives us the power series expansion for the Bessel function of the first kind

$$\boxed{J_n(x) = (\frac{x}{2})^n \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m}}{m!(n+m)!}}$$

a result which agrees with one stated without proof in section 5.3.

CONVERGENCE

In order to check if the power series for $J_n(x)$ is convergent we can use the ratio test of section 1.4. We have

$$\left| \frac{x^{m+2} c_{m+2}}{x^m c_m} \right| = \frac{x^2}{4(m+1)(n+m+1)}$$

For any *finite* x the ratio will always be less than 1 if m is large enough. *The series is therefore convergent for all finite values of* x.

FORMULAS FOR DERIVATIVES

The power series formula for the Bessel functions can be used to derive a number of exact relationships between functions of different order. As a first example let us consider the derivative of $J_0(x)$. We have

$$J_0(x) = \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m}}{m!m!}$$

$$\frac{d}{dx} J_0(x) = -\sum_{m=1}^{\infty} \frac{(-1)^{m-1} m (\frac{x}{2})^{2m-1}}{m!m!}$$

Introducing the new "dummy index" $k = m - 1$ we find

$$\frac{d}{dx} J_0(x) = -\frac{x}{2} \sum_{k=0}^{\infty} \frac{(-1)^k (\frac{x}{2})^k}{k!(k+1)!} = -J_1(x)$$

Hence

$$\boxed{\frac{d}{dx} J_0(x) = -J_1(x)}$$

A further example is

$$\boxed{\frac{d}{dx}(x^n J_n(x)) = x^n J_{n-1}(x)}$$

To prove this let us note that the left hand side can be written

$$2^n \frac{d}{dx} \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m+2n}}{m!(n+m)!} = 2^n \sum_{m=0}^{\infty} \frac{(-1)^m (\frac{x}{2})^{2m+2n-1}}{m!(n+m-1)!} = x^n J_{n-1}(x)$$

q.e.d.
A special case is

$$\frac{d}{dx}(x J_1(x) = x J_0(x)$$

We can use this result to evaluate

$$\int_0^{\alpha_{0i}} J_0(x) x dx = x J_1(x)|_0^{\alpha_{0i}} = \alpha_{0i} J_1(\alpha_{0i})$$

where $\alpha_{0i}$is the $i-$th root of $J_0(x) = 0$. This is a result which we used in section 5.2.

SUMMARY
We have derived a power series expression for the Bessel functions of integer order. This formula was used to derive a number of relationships between Bessel functions and their derivatives. It is not too difficult to derive many other such relationship.

PROBLEMS

Problem 5.6:1
Using the infinite series representation for the Bessel function verify that

$$\frac{d}{dx}(x^{-n}J_n(x)) = -x^{-n}J_{n+1}(x))$$

5.7 Separation of variables in spherical coordinates

LAST TIMES
Solved boundary value problems in cylindrical and polar coordinates.

TODAY
Consider similar problems in spherical coordinates.
The Laplacian in spherical coordinates can be written

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta}) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 u}{\partial\phi^2}$$

Typical boundary value problems that we wish to be able to solve are on the form

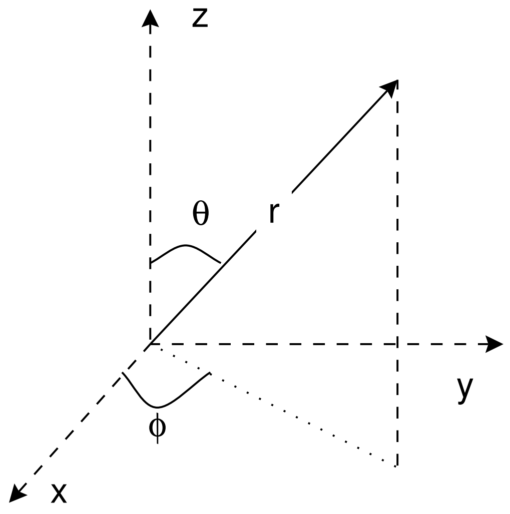
$$\nabla^2 u = -\lambda^2 u$$

arising from either the heat or wave equation, or

$$\nabla^2 u = 0$$

from a potential problem in a source free region, or

$$\nabla^2 u = g(\vec{r})$$



from a potential problem with sources. The general problem is notationally rather complicated. The choice of using spherical coordinates is often motivated by a problem where for reasons of symmetry the solution does not depend on all three coordinates r, θ, ϕ.

EXAMPLE
"Spherical roast"
We wish to solve

$$\nabla^2 u = \frac{1}{k}\frac{\partial u}{\partial t} \tag{30}$$

Suppose that for reasons of symmetry, we know that $u = u(r,t)$ depends only on r and t, not on θ and ϕ. We wish to solve the following boundary/ initial value problem

$$u(a,t) = 0,\ u(r,0) = f(r)$$

With the symmetry assumption the Laplacian simplifies to

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r})$$

We attempt to find solutions to (30) be the method of separation of variables

$$u = \rho(r)\tau(t)$$

We find

$$\frac{\nabla^2\rho}{\rho} = \frac{1}{k\tau}\frac{d\tau}{dt} = -\lambda^2 = 0$$

We have

$$\tau(t) = const.e^{-k\lambda^2 t}$$

while the differential equation for ρ is

$$\frac{d^2\rho}{dr^2} + \frac{2}{r}\frac{d\rho}{dr} + \lambda^2\rho = 0$$

This equation simplifies if we make the substitution

$$\rho = \frac{R(r)}{r}$$

We find

$$\frac{d^2R}{dr^2} + \lambda^2 R = 0$$

with solution

$$R(r) = A\sin(\lambda r) + B\cos(\lambda r)$$

giving

$$\rho(r) = \frac{1}{r}(A\sin(\lambda r) + B\cos(\lambda r))$$

We require that ρ is bounded at $r = 0$, hence $B = 0$. We also require that $\rho(a) = 0$, giving

$$\lambda = \frac{n\pi}{a},\ n = integers$$

Combining terms we write the solution to our boundary value problem

$$\boxed{u(r,t) = \sum_{n=1}^{\infty} \frac{\alpha_n}{r}\sin(\frac{n\pi r}{a})\exp(-\frac{n^2\pi^2 kt}{a^2})}$$

The coefficients α_n can be determined by the initial condition

$$f(r) = \sum_{n=1}^{\infty} \frac{\alpha_n}{r}\sin(\frac{n\pi r}{a})$$

giving

$$\boxed{\alpha_n = \frac{2}{a}\int_0^a r dr f(r) \sin\frac{n\pi r}{a}}$$

We now apply the method of separation of variables to the slightly more complicated problem of finding solutions to

$$\nabla^2 u = \lambda^2 u$$

which depend on r and θ but not on ϕ. In this case the Laplacian becomes

$$\nabla^2 u = \frac{1}{r^2}\frac{\partial}{\partial r}(r^2\frac{\partial u}{\partial r}) + \frac{1}{r^2\sin\theta}\frac{\partial}{\partial\theta}(\sin\theta\frac{\partial u}{\partial\theta})$$

We attempt to find solutions on the form

$$u = \rho(r)T(\theta)$$

We find

$$\frac{1}{\rho}[\frac{d}{dr}(r^2\frac{d\rho}{dr}) + r^2\lambda^2\rho] = -\frac{1}{T\sin\theta}\frac{d}{d\theta}[\sin\theta\frac{dT}{d\theta}] = \nu = constant$$

The differential equation for ρ is

$$\frac{d}{dr}(r^2\frac{d\rho}{dr}) + r^2\lambda^2\rho - \nu\rho = 0$$

The solution this equation are called spherical Bessel functions and their properties will be discussed in section 5.8.

The differential equation for T

$$\frac{d}{d\theta}(\sin\theta\frac{dT}{d\theta}) + \nu\sin\theta T = 0$$

becomes somewhat easier to handle if we use $x = \cos\theta$ rather than θ as independent variable. Let us write

$$y(x) = T(\theta)$$

We have

$$\frac{dx}{d\theta} = -\sin\theta$$

$$\frac{dT}{d\theta} = -\sin\theta\frac{dy}{dx}$$

$$\frac{d}{d\theta}(\sin\theta\frac{dT}{d\theta}) = \cos\theta\frac{dT}{d\theta} + \sin\theta\frac{d^2T}{d\theta^2}$$

$$\frac{d^2T}{d\theta^2} = -\cos\theta\frac{dy}{dx} - \sin\theta\frac{d^2y}{dx^2}\frac{dx}{d\theta} = -\cos\theta\frac{dy}{dx} + \sin^2\theta\frac{d^2y}{dx^2}$$

Substituting these results into the differential equation gives

$$\sin^3\theta\frac{d^2y}{dx^2} - 2\sin\theta\cos\theta\frac{dy}{dx} + \nu y\sin\theta = 0$$

Dividing both sides by $\sin\theta$ gives rise to *Legendre's differential equation*

$$\boxed{(1-x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu y = 0}$$

The differential equation is singular at $x = \pm 1$. We shall require that the solution is nonsingular at both $x = 1$ and $x = -1$. We shall see in the next lecture that this is only possible for some values of ν and gives rise to an eigenvalue problem.

PROBLEMS

Problem 5.7:1
a:
For what values of the constants a and b will

$$u(r,t) = t^b\exp(-ar^2/t)$$

be a solution to the two dimensional heat equation

$$\nabla^2 u = \frac{\partial u}{k\partial t}$$

in polar coordinates.
b:
For what values of the constants a and b will

$$u(r,t) = t^b \exp(-ar^2/t)$$

be a solution to the three dimensional heat equation

$$\nabla^2 u = \frac{\partial u}{k\partial t}$$

in spherical polar coordinates.

Problem 5.7:2
a:
Find solutions to the three dimensional wave equation

$$\nabla^2 u(r,t) = \frac{1}{c^2}\frac{\partial^2 u(r,t)}{\partial t^2}$$

with no angular dependence satisfying the condition that $u(r,t)$ is finite at $r=0$ and

$$u(R,t) = 0$$

b:
Show that the three dimensional wave equation is satisfied by

$$u(r,t) = \frac{1}{r}(\phi(r-ct) + \psi(r+ct))$$

for arbitrary functions ϕ and ψ as long as they can be differentiated twice with respect to their arguments.
c:
What are the restrictions on ϕ and ψ in part **b:** for the solution to be bounded $(<\infty)$ at the origin $r=0$ for all times.

5.8 Legendre polynomials. Spherical Bessel functions

LAST TIME
Considered the partial differential equation

$$\nabla^2 u + \lambda^2 u = 0$$

in spherical coordinates (r, θ, ϕ). We assumed that the solution could depend on r and θ, but not on ϕ, and attempted to find solution by the method of separation of variables:

$$u = \rho(r)T(\theta)$$

This procedure lead to the differential equation for T

$$\frac{d}{d\theta}(\sin\theta\frac{dT}{d\theta}) + \nu\sin\theta T = 0$$

where ν was a separation of variables constant. We argued that this equation became easier to handle if we introduced the new independent variable

$$x = \cos\theta$$

We defined $y(x) = T(\theta)$ and derived Legendre's differential equation

$$(1 - x^2)\frac{d^2y}{dx^2} - 2x\frac{dy}{dx} + \nu y = 0 \tag{31}$$

We next attempt to solve (31) by making a power series expansion

$$y = \sum_{m=0}^{\infty} a_n x^m$$

We have

$$\frac{d^2y}{dx^2} = \sum_{n=2}^{\infty} a_n n(n-1)x^{n-2} = \sum_{m=0}^{\infty} a_{m+2}(m+2)(m+1)x^m$$

$$-x^2\frac{d^2y}{dx^2} = -\sum_{m=0}^{\infty} a_m m(m-1)x^m$$

$$-2x\frac{dy}{dx} = -\sum_{m=0}^{\infty} 2m a_m x^m$$

$$\nu y = \sum_{m=0}^{\infty} \nu a_m x^m$$

Substituting into the differential equation and collecting terms we find

$$\sum_{m=0}^{\infty} x^m[a_{m+2}(m+2)(m+1) - a_m(m(m-1) + 2m - \nu)] = 0$$

This equation must hold for all x. This gives us the *recursion relation*

$$a_{m+2} = \frac{m(m+1) - \nu}{(m+2)(m+1)} a_m \tag{32}$$

If we specify an initial value of y

$$a_0 = y(0)$$

equation (32) will allow us to compute $a_2, a_4, a_6 \cdots$. Similarly, if we know the value of dy/dx at $x = 0$

$$a_1 = \frac{dy}{dx}|_{x=0}$$

we can use the recursion relation to determine $a_3, a_5 \cdots$. The general solution to Legendre's equation (31) can thus be written

$$y(x) = a_0 b(x) + a_1 c(x)$$

where $b(x)$ is an *even* function of x, $b(x) = b(-x)$, with only even powers of x in its power series expansion. The other term $c(x)$ is an odd function of x, $(c(x) = -c(-x))$, with a power series expansion with only odd terms of x.

Will the power series expansion that we have found be convergent?
We notice that for large values of m we have

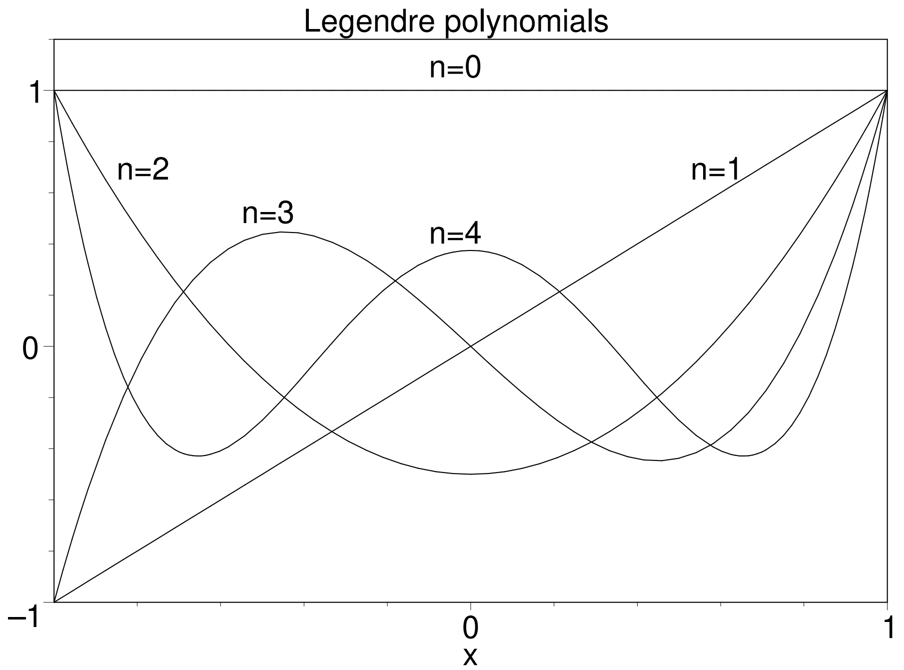
$$\frac{a_{m+2}}{a_m} \Rightarrow 1, \ as\ m \to \infty$$

hence the *radius of convergence* of the power series is $x = 1$. If we require that the solution $y(x)$ be nonsingular at $x = \pm 1$ *the series must break off so that y becomes a polynomial of finite order!.*

If we choose $\nu = n(n+1)$ where n is an even integer the even series will break off at n and

$$a_{n+2} = a_{n+4} \cdots = 0$$

and if n is odd the odd series breaks off after n. The condition that the solution should be non-singular at $x = \pm 1$ thus gives us the *eigenvalues* $\nu = n(n+1)$ with the polynomial solutions as *eigenfunctions.*



It is conventional to label the *Legendre polynomials* which are solutions to Legendre's equation as $P_n(x)$ and to normalize them so that $P_n(x=1)=1$. From the recursion relation (32) it is easy to show that

$$P_0(x) = 1$$

$$P_1(x) = x$$

$$P_2(x) = \frac{1}{2}(3x^2 - 1)$$

$$P_3(x) = \frac{1}{2}(5x^3 - 3x)$$

We next note that the differential equation (31) can be rewritten

$$\frac{d}{dx}\left((1-x^2)\frac{dy}{dx}\right) + \nu y = 0$$

which is on the Sturm-Liouville form (section 4.5)

$$\frac{d}{dx}(s(x)\frac{dy}{dx}) + \lambda^2 p(x)y = 0$$

with $p(x) = 1$, $s(x) = 1 - x^2$, $\nu = \lambda^2 = n(n+1)$. It can be shown that the Legendre polynomials are *orthogonal* i.e.

$$\int_{-1}^{1} dx P_n(x) P_m(x) = 0 \; if \; n \neq m$$

One can also show that with the normalization that we have chosen

$$\int_{-1}^{1} dx P_n(x)^2 = \frac{2}{2n+1} \tag{33}$$

As an example let us evaluate

$$\int_{-1}^{1} dx P_2(x)^2 = \int_{-1}^{1} dx \frac{1}{4}(3x^2 - 1)^2 = \frac{2}{5}$$

in agreement with (33).
The Legendre series can be used to expand an arbitrary function $f(x)$ defined for $1 \leq x \leq 1$ can be expanded in a Legendre series

$$f(x) = \sum_{n=0}^{\infty} \alpha_n P_n(x)$$

$$\alpha_n = \frac{2n+1}{2} \int_{-1}^{1} f(x) P_n(x) dx$$

In the Maple worksheet
http://www.physics.ubc.ca/˜birger/p312leg.mws (or .html)
we discuss further some of the properties of the Legendre Polynomials.

SPHERICAL BESSEL FUNCTIONS
When we separated the variables

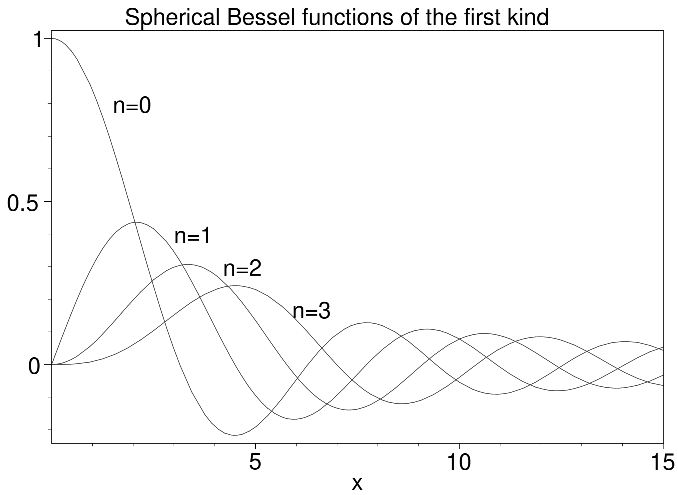
$$u = \rho(r) T(\theta)$$

in

$$\nabla^2 u = \lambda^2 u$$

(section 5.7), the differential equation for ρ was shown to be

$$\frac{d}{dr}(r^2 \frac{d\rho}{dr}) + r^2 \lambda^2 \rho - n(n+1)\rho = 0$$



The solution to this equation can be expressed in terms of *spherical Bessel functions*. To see the relation to Bessel's equation we make the substitutions $x = \lambda r$ and

$$\rho(r) = \frac{R(\lambda r)}{\sqrt{\lambda r}} = \frac{R(x)}{\sqrt{x}}$$

After some algebra we obtain

$$x^2 \frac{d^2 R}{dx^2} + x \frac{dR}{dx} + (x^2 - (n + \frac{1}{2})^2) R = 0$$

which is of the same form as Bessel's equation of *half integer order* (section 5.6)! It is customary to define the *spherical Bessel functions of the first kind* as

$$j_n(x) = \sqrt{\frac{\pi}{2x}} J_{n+1/2}(x)$$

and the *spherical Bessel functions of the second kind* as

$$y_n(x) = \sqrt{\frac{\pi}{2x}} Y_{n+1/2}(x)$$

These functions can actually be shown to be trigonometric functions multiplied by rational functions. $j_n(x)$ is well behaved near $x = 0$ while $y_n(x)$ is not. In the Maple worksheet
http://www.physics.ubc.ca/˜birger/p312leg.mws (or .html)
we discuss some of the properties of the spherical Bessel functions. The

spherical Bessel functions satisfy orthogonality relations, just as the Legendre polynomials, and we can expand an arbitrary function in spherical Bessel functions.

SUMMARY

- We have solved Legendre's differential equation by making a power series expansion.
- The requirement that the solution be nonsingular gave rise to the eigenvalues $\nu = n(n+1)$ where n is an integer.
- We found a simple recursion relation which allowed us to compute the Legendre polynomials.
- We showed that the Legendre polynomials were orthogonal and we argued that an arbitrary function $f(x)$ defined for $-1 \leq x \leq 1$ could be expanded in a Legendre series in much the same as it could be expanded in a Fourier or Bessel series.
- We also discussed briefly spherical Bessel functions.

Our next step in separating the variables in spherical coordinates would be to consider cases where the solution depends on all three coordinates r, θ, ϕ. We will not have time to do this, so if you need the solution you will have to look up *spherical harmonics* and *associated Legendre polynomials* in e.g. the books by Riley *et al.* or Arfken and Weber.

6 Fourier transforms

6.1 Fourier integral

SOME TIME AGO (section 4.3)we

- Introduced the complex Fourier series.

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i2\pi nx/a}$$

$$c_n = \frac{1}{a}\int_0^a f(x) e^{-i2\pi nx/a} dx$$

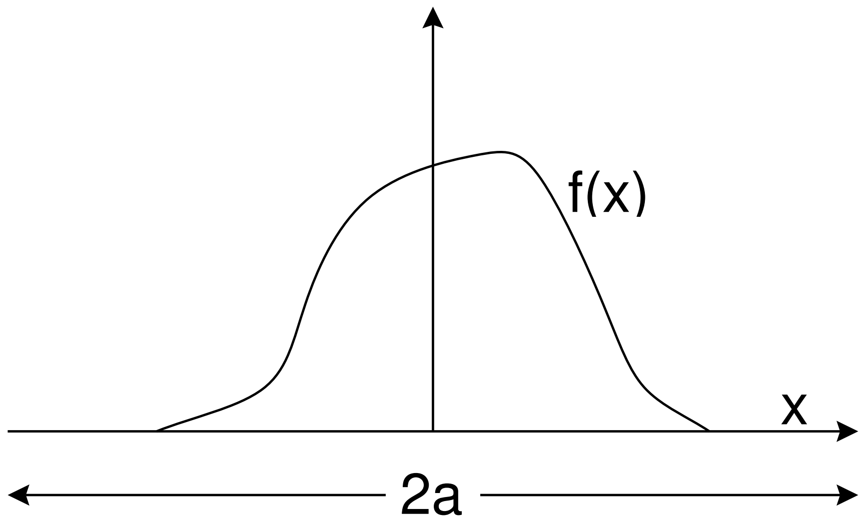
- showed by examples how Fourier series coefficients could be evaluated by Maple.

- made some comments about convergence.

TODAY

- Change range to $-a < x < a$

- Consider Fourier integral as limit of series as $a \to \infty$.



Let us first consider some consequences of taking the limit of infinite range. When we make a Fourier series expansion with in the range $-a < x < a$ we replace the function outside the range by its periodic extension. This is harmless, if function falls off rapidly for large a, and we are only interested in what happens inside the range $-a < x < a$. *We now wish to consider a non-periodic function $f(x)$ which vanishes for large $|x|$. the Fourier integral is the limit of its Fourier series as $a \to \infty$*

COMPLEX FOURIER SERIES $-a < x < a$.
It is convenient to make the range symmetric about origin and let the period

be $2a$. The complex series expansion becomes

$$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{i\pi nx/a}$$

$$c_n = \frac{1}{2a}\int_{-a}^{a} dx\, f(x) e^{-i\pi nx/a}$$

We now:

- replace n by the new variable
$$\lambda = \frac{\pi n}{a}$$
- note that λ changes by
$$\Delta\lambda = \frac{\pi}{a}$$
when n is incremented by one.
- Replace sum over n by integral
$$\sum_n = \frac{a}{\pi}\int d\lambda$$

Our final step is to define $F(\lambda)$ as

$$F(\lambda) = 2ac_n$$

and take limit $a \to \infty$. Get

$$\boxed{\begin{aligned} f(x) &= \int_{-\infty}^{\infty} F(\lambda) e^{i\lambda x} \frac{d\lambda}{2\pi} \\ F(\lambda) &= \int_{-\infty}^{\infty} f(x) e^{-i\lambda x} dx \end{aligned}}$$

COMMENTS ON NOTATION

- When x is a spatial variable λ is a $wave - vector$ and the symbol k or q is commonly used.

- When x is a time λ is a frequency and the symbol ω is often used with the opposite sign.

- Some people prefer symmetric form in which both integrals have prefactor $1/\sqrt{2\pi}$.

- Others put the prefactor $\frac{1}{2\pi}$ in front of the second integral not the first. Our notation agrees with the one used by Maple.

- $F(\lambda)$ is called the *Fourier transform* of $f(x)$.

SINE and COSINE TRANSFORMS
Recall that

$$e^{-i\lambda x} = \cos(\lambda x) - i\sin(\lambda x)$$

Assuming that $f(x)$ is real we find for the real and imaginary part of $F(\lambda)$ of $F(\lambda)$

$$\mathcal{R}F(\lambda) \equiv A(\lambda) = \int_{-\infty}^{\infty} f(x)\cos(\lambda x)dx$$

$$\mathcal{I}F(\lambda) \equiv -B(\lambda) = -\int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx$$

If we substitute

$$e^{+i\lambda x} = \cos(\lambda x) + i\sin(\lambda x)$$

into expression for $f(x)$ get

$$\boxed{\begin{aligned} f(x) &= \frac{1}{\pi}\int_{0}^{\infty}[A(\lambda)\cos(\lambda x) + B(\lambda)\sin(\lambda x)]d\lambda \\ A(\lambda) &= \int_{-\infty}^{\infty} f(x)\cos(\lambda x)dx \\ B(\lambda) &= \int_{-\infty}^{\infty} f(x)\sin(\lambda x)dx \end{aligned}}$$

Again we note that different authors places the factors of π differently in the transform integrals.
Note that
$$A(\lambda) = A(-\lambda)$$
$$B(\lambda) = -B(-\lambda)$$
The sine&cosine transforms most useful if $f(x)$ is either an even or odd function of x. In the former case $B(\lambda) = 0$, in the latter $A(\lambda) = 0$.

Can also consider functions defined for
$$0 < x < 0$$
Then the sine transform is the odd extension, while the cosine transform is the even extension.

We illustrate this by some examples in the Maple worksheet at
http://www.physics.ubc.ca/~birger/p312l7.mws (or .html)

SUMMARY
We

- introduced the complex Fourier integral and transform
- defined the Fourier sine and cosine transform
- did this by taking the limit $a \to \infty$ for the period of the function

6.2 Dirac $\delta-$ function. LRC-circuit with non-periodic forcing

.

LAST TIME

- Introduced the Fourier transform
- We also used a Maple worksheet to work out some examples.

TODAY

- As an example discuss the LRC-circuit with non periodic forcing.
- First make a detour to discuss the Dirac $\delta-$function.

DIRAC $\delta-$FUNCTION
Suppose a function $f(x)$ has the Fourier transform

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx$$

Substitute $F(\lambda)$ into the Fourier integral

$$f(y) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\lambda)e^{i\lambda y}d\lambda$$

get

$$f(y) = \int_{-\infty}^{\infty} f(x)dx\frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{i\lambda(y-x)}$$

The last integral on the right hand side is not mathematically well defined, unless we specify more carefully how to take the limits of integration.

The problem is that when substituting $F(\lambda)$ we also changed the order of integration-an operation which is not always mathematically legal.

Being physicists not mathematicians we ignore this.

Define the Dirac $\delta-$function

$$\boxed{\delta(y-x) = \frac{1}{2\pi}\int_{-\infty}^{\infty} d\lambda e^{i\lambda(y-x)}}$$

$\delta(y-x)$ is strictly speaking not a function since the integral is not well defined except under the integral sign where it has the property

$$\int_{-\infty}^{\infty} dx\ f(x)\delta(y-x) = f(y)$$

for any *arbitrary function* f.

To see why this works let is imagine that the Fourier transform of a function $f(x)$ has the property that $F(\lambda) = 0$ for $|\lambda| > \Lambda$.

$$f(y) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} F(\lambda)e^{i\lambda y}d\lambda$$

Substitute into

$$F(\lambda) = \int_{-\infty}^{\infty} f(x)e^{-i\lambda x}dx$$

get

$$f(y) = \int_{-\infty}^{\infty} f(x)dx\frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\lambda e^{i\lambda(y-x)}$$

which is can be evaluated. Define

$$\delta_\Lambda(y-x) = \frac{1}{2\pi}\int_{-\Lambda}^{\Lambda} d\lambda e^{i\lambda(y-x)} = \frac{\sin(\Lambda(y-x))}{\pi(y-x)}$$

and consider the $\delta-$function as limit

$$\delta(x) = \lim_{\Lambda\to\infty}\delta_\Lambda(x)$$

It follow that the $\delta-$function is *even*

$$\delta(x) = \delta(-x)$$

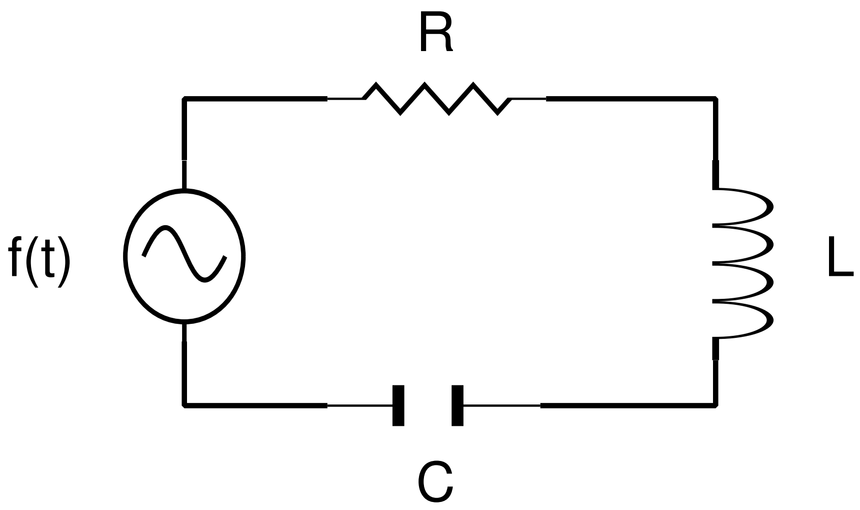
The procedure is illustrated on a Maple worksheet at
http://www.physics.ubc.ca/~birger/p312l9.mws (or .html)

LRC CIRCUIT REVISITED

$$L\frac{d^2q}{dt^2} + R\frac{dq}{dt} + \frac{1}{C}q = V(t)$$

With

$$\tau = \frac{t}{\sqrt{LC}};\; y = \frac{q}{C}$$



$$f(\tau) = V(t); 2\alpha = R\sqrt{\frac{C}{L}}$$

The differential equation in the new variables was

$$\frac{d^2y}{d\tau^2} + 2\alpha\frac{dy}{d\tau} + y = f(\tau)$$

NON PERIODIC FORCING

Fourier transform the forcing term

$$F(\omega) = \int_{-\infty}^{\infty} f(\tau)e^{-i\omega\tau}d\tau$$

where the forcing terms is given by the Fourier integral

$$f(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} F(\omega')e^{i\omega'\tau}d\omega'$$

We assume that the solution has the Fourier integral

$$y(\tau) = \frac{1}{2\pi}\int_{-\infty}^{\infty} Y(\omega')e^{i\omega'\tau}d\omega'$$

Substituting the Fourier integrals into the differential equation:

$$\int_{-\infty}^{\infty} d\omega'(-(\omega')^2 + 2i\alpha\omega' + 1)Y(\omega')e^{i\omega\prime\tau}$$

$$= \int_{-\infty}^{\infty} d\omega' F(\omega') e^{i\omega'\tau}$$

Multiply both sides by

$$\frac{1}{2\pi} e^{-i\omega\tau}$$

and integrating over τ using the properties of $\delta-$function,

$$\delta(\omega - \omega') = \frac{1}{2\pi} \int_{-\infty}^{\infty} d\lambda e^{i\lambda(\omega-\omega')}$$

we find

$$Y(\omega) = \frac{F(\omega)}{-\omega^2 + 2i\alpha\omega + 1}$$

which has to be substituted back into Fourier integral.

$$y(\tau) = \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{i\omega\tau} \frac{F(\omega)}{-\omega^2 + 2i\alpha\omega + 1} d\omega$$

To get further we need to specify the forcing function $f(\tau)$. In some cases the integral can be evaluated analytically using the method of residues from the theory of complex variables. For numerical work the *fast Fourier transform* method is usually more convenient than numerical integration of the Fourier integrals.

SUMMARY
We have

- defined the Dirac $\delta-$function by a limiting procedure.
- found that $\delta-$function only well defined under integral sign.
- Used Fourier transformation to analyze an LRC circuit with non periodic forcing.

6.3 Heat conduction in semi- infinite and infinite media.

LAST TIME

- Continued our discussion of Fourier transforms and discussed the Dirac $\delta-$function.
- Discussed, as an example, the LRC-circuit with non-periodic forcing.

TODAY
Consider some more applications of the use of Fourier transforms.

PENETRATION OF SEASONAL HEAT FLUCTUATIONS INTO SOIL
Soil is a poor conductors of heat, hence seasonal fluctuations in surface temperature do not penetrate more than a few meters down. We illustrate this by considering the following idealized problem:

Assume that the surface of a thick layer of soil is exposed to the elements and that the temperature at the surface varies as

$$T = T_0 + T_1 \cos(\omega t) = T_0 + T_1 \Re e^{i\omega t}$$

where $\omega = 2\pi/1year$, and the symbol $\Re$ stands for real part. We also assume that deep down (deep here means more that a couple of meters) the temperature is approximately constant and equal to T_0.

We assume that the climate has stayed the same for a sufficient number of years that the temperature fluctuations to be proportional to T_1. Define

$$u(x,t) = T(x,t) - T_0$$

where x is depth below the surface. We look for a solution

$$u(x,t) = \Re(\phi(x)e^{i\omega t})$$

and find

$$\phi'' = \frac{i\omega}{k}\phi$$

Noting that

$$\sqrt{i} = \pm\frac{1+i}{\sqrt{2}}$$

we find

$$u(x,t) = \Re\{c_1 \exp(-x\sqrt{\frac{\omega}{2k}}) \exp(i(\omega t - x\sqrt{\frac{\omega}{2k}})) + c_2 \exp(x\sqrt{\frac{\omega}{2k}}) \exp(i(\omega t + x\sqrt{\frac{\omega}{2k}}))\}$$

We assume that $u \to 0$ for large x hence we must have $c_2 = 0$ and find

$$\boxed{T(x,t) = T_0 + T_1 \exp(-x\sqrt{\frac{\omega}{2k}}) \cos(\omega t - x\sqrt{\frac{\omega}{2k}})}$$

We notice that as the heat penetrates deeper into the soil there is a *phase lag* so that at a depth

$$x = \pi\sqrt{\frac{2k}{\omega}}$$

the coldest time is the middle of the summer! However, at such depth the temperature fluctuations are severely damped.

INFINITE ROD

No rod is, of course, infinitely long, but sometimes from the point of view of what happens near the center, the effects of the endpoints can be neglected. As an example let us solve the following problem:

Solve

$$\frac{\partial^2 u(x,t)}{\partial x^2} = \frac{1}{k}\frac{\partial u}{\partial t}$$

subject to the *initial condition*

$$u(x,0) = f(x)$$

is a known function with Fourier transform

$$F(\lambda) = \int_{-\infty}^{\infty} dx f(x) e^{-i\lambda x} \tag{34}$$

Also, assume that $u(x,t)$ is *bounded* i.e. stays finite as $x \to \pm\infty$.

The solution to this problem is (as can be seen by substitution into the differential equation, and noting that the boundary condition is satisfied)

$$u(x',t) = \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} F(\lambda) \exp(-\lambda^2 kt + i\lambda x')$$

We substitute (34) back into the integral to get

$$u(x',t) = \int_{-\infty}^{\infty} dx f(x) \int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp(-\lambda^2 kt - i\lambda(x - x'))$$

The integral over λ can be carried out analytically, and it can be shown that

$$\int_{-\infty}^{\infty} \frac{d\lambda}{2\pi} \exp(-\lambda^2 kt - i\lambda x') = \frac{1}{\sqrt{4\pi kt}} \exp(-\frac{-(x-x')^2}{4kt})$$

The resulting function is a Gaussian with variance

$$\sigma^2 = 2kt$$

that grows linearly in time. The width of the Gaussian is proportional to the square root of the variance. Hence, the width spreads as the square root of the time. The solution to our problem can be expressed as the *convolution integral*

$$\boxed{u(x', t) = \frac{1}{\sqrt{4k\pi t}} \int_{-\infty}^{\infty} dx f(x) \exp(-\frac{(x-x')^2}{4kt})}$$

SUMMARY

- We have concluded our discussion of the one-dimensional heat conduction problem by considering heat conduction problems in semi-infinite and infinite media.
- In both cases the absence of a finite boundary simplified the problem, and we could apply Fourier methods fairly directly.
- In the case of the infinite medium we found that a Gaussian convolution integrals played a fundamental role.

References

[1] George B. Arfken and Hans J. Weber [1995], *Mathematical method for physicists* Academic Press, 4th edition.

[2] W.E. Boyce and R.C. DiPrima [1996], *Elementary differential equations and boundary value problems* 6th edition (earlier editions are also OK).

[3] Richard H. Enns and George C. McGuire [2000], *Nonlinear physics with Maple for scientists and engineers*, 2nd Edition, Birkhäuser.

[4] Jens Feder [1988] *Fractals*, Plenum.

[5] Michael Kofler [1997], *MAPLE, an introduction and reference*, Addison Wesley.

[6] Stephen Lynch [2001], *Dynamical systems with applications using Maple*, Birkhäuser.

[7] M.B. Monagan, K.O. Geddes, K.M. Heal, G. Labahn and S.M. Vorkoetter [1998], *Maple V, release 5, programming guide*, Springer.

[8] David L. Powers, *Boundary value problems*, Saunders 3rd ed. Saunders 1985.

[9] William H. Press, Saul A. Teukolsky, William T. Vetterling and Brian P. Flannery, *Numerical recipes in C, the art of scientific computing*, second ed. 1992 Cambridge University Press.

[10] Ken F. Riley, Michael P. Hobson and Stephen J. Bence [2002], *Mathematical methods for physics and engineering*, Cambridge University Press 2nd edition. The first edition from 1998 is also still useful.

[11] Stephen Wolfram [1999], *Mathematica*, Fourth edition, Addison Wesley.