The Totally Asymmetric Simple Exclusion Process (TASEP) and related models: theory and simulation results

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Synopsis

Simple exclusion processes

- The Bethe ansatz

TASEP

- Definition
- Exact density profile
- Phase behaviour
- Simulation results

Simple exclusion processes

- The simple exclusion process refers to a family of closely related, simple models for 1D particle transport.
- There is a lattice of L sites, each site is either singly occupied or empty.
- The dynamics must be specified. Two rates are defined, γ and δ, which determine the chance of a particle moving one site to the left or the right.
- If $\gamma = \delta$, this is the symmetric simple exclusion process (SSEP). If $\gamma \neq \delta$, it's the asymmetric simple exclusion process (ASEP).
- If $\gamma = 0$ and $\delta = 1$, this is the totally asymmetric simple exclusion process (TASEP).
- The boundary conditions must also be specified. Closed and periodic boundaries are often used, as well as open boundaries with specified entry and exit rates α and β.

The Bethe ansatz A commonly used technique for solving problems of this sort is to translate everything into the language of quantum spin Hamiltonian models:

- Instead of having sites occupied or unoccupied, we can simply think of them as being in one of two states.
- The dynamics of the particular process being studied can be treated by choosing the proper transition matrix between states.
- The Bethe ansatz is a particular technique for extending results for two coupled spins to the many-body system. First introduced by Hans Bethe in 1931 in his solution of the 1D Heisenberg model (like Ising model but with spins described by Pauli matrices), though he never extended the idea to stochastic processes. It has been very useful in studying SSEP and ASEP with closed or periodic boundaries, and other integrable systems.

TASEP

Definition The TASEP with open boundaries as studied by Derrida *et al.* is defined as follows:

- A 1D lattice of length L, with an entry rate α, exit rate β, and a rate of 1 for particles to move to the right.
- At each timestep t, pick a random lattice site i between 0 and L. If $i \neq 0$ and $i \neq L$, and site i is occupied, and site $i + 1$ is not occupied, then the particle moves from i to $i + 1$
- If $i = 0$, and site 1 is not occupied, then a new particle moves into site 1 with probability α.
- If $i = L$, and the last site is occupied, it becomes empty with probability β.

Exact density profile

- The Bethe ansatz is not well suited to the TASEP with open boundaries because the model violates conservation of particle number.

- The TASEP just described was solved exactly by Derrida *et al.* for $\alpha = \beta = 1$, soon afterwards Schütz and Domany solved it for all α and β.

- The solution consists of defining the master equation for the process, then using generating function techniques to find the final density profile.

- Well, sort of. What they *actually* did was, they solved the master equation by hand for small L on a computer, looked at the final results and made some guesses about what the solution might be for general L.

The exact density profile for $\alpha = \beta = 1$ at site K is given by

$$\langle \rho_K \rangle = \frac{1}{2} + \frac{(2K)!(L!)^2(2L-2K+2)!(L-2K+1)}{4(K!)^2(2L+1)![(L-K+1)!]^2}$$

Phase behaviour This simple model shows a surprisingly rich phase behaviour! This can best be understood by defining the lattice derivative $t_K = \rho_{K+1} - \rho_K$. It turns out that t_K can be expressed as a product of a function of α and a function of β, which means that we should expect phase transitions to occur for some critical value of α, independent of β (and vice-versa). This critical value turns out to be $1/2$.

We want to define a length scale $\xi_\sigma^{-1} = -\ln(4\sigma(1-\sigma))$, where $\sigma = \alpha$ or β. Note that ξ_σ diverges as $\sigma \to 1/2$. We will also see a combined length scale defined as $\xi^{-1} = \xi_\alpha^{-1} - \xi_\beta^{-1}$.

Now, the behaviour of the system can be completely specified by finding the expression for the lattice derivative t_K in each phase as well as the boundary densities $\langle\rho_1\rangle$ and $\langle\rho_L\rangle$. We can also get the current $j = \langle\rho_k(1-\rho_{k+1})\rangle$.

We start our tour of the phase diagram by considering the situation if $\beta < \alpha$ and $\alpha < 1/2$. Here both length scales contribute and we find an exponential decay of the density profile. We have $j = \beta(1-\beta)$, $\langle\rho_1\rangle = 1 - \frac{\beta}{\alpha}(1-\beta)$ and $\langle\rho_L\rangle = 1-\beta$. Also,

$$t_K = (1-2\alpha)(1-e^{-1/\xi})e^{-K/\xi}$$

This is a high density phase, the particles enter more quickly than they can leave. $\rho_{\rm bulk} = 1-\beta$.

If β remains small but $\alpha \to 1/2$, then ξ_α diverges and the exponential decay depends only on β. We get for the slope of the profile

$$t_K = \frac{(1-2\beta)^2}{2\sqrt{\pi}} K^{-1/2} e^{-K/\xi_\beta}$$

This is still in the high density phase, the current and boundary densities are identical to those obtained above. If we continue to increase α past the critical value the slope changes again:

$$t_K = \frac{(1-\alpha-\beta)(\alpha-\beta)}{(1-2\alpha)^2\sqrt{\pi}} K^{-3/2} e^{-K/\xi_\beta}$$

The slope changes sign for $\alpha = \rho_{\mathrm{bulk}} = 1-\beta$.

Keeping $\alpha > 1/2$, we now increase β to its critical value of $1/2$. Now ξ_β diverges, and we get

$$t_K = -\frac{1}{4\sqrt{\pi}}(1 - \frac{K}{L})^{-1/2} K^{-3/2}$$

Near the left boundary, we see a power-law decay as the $K^{-3/2}$ term dominates. Near the right side, except for terms of order $1/L$ the profile is flat, and approaches the bulk density $\langle \rho_L \rangle = \rho_{\text{bulk}} = 1/2$. We also have $\langle \rho_1 \rangle = 1 - \frac{1}{4\alpha}$. The current reaches its maximum value, $j = 1/4$.

If we now continue to increase β, we enter a phase characterized by the fact that the current is a maximum throughout the phase, with $j = j_{\text{max}} = 1/4$. The slope of the profile changes slightly,

$$t_K = -\frac{1}{4\sqrt{\pi}}(1 - \frac{K}{L})^{-3/2} K^{-3/2},$$

so the density goes like $K^{-1/2}$ near the left side and $(L-K)^{-1/2}$ near the right side. The boundary densities are $\langle \rho_1 \rangle = 1 - \frac{1}{4\alpha}$ and $\langle \rho_L \rangle = \frac{1}{4\beta}$.

Finally, we come full circle and keeping $\beta > 1/2$, we lower α past its critical value. We now enter a low density phase. Due to the particle-hole symmetry of this model, the low density phase can simply be seen as a mirror image of the high density phase where α and β switch roles and the density of holes is analogous to the density of particles.

I saved the best for last! What happens between the high density phase and the low density phase? On the line $\alpha = \beta < 1/2$, both correlation lengths ξ_α and ξ_β remain finite, however since they are equal, the effective length scale ξ diverges, since $\xi^{-1} = \xi_\alpha^{-1} - \xi_\beta^{-1}$! The density profile is linear with slope

$$t_K = (1 - 2\alpha)/L$$

The current is $j = \alpha(1 - \alpha)$, and the boundary densities are $\langle \rho_1 \rangle = \alpha$ and $\langle \rho_L \rangle = 1 - \alpha$.