The Department of Physics & Astronomy

The University of British Columbia

PHYSICS 314

FLUIDS

Summary of lectures by

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Part I
INTRODUCTION

1 Course organization

The purpose of the course is to present a survey of the properties of *fluids*, a term which describes both *gases* and *liquids*. We start by describing the different forms of matter and common types of interaction between atoms and molecules. We then move on to kinetic theory. Observable quantities, such as thermodynamic variables, are interpreted as averages over probability distributions. We present, for gases, the Maxwell-Boltzmann velocity distribution, and the more general idea of a Boltzmann factor. Results for diffusion, thermal conduction, and viscosity are derived from the concepts of collision cross section and mean free path. We then show how these ideas are modified when working with dense fluids. Generally, *diffusion* has the effect of smoothing out concentration inhomogeneities. However, diffusive processes can also lead to *pattern formation* with important consequences in geology and biology.

Most of the rest of the course is spent on an introduction to hydrodynamics. We start out with laminar flow, including a discussion of non-Newtonian fluids, continue with the Bernoulli equation, and move on to convection, and turbulence. We will also discuss waves in fluids, including capillary and gravity waves, tidal waves in a shallow channel, sound waves and shocks. In the case of flow problems one often finds that it is infeasible to work all the way from first principles, and there will be some discussion of modeling and the use of dimensional analysis. An attempt has been made to stick with SI units throughout the course, although I have not been completely successful in this respect. I have also included a rather long review section consisting of old exam problems. Some of these come with solutions, others don't!

It is expected that students taking this course will have had a course in thermal physics such as PHYS 313 (or an equivalent course in Chemistry or Geophysics). The way it is commonly expounded, the theory of fluids will present formidable mathematical obstacles. I try in this course to emphasize qualitative aspects avoiding excessive formalism. Nevertheless, use of the material in the standard second year mathematics courses (MATH 200, 221, 215) cannot be avoided. It would also be an advantage, but it is not required, for students to have had a second year mechanics course such as PHYS 216.

There will be 6 regular assignments, a midterm and a final exam. I will also require a short term paper. The basic marking scheme is 20% assignments, 20% term paper, 20% midterm and 40% final.

I don't know of any available text that covers all the material in this course. For an elementary account of the properties of matter and kinetic theory I recommend the books by Flowers and Mendoza [23] and Tabor [51]. The student may also find the books by Gopal [27], Kittel and Krömer [34] and the more advanced book by Reichl [46] useful for the kinetic theory part. When it comes to hydrodynamics I found that chapters 40 and 41 in Volume II of the Feynman lectures [22] offer a fresh point of view. The book by Massey [38] contains much information and is easy to read, and Pnueli and Gutfinger [43] and Sabersky et al. [48] are also helpful. The text by Main [36] contains an excellent discussion of waves. Other more specialized references will be given when appropriate.

I have "borrowed" freely from old exams prepared by my colleagues David Balzarini, Meyer Bloom, Roland Cobb, Harold Davis, Dan Murray and Murray Neuman when selecting problems and examples. I am grateful to Michael Craddock for bringing errors and typos to my attention.

The course description in the UBC Calendar is

> PHYS 314 (3) Fluids. Kinetic theory: Diffusion, viscosity and sound waves. Introduction to hydrodynamics: Laminar flow, capillary and gravity waves, convection and turbulence. Dimensional analysis. Pre-requisite: 1 of PHYS 203, PHYS 313. Corequisite: MATH 215. [3-0-0]

2 Gases, liquids and solids.

The materials which we meet in our daily lives, air water, metals semiconductors, rocks, plastics, can all be classified as being either gases, liquids or solids. In this course we generally consider matter to be made up of *molecules.* These are not the fundamental building blocks of nature since molecules in turn are made up of atoms, which contain electrons, protons and neutrons. The latter are composites of quarks and gluons etc. The reason we work with molecules as the basic building block is that they tend to stay intact over a reasonable time under the conditions encountered in the study of *materials.* We shall be concerned with the forces between molecules in a fairly general way. The origin of the forces lies in the behavior of the electronic charge clouds around the nuclei. For this reason *quantum mechanics* is required to compute such forces, but once the forces are known one can often to a good approximation use Newtonian mechanics to work out their effect. We will in this course concentrate on achieving a qualitative understanding of the properties of fluids, and quite idealized models are often adequate for this purpose. It is not only very difficult to obtain a quantitative understanding of detailed *macroscopic* properties from *microscopic* forces, but when the goal can be achieved, the calculations are often quite tedious.

A distinction between *gases, liquids and solids* can be made, because sharp transitions take place between them. For example, the substance water can freeze to form ice or boil to form a vapor, and the processes can be reversed through melting or condensation. Our climate is to a very large extent determined by the transitions between these phases.

The least ordered phase is the *gas*. For the present purposes a gas is a substance which fills the space available. Gases have relatively low density and high *compressibility*. A gas has no *rigidity*, i.e. there is no resistance to changes in shape if the change is carried out slowly. The *viscosity*, or resistance to *shear*, is low. The molecules are usually far apart compared to their diameter, and they can move over distances corresponding to many diameters without experiencing collisions. The spatial distribution of molecules is close to random. The gas phase occurs at low pressures and high temperatures. At ambient pressures all known substances condense if cooled to low temperatures. The *kinetic theory* of gases is a favorite tool to describe the equilibrium properties of gases. As we shall see, we can also use this theory to derive expressions for important quantities such as the, *thermal conductivity, viscosity* and *diffusivity which describe how gases respond when disturbed from equilibrium.*

When a gas is cooled at ambient pressures it will as a rule condense to form a *liquid*. This phase is characterized by a higher density than the gas phase. At atmospheric pressure a density difference of a factor of 10^3 is common when a vapor condenses to form a liquid. If the pressure is lowered a liquid may evaporate or boil, but it is a *self-bound system*, which cannot expand to fill an arbitrarily large volume. Since the density is not strongly dependent on the pressure the compressibility is low. The molecules in a liquid are densely packed, one cannot describe a liquid as a collection of independent particles that only occasionally interact through collisions. Instead there is considerable amount of local molecular order caused by the dense packing. Liquids have no rigidity . A liquid can change its shape without any cost of bulk energy if the change is done slowly enough. If we generate flow with nonuniform flow velocity there will be frictional or *viscous* forces which we characterize by the *viscosity.* Liquids generally have a higher viscosity than gases.

The transition from a gas to a liquid can take place suddenly by a *phase transition.* At the transition liquid and vapor coexist. For mechanical stability the coexisting phases must have the same pressure so at a given temperature the liquid has a certain *vapor pressure.* There will also be a *latent heat* associated with the change of phase. As the temperature is increased the vapor pressure will increase, the latent heat will decrease and the discontinuity in density will decrease. Finally, at the *critical point* the latent heat and the density discontinuity vanishes and there is a continuous change between a liquid and a gas. Thus, apart from the phase transition, which only occurs in a certain temperature range, the difference between a liquid and a gas is not fundamental; they are both *fluids.* A liquid is a high density fluid, while a gas is a low density fluid, and a vapor is a gas whose temperature is below the liquid gas critical point.

When a fluid is cooled it will eventually solidify. (An interesting exception is helium, which at low pressures stays fluid down to the lowest recorded temperatures, and instead undergoes a transition to a superfluid.) *Solids* tend to have densities and compressibilities which are similar to those of liquids, and the latent heat in a transition from a liquid to a solid is usually less than for the liquid gas transition. However, solids are rigid and do not change their shape easily under the action of small forces, while a liquid shows no resistance to shear if applied slowly enough.

Glasses are forms of matter intermediate between solids and liquids. They are hard and brittle, i.e. their mechanical properties are those of a solid. Atomically they are non crystalline *supercooled liquids*, and they are generally believed to be only *metastable* and should eventually crystallize. However, the time required may be extremely long.

For a pure substance, the pressure P is in principle determined, if the number of particles N, per unit volume V, and the temperature T are known. The functional form of P is known as an *equation of state* and can be plotted as an equation of state surface (see figure 1). Ideally, one would like to be able to compute the equation of state, and to predict the location of interesting points such as the critical point where the vapor liquid coexistence line ends, and the triple point where the solid, liquid and vapor phases coexist. This will require a knowledge of the interaction between the molecules. If the molecules are *ions* the dominant forces are electrostatic in origin, and the interaction energy between two charges q_1 and q_2 a distance

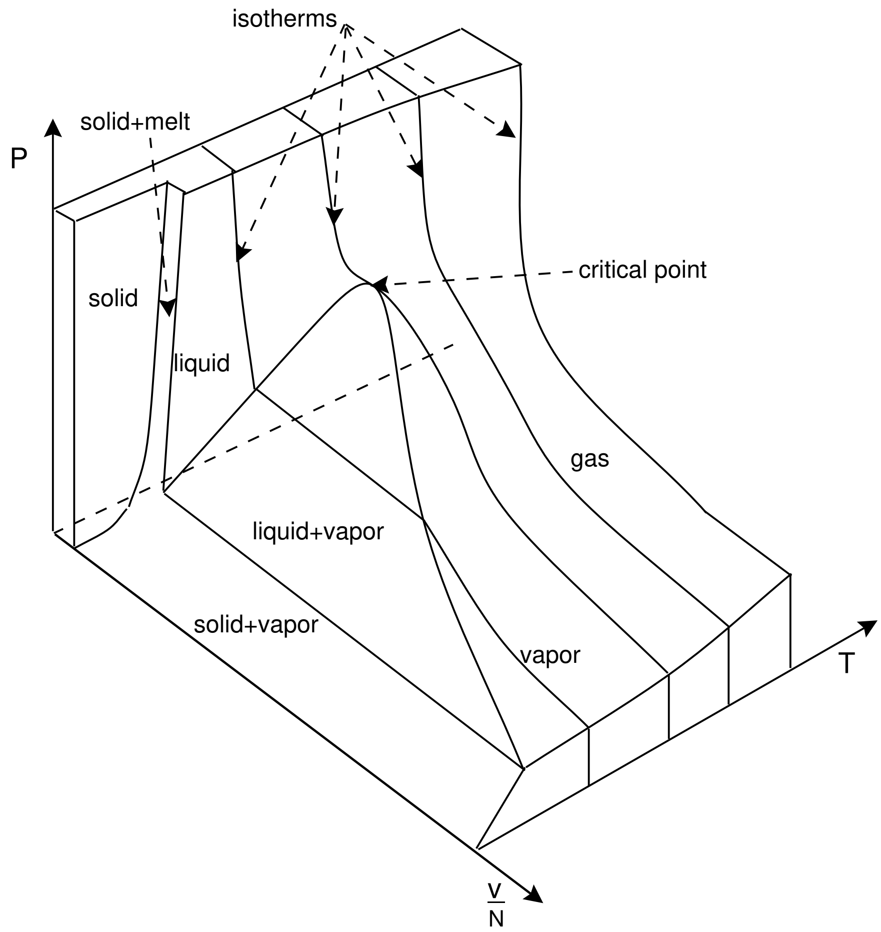


Figure 1: Equation of state surface $P(\frac{V}{N}, T)$.

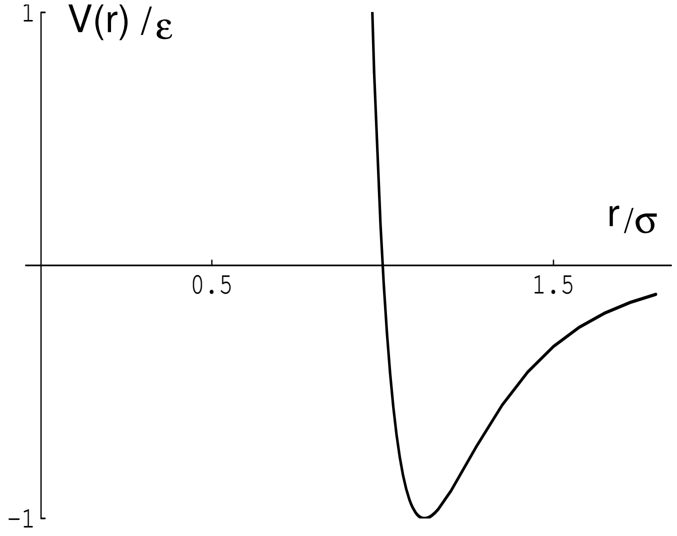


Figure 2: Lennard-Jones potential.

r apart is given by the *Coulomb potential*

$$V(r) = \frac{q_1 q_2}{4\pi\epsilon_o r} \tag{1}$$

The interaction between the atoms within a molecule is typically *covalent*. The covalent energy is quantum mechanical in origin and comes about because electrons occupy orbitals that are shared between the different atoms of a molecule. Covalent forces can be very strong and they are directional, i.e. they depend not only on the distance between atoms but on bond angles. Covalent forces are usually not important between molecules.

There are two main contributions to the potential energy $V(r)$ associated with two neutral molecules a distance r apart. If the two molecules get so close together that the electronic clouds of the nuclei significantly overlap, there is a strong *repulsion* due to the *Pauli exclusion principle*. At larger separation the dominant contribution comes from *van der Waals attraction*, which is caused by fluctuating dipole forces, and the van der Waals potential energy is proportional to the inverse sixth power of the separation. A commonly used interaction energy for spherical molecules is the *Lennard-Jones potential*

$$V(r) = 4\epsilon \left[\left(\frac{\sigma}{r}\right)^{12} - \left(\frac{\sigma}{r}\right)^{6} \right] \tag{2}$$

This potential is shown in figure 2. It is easy to verify that the potential has a minimum $-\epsilon$ at separation $r_0 = 2^{1/6}\sigma$. The interaction is repulsive for $r < r_0$ and is attractive for $r > r_0$. There is no particular reason to use the value 12 for the exponent associated with the repulsive term, and other forms are sometimes employed. The $1/r^6$ behavior of the attractive part of the interaction is, however, realistic for neutral molecules. The force $\mathbf{F}$ between two spherical molecules can be expressed in terms of the gradient of the potential

$$\mathbf{F}(\mathbf{r}) = -\hat{\mathbf{r}}\frac{dV(r)}{dr} = -\nabla V(\mathbf{r}) \tag{3}$$

Here $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.

Computer simulations using *Molecular dynamics* (see e.g. Verlet [56, 57]) have demonstrated that the Lennard-Jones potential (2) can give a quite adequate description of equation of state related properties of "simple" substances such as the rare gas elements argon, krypton and xenon.

For molecules, more complicated potentials than (2) are required, but molecular dynamics simulations still often work quite well for equilibrium properties. When it comes to dynamical properties one faces the formidable obstacle that even a "state of the art" simulation describing millions of collisions per molecule will correspond to only a fraction of a microsecond in real time. For this reason we will not attempt a completely microscopic description of fluids in this course, but we will instead take a more phenomenological approach. We will not be interested in the equilibrium properties of materials as much as in how a system responds when disturbed from equilibrium. A major theme will be to relate material properties such as diffusivity and viscosity to macroscopic flow properties.

For an overview of condensed matter physics at an introductory level I recommend the text by Barber and Loudon [4].

2.1 Problems

Problem 2-1.

> When solving problems it is frequently practical to work in reduced units in order to identify characteristic scales of energy, length and time. To illustrate this point consider two small magnets of mass m, arranged so that they repel each other, as shown in figure 3. The magnets are restrained from moving horizontally.

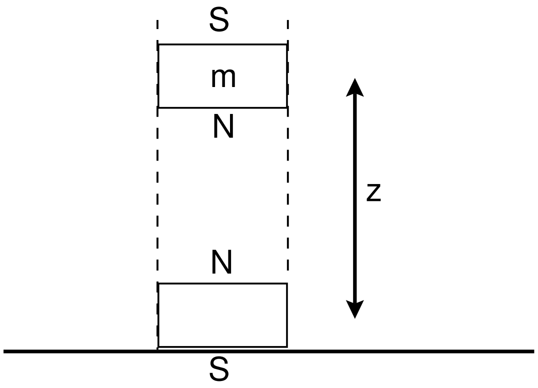


Figure 3: Figure to Problem 2.1

The interaction energy between two magnetic dipoles is proportional to the inverse third power of the distance. Write c/z^3 for the magnetic interaction energy, with c a constant.

(a) Plot the magnetic, gravitational and total potential energy against height z. Use $(mg)^{3/4}c^{1/4}$ as unit of energy and $(c/mg)^{1/4}$ as unit of height.

(b) Find the equilibrium height of the top magnet.

(c) Calculate the frequency of small oscillations about the equilibrium position.

Problem 2-2.

The two columns to the left in table 1 show data for the density of water in the range 0 to 10°C, while the two columns on the right present more detailed data in the range 0 to $0°C$.

(a) Fit the two data sets to the expression

$$\rho(T) = a + bT + cT^2$$

where T is the temperature in °C and a, b, c are constants.

(b) Make separate plots for the two temperature regions and show the data points and the two fits.

You should find that both fits appear to work well in their respective regions. However, the overall fit will not agree well with data in the smaller region. Similarly, the fit to the smaller region doesn't give a very good overall fit.

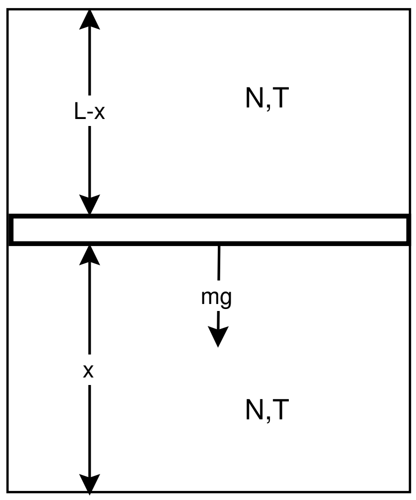


Figure 4: Figure to Problem 2.3

Problem 2-3.

A sealed cylindrical vessel (see figure 4) is in contact with a heat bath at constant temperature T. A friction-less airtight piston of weight mg divides the container into two volumes V_1 and $V_2 = V - V_1$. There are N ideal gas atoms in both the top and in the bottom partitions.

(a). Find the equilibrium height of the piston.

(b). The ideal gas in the two partitions is replaced by steam. At a certain temperature the bottom partition is found to contain a puddle of water coexisting with its vapor. Which of the following statements may be true at equilibrium:

1. The top partition contains water in coexistence with its steam.
2. The top partition contains only steam.
3. The top partition contains only liquid water.

Table 1: Data for the density of water

$T[^\circ\mathrm{C}]$	$\rho[\mathrm{kg\ m}^{-3}]$	$T[^\circ\mathrm{C}]$	$\rho[\mathrm{kg\ m}^{-3}]$
0	999.841	0	999.841
10	999.700	1	999.902
20	998.203	2	999.941
30	995.646	3	999.965
40	992.21	4	999.973
50	988.04	5	999.965
60	983.21	6	999.941
70	977.79	7	999.902
80	971.80	10	999.700
90	965.31		
100	958.35		

3 Probabilities and averages.

Because of the important role of statistics, we will need to review some needed concepts before we can proceed with our main subject.

There are two main approaches to the problem of estimating how often possible outcomes of random events will occur. We may predict the frequency of allowed outcomes in repeated experiments from *a priori* (first principles) knowledge of the probabilities of contributing factors, making use of properties of permutations, combinations and binomial coefficients. Alternatively, we may be able to observe the possible outcomes as they occur, and we may wish to estimate *a posteriori* (after the fact) their probability from measured frequencies of occurrence.

In the latter case, one way to proceed is to plot the data in a bar chart (or histogram). As an example let us consider the distribution of *speeds* v of a system of N particles. We divide the range of v into intervals or *bins* of width Δ. The speed of a given particle lies in the i'th bin if its speed is between v_i and $v_i + \Delta$. We then count the number n_i in each bin and plot the result. If we only include a moderate number of particles, the resulting bar chart will typically have a ragged shape as in figure 5 a. On the other hand if we measure the speeds of a much larger number of particles it is likely to have a more regular shape as indicated in figure 5 b.

According to the *law of large numbers* the relative frequency after many measurements will almost always be close to a limiting value. This limiting value is the *probability* p_i, of any given particle being in the i-th bin

$$p_i = \lim_{N\to\infty} \frac{n_i}{N} \tag{4}$$

An important property of probabilities is the fact that if two possible events i and j are *exclusive* the probability that one or the other happens is

$$p(i.or.j) = p(i) + p(j) \tag{5}$$

Similarly if two possible events i and j are *independent* the probability that they both happen is

$$p(i.and.j) = p(i)p(j) \tag{6}$$

If the two events are not independent they are said to be *correlated*. The probability distribution for exclusive events satisfies the normalization condition

$$1 = \sum_i p_i \tag{7}$$

The histogram represents a *discrete* distribution. The particle speed can in general take on a continuous range of values. We obtain the *continuous probability distribution* by taking the limit

$$p(v_i) = \lim_{\Delta\to 0} \frac{p_i}{\Delta} \tag{8}$$

The continuous probability distribution associated with the particle speed may then look like figure 8(a) in the next chapter. The probability that the speed of a particle is between v_i and $v_i + dv$ is p$(v_i)dv$, and the continuous probability distribution satisfies the normalization condition

$$\int p(v)dv = 1 \tag{9}$$

Equation (9) is a statement of the fact that since all the particles must have some speed, the probabilities will add up to one. The *mean speed* is given by

$$\langle v\rangle = \int vp(v)dv \tag{10}$$

Let $f(v)$ be some property associated with the speed of a particle (e.g. the particle kinetic energy is given by $f(v) = \frac{1}{2}mv^2$, where m is the particle mass). The *average* value of $f(v)$ is then

$$\langle f(v)\rangle = \int f(v)p(v)dv \tag{11}$$

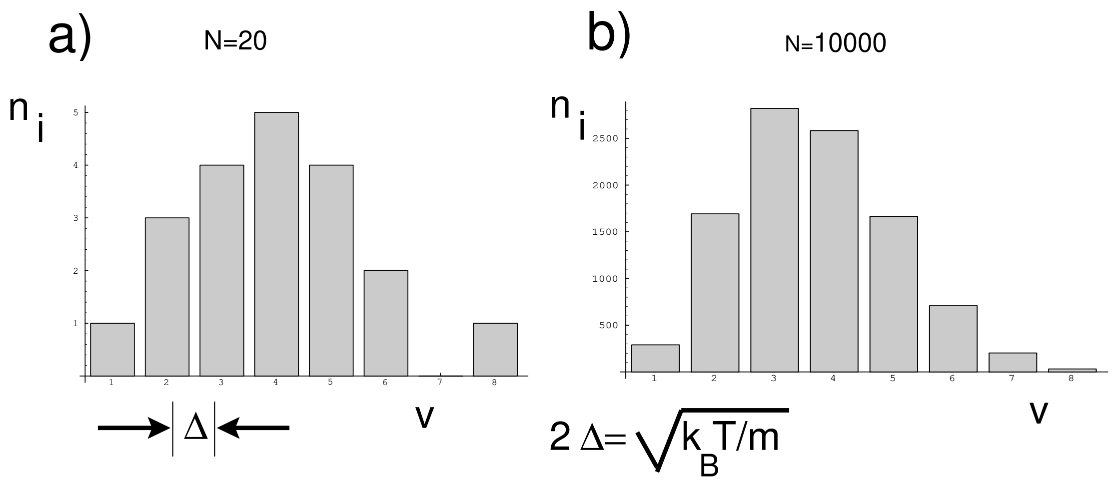


Figure 5: Speed distribution for (a) a large (b) a small number of particles.

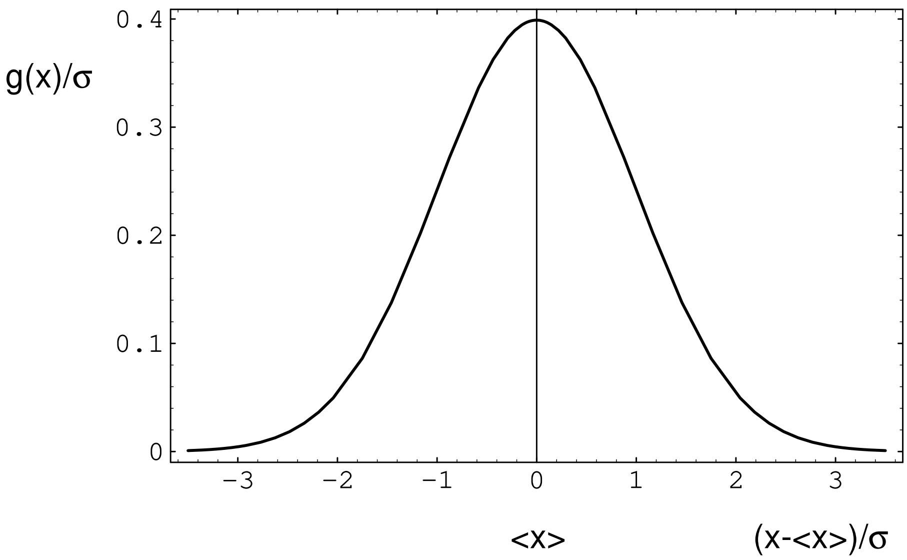


Figure 6: The normal distribution.

In general, we will use angular brackets $\langle\ \rangle$ to describe averages. When applied to a probability distribution, the mean is the average outcome. Since we are dealing with probabilities, actual outcomes will typically deviate or *fluctuate* from the average. Let x be a *stochastic variable* , i.e. a quantity that can take on different values with some probability. A useful estimate of a typical deviation of the value of x from the mean is the *standard deviation* σ or root mean square fluctuation

$$\sigma = \sqrt{\langle (x - \langle x \rangle)^2 \rangle} = \sqrt{\langle x^2 \rangle - \langle x \rangle^2} \tag{12}$$

A closely related quantity is the *variance:* $var = \sigma^2$. A probability distribution which occurs in many context is the *Gaussian* or *normal distribution,* which can be expressed in terms of the mean and variance as

$$g(x) = \frac{1}{\sqrt{2\pi(var)}} \exp\left[\frac{-(x - \langle x \rangle)^2}{2(var)}\right] \tag{13}$$

The normal distribution is plotted in figure 6. The importance of this distribution is related to the *central limit theorem* of statistics:

> Suppose x_i is a random variable and X is the sum of N independent such variables all with the same mean $\langle x \rangle$ and variance

$var = \langle x^2 \rangle - \langle x \rangle^2$ In the limit that N gets very large the probability distribution for X will be Gaussian with mean $N\langle x \rangle$ and variance $V = N\ var$.

Note that we are not assuming that the probability distribution for x is Gaussian - only that it has a finite mean and a variance.

3.1 Outcome of coin tosses

A coin is tossed N times. We associate the value $x = +1$ with "heads" and $x = 0$ with "tails". The average value $\langle x \rangle$ of x is then $\frac{1}{2}$ if the coin is unbiased. The average value of x^2 is also $\langle x^2 \rangle = \frac{1}{2}$. The variance associated with a single toss is thus $var = \langle x^2 \rangle - \langle x \rangle^2 = \frac{1}{4}$. After N tosses the number of times heads has come up will be X. By the central limit theorem the mean value of X will be $\langle X \rangle = N/2$, the variance will be $V = N/4$, and the probability distribution in the limit of large N will be

$$p(X) = \sqrt{\frac{2}{\pi N}} \exp\left[\frac{-(2X-N)^2}{2N}\right] \tag{14}$$

This result can be derived independently using the *binomial distribution*. There are two possible outcomes of each toss - heads or tails. After N tosses there are thus 2^N possible outcomes. Of these

$$\frac{N!}{X!(N-X)!} \tag{15}$$

correspond to the outcome X. The probability distribution for X is thus in general

$$p(X) = \frac{N!}{2^N X!(N-X)!} \tag{16}$$

If N is very large we can use the *Stirling approximation*

$$\ln(z!) \simeq \frac{1}{2}\ln(2\pi) + (z + \frac{1}{2})\ln z - z \tag{17}$$

as $z \to \infty$. It is left as an exercise for the reader to use (16) and (17) to derive (14).

Not all distributions have a finite mean or a variance. A famous counterexample, discussed already in the early days of probability theory, is Bernoulli's *St. Petersburg paradox*. Imagine that you are allowed to play the following game against "the bank". The banker tosses a fair coin. If the outcome is "tails" you are given a ducat (or whatever they used for money in St. Petersburg towards the end of the 18th century). If the outcome is "heads" the coin is tossed again until the outcome is "tails". If tails come up in the second trial you get 2 ducats. If tails only comes up after n trials you get 2^{n-1} ducats.

The probability that tails comes up for the first time after n trials is the probability 2^{-n+1} that the first $n-1$ trials yielded "heads" times the probability $\frac{1}{2}$ that the next trial gives heads, or 2^{-n}. The mean profit is then

$$\langle \$ \rangle = \frac{1}{2} + \frac{1}{4}2 + \ldots \frac{1}{2^n}2^{n-1} + \ldots = \frac{1}{2} + \frac{1}{2} + \frac{1}{2} + \ldots = \infty \tag{18}$$

Thus, although the probability distribution is well defined with a finite value for each possible outcome, the mean is infinite! How much would you be willing to pay to be allowed to take part in this game?

In what follows we will not encounter further "pathological" probability distributions of this type and the central limit theorem will be "safe".

3.2 Some properties of Gaussians

When dealing with Gaussian distributions the following integrals are useful

$$\int_{-\infty}^{\infty} \exp(-ax^2)dx = 2\int_{0}^{\infty} \exp(-ax^2)dx = \sqrt{\frac{\pi}{a}} \tag{19}$$

$$\int_{-\infty}^{\infty} x^2\exp(-ax^2)dx = 2\int_{0}^{\infty} x^2\exp(-ax^2)dx = \frac{1}{2a}\sqrt{\frac{\pi}{a}} \tag{20}$$

$$\int_{-\infty}^{\infty} x\exp(-ax^2)dx = 0 \tag{21}$$

$$\int_{0}^{\infty} x\exp(-ax^2)dx = \frac{1}{2a} \tag{22}$$

The *error function* is defined as

$$\mathrm{erf}(x) = \frac{1}{\sqrt{\pi}}\int_{-x}^{x} \exp(-s^2)ds \tag{23}$$

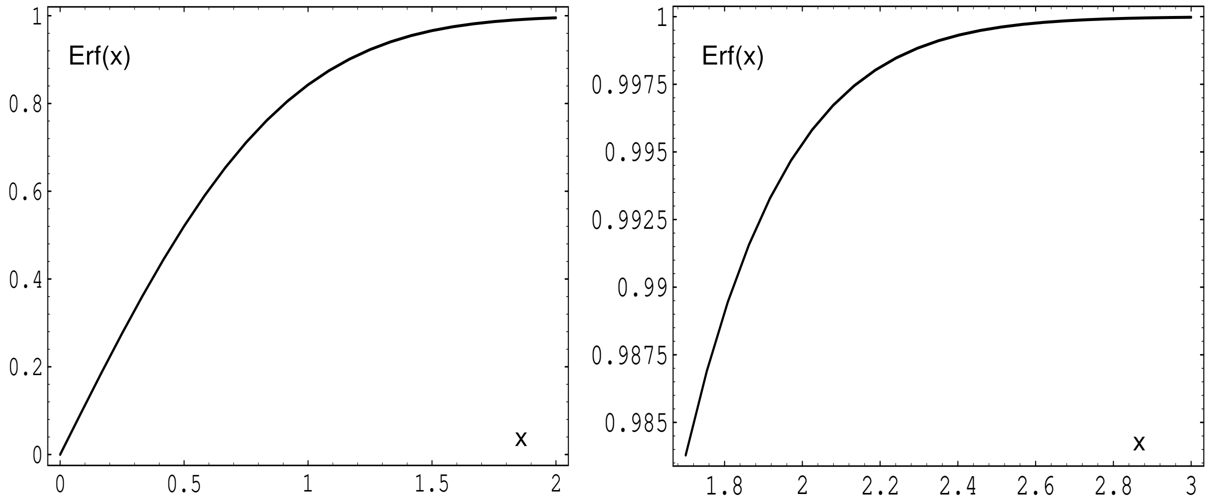


Figure 7: Error function

and has the properties

$$\mathrm{erf}(0) = 0; \quad \mathrm{erf}(1) = 1; \quad \mathrm{erfc}(x) \equiv 1 - \mathrm{erf}(x) \tag{24}$$

where $\mathrm{erfc}(x)$ is called the complementary error function. Tables of the error function are readily available. Plots of the error function is given in figure 7.

3.3 Problems

Problem 3-1.

a). Plot a histogram for the probabilities of the outcome of 5 tosses of a fair coin (5 heads, four heads one tail, etc.). Are there any significant differences between results obtained using the binomial distribution and the Gaussian approximation?

b). Estimate the probability of 1050 heads in 2000 tosses of a fair coin.

c). Estimate the probability of more than 1050 heads in 2000 tosses of a fair coin.

Problem 3-2.

Derive (14) from (16) using the Stirling formula (17) assuming N to be large.

Problem 3-3.

(a). A person puts with equal probability either a red or a white marble into an urn. Subsequently a red marble is added to the urn. The urn is shaken and one of the two marbles is selected randomly and picked up from the urn. The selected marble is red. What is the probability that the remaining marble is red?

Problem 3-4.

In a television game show the player is presented with three doors and told that there is a brand new luxury car behind one of them. Behind the other two doors there is nothing of value. The player is asked to guess behind which of the three doors the car is hidden and will win the car if the guess is correct. The player will tell the guess to the host. The host will then open one of the other doors. There is nothing of value behind that door. Finally the host asks if the player will change her or his guess. Is this a good idea for the player to do?

Part II
Kinetic theory

Thermal energy is associated with the random motion of molecules. Let us first consider a gas of low density. The molecules will move in straight trajectories until they collide with other molecules or the walls of the gas container. After a few collisions it becomes practically impossible to relate the velocity and position of the molecules to the corresponding quantities at an earlier time. The difficulty is not just the enormous amount of data required to describe a large number of particles. A more fundamental problem is the fact that after a few collisions the positions and the velocities of the particles become extremely sensitive to the initial conditions. A very similar situation occurs when throwing an unbiased die or tossing a coin. In principle, it should be possible to predict the outcome of the toss using Newton's laws and the initial velocity and position. In practice, the calculation will not be able to predict the behavior of real coins, because initial conditions that give rise to radically different outcomes are so close together that it is difficult to specify the initial conditions and parameters of the problem with sufficient accuracy. This type of motion has been described as *chaotic*. Each particle is just as likely to move in any direction as in any other, and the speed of the particles is frequently changing. Therefore, if we wish to describe the properties of a gas in terms of the interaction between molecules, a statistical description is essential.

We distinguish between the random motion of a molecule and bulk (ordered) movement. An example of the latter is the flight of a solid object such as a pebble thrown in the air. A situation which is more difficult to analyze is the flow of a liquid. A "bulk" velocity can then be defined for each small region of the fluid. In all cases the separation between bulk and random motion can be made by an averaging process. The bulk motion will survive an average over molecular velocity, while the random motion averages out as noise. This averaging is reasonably straightforward when the flow is laminar (streamlined). However, when a flow is turbulent what appears as bulk motion on one length scale may be random on a larger scale.

We now assume that we can distinguish bulk and random motion, and we associate the concept of temperature with the random motion. A molecule will in general possess both kinetic and potential energy. In a gas, at low density, the molecules spend most of their time far apart. The interaction energy falls off rapidly with distance and the potential energy can often be neglected. In a solid or a liquid the potential energy plays a key role in holding the substance together, and the potential energy is often much larger in magnitude than the kinetic energy. Only the kinetic energy contributes to the temperature, however. Also, it must be stressed that not all forms of motion can be considered to be thermal, any bulk motion must be eliminated by transforming to a frame of reference in which the average velocity is zero.

Although only the random motion contributes to the temperature, bulk motion can be converted or dissipated into heat, and thus cause a rise in the temperature. From the point of view of thermodynamics the state with bulk motion is then a *non-equilibrium state*, and the degradation of ordered motion into heat through frictional forces constitutes *approach to equilibrium*. Kinetic theory is concerned with a number of *dissipative processes* such as diffusion, thermal conduction and viscous flow.

4 Maxwell velocity distribution.

The foundation of statistical physics was laid about 100 years ago. Important names[1] in this context are Gibbs, Boltzmann and Maxwell. Boltzmann and Maxwell were among other things interested in the probability distribution for velocities in an ideal gas. Let $\mathbf{v}$ be the vector velocity of a particle with speed $v = |\mathbf{v}|$. If the gas is at rest, and in equilibrium with respect to its container, they found that the probability distribution for the velocity of each particle is given by

$$p(\mathbf{v}) = \left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{mv^2}{2k_B T}\right) \tag{25}$$

We interpret this formula by saying that the probability that the velocity $\mathbf{v} = \{v_x, v_y, v_z\}$ has components between v_x and $v_x + dv_x$, v_y and $v_y + dv_y$, v_z and $v_z + dv_z$ is

$$p(\mathbf{v})dv_x dv_y dv_z \equiv p(\mathbf{v})d^3v \tag{26}$$

The probability distribution (25) is normalized

$$\int d^3v\ p(\mathbf{v}) = 1 \tag{27}$$

as can be seen by substituting $a = \frac{m}{2k_BT}$ into (25) and using (20) to get

$$\int d^3v\ \exp\left(-\frac{mv^2}{2k_BT}\right) = 4\pi \int_0^\infty v^2 dv\ e^{-av^2} = \left(\frac{2\pi k_B T}{m}\right)^{\frac{3}{2}} \tag{28}$$

If there are N particles in the gas the average number of particles with velocity in an interval d^3v surrounding $\mathbf{v}$ is $f(\mathbf{v}) = Np(\mathbf{v})$.

Before deriving the Maxwell velocity distribution let us introduce some related quantities. First let us consider the distribution of *speeds* (magnitude of the velocity). All velocities in a shell of "volume"

$$d^3v = 4\pi v^2 dv$$

have speeds between v and $v + dv$. Since the same number of particles are counted in each case we must have for the speed distribution $g(v)$

$$g(v)dv = f(\mathbf{v})d^3v$$

[1]Readers who are interested in the history of thermodynamics will enjoy the article by M. J. Klein [35] on the US physicist Josiah Willard Gibbs 1839-1903. I also recommend Boltzmann's entertaining description of his "sabbatical" in California [12].

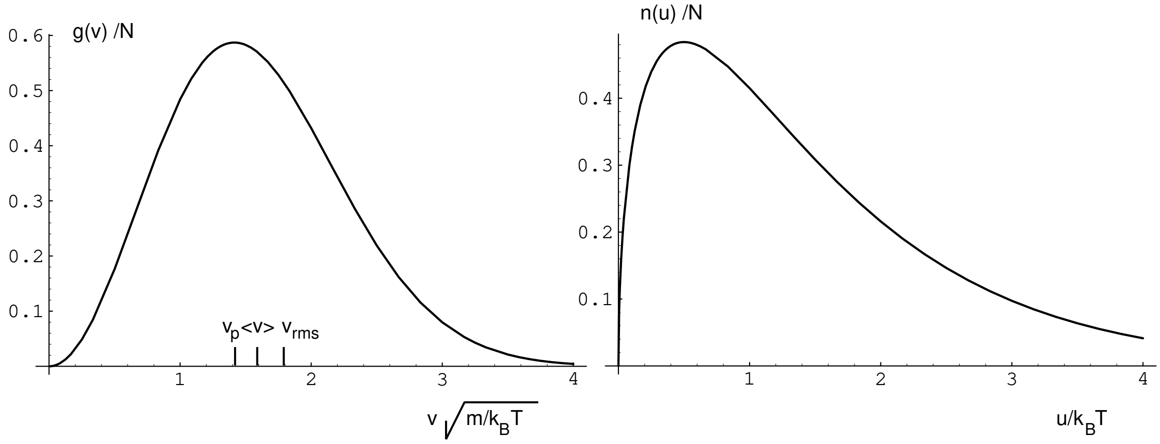


Figure 8: a) Maxwell speed distribution, b) energy distribution.

or

$$g(v) = 4\pi v^2 f(\mathbf{v}) = N v^2 \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T}\right)^{\frac{3}{2}} \exp\left(-\frac{m v^2}{2 k_B T}\right) \tag{29}$$

Similarly, the distribution $n(u)$ of energies can be obtained by substituting $u = \frac{1}{2} m v^2$ into

$$n(u) du = g(v) dv$$

giving the energy distribution

$$n(u) = g(v) \frac{dv}{du} = \frac{2\pi N}{(\pi k_B T)^{\frac{3}{2}}} \sqrt{u} \exp(-\frac{u}{k_B T}) \tag{30}$$

The Maxwell speed and energy distribution are plotted in figure 8. The most probable speed v_p is the one for which g(v) has a maximum. If we differentiate (29) we find that $dg/dv = 0$ for

$$v_p = \sqrt{\frac{2 k_B T}{m}} \tag{31}$$

The *average speed* can be obtained by using (22) to evaluate the integral

$$\langle v \rangle = \frac{1}{N} \int_0^\infty v \; g(v) dv = \sqrt{\frac{8 k_B T}{\pi m}} \tag{32}$$

Similarly, the mean square speed is

$$\langle v^2 \rangle = \frac{1}{N} \int_0^\infty v^2 g(v) dv = \frac{3k_B T}{m} \tag{33}$$

¿From (33) we see that the mean kinetic energy is given by

$$\frac{1}{2} m \langle v^2 \rangle = \frac{3}{2} k_B T \tag{34}$$

This result is familiar from the thermodynamics of the ideal gas. The r.m.s. (root mean square) speed is commonly used in estimates of "typical" speeds and is given by

$$v_{rms} = \sqrt{\frac{3k_B T}{m}} \tag{35}$$

We observe that the Maxwell speed distribution is *skew* and that there are slight differences between the most probable, the average and the r.m.s. speeds. As an example we next evaluate the average value of $1/v$:

$$\langle \frac{1}{v} \rangle = \frac{1}{N} \int_0^\infty dv \frac{1}{v} g(v) = \sqrt{\frac{2}{\pi}} \left(\frac{m}{k_B T} \right)^{\frac{3}{2}} \int_0^\infty dv v \exp\left(-\frac{mv^2}{2k_B T} \right) \tag{36}$$

which we evaluate using (22) to

$$\langle \frac{1}{v} \rangle = \sqrt{\frac{8m}{\pi k_B T}} \int_0^\infty dx x e^{-x^2} = \sqrt{\frac{2m}{\pi k_B T}} \tag{37}$$

note that $\langle 1/v \rangle$ is not the same as $1/\langle v \rangle$! The fact that the average value $\langle f(v) \rangle$ of some function f of the velocity is not equal to the average velocity $f(\langle v \rangle)$ will be a source of trouble when we derive formulas for transport properties such as diffusivity, viscosity and thermal conductivity. In "hand waving" derivations of these quantities the above distinction is often forgotten and formulas found in the literature sometimes differ by factors of, say, 2 or $\sqrt{\pi}$.

4.1 Most probable distribution, Boltzmann factor

We now derive the result (25). To do this we assume that all velocities are *a priori* equally probable, except that there is an overall restriction:

for an ideal gas at temperature T the average translational kinetic energy should be given by (34). We shall see that the Maxwell velocity probability distribution represents the *most probable distribution of velocities.* Here "most probable" means that it is the distribution that can be achieved in *the most possible different ways*, subject to our assumptions. In order to avoid the complication of continuous distributions we first consider the case where there are r possible outcomes and N trials. In a typical experiment or realization, the i'th outcome will occur n_i times. The number of ways this result can be achieved is

$$W(n_1...n_i...n_r) = \frac{N!}{n_1!n_2!...n_i!...n_r!} \tag{38}$$

We wish to find $n_i,\ i = 1, 2..r$, that maximize (38) subject to the constraint that

$$\sum_{i=1}^{r} n_i = N \tag{39}$$

We will also associate an energy u_i with the i'th outcome and assume that

$$\sum_{i=1}^{r} n_i u_i = U_{tot} \tag{40}$$

where U_{tot} is related to the *temperature*. To facilitate the algebra we introduce

$$S = k_B \ln W \tag{41}$$

S has a most important physical interpretation as the *entropy*[2]. Whenever W has a maximum as a function of the arguments so has S. The constraints (39), (40) can be incorporated using the method of *Lagrange's multipliers* and our task is to maximize

$$S - \sum_{i=1}^{r} (\lambda + \alpha u_i) n_i \tag{42}$$

where λ and α must be chosen so as to satisfy the constraints. Differentiation with regards to n_i then gives using (38), (41)

$$\frac{\partial S}{\partial n_i} = -k_B \frac{\partial ln(n_i!)}{\partial n_i} = \lambda + \alpha u_i$$

[2]In statistical mechanics (41) can be thought of as the definition of entropy. The realization that the entropy defined this way agrees with the thermodynamic entropy was a major discovery by Boltzmann. In recognition of this, the formula is inscribed on Boltzmann's tombstone in Vienna.

Stirling's formula (17) (assuming n_i to be large) gives

$$\ln n_i + \frac{1}{2n_i} + ... = -(\lambda + \alpha u_i)/k_B$$

and neglecting the second term on the left hand side leads to

$$n_i = \exp\left(-\frac{\lambda}{k_B}\right) \exp\left(-\frac{\alpha u_i}{k_B}\right) \tag{43}$$

Returning to the velocity distribution we let n_i be the number of particles whose velocities are within d^3v of $\mathbf{v}$ and have $u_i = \frac{1}{2}mv^2$. If we choose λ so that

$$N\left(\frac{m}{2\pi k_B T}\right)^{\frac{3}{2}} = e^{-\lambda/k_B}$$

and let $\alpha = 1/T$, we recover the Maxwell velocity distribution (25),

$$n_i = f(\mathbf{v}) = Np(\mathbf{v})$$

Since we have already shown that the Maxwell energy distribution is correctly normalized and gives the correct average kinetic energy, this concludes the proof.

The identification of the Lagrange multiplier α with the inverse temperature $1/T$ is a very important result and allows us to generalize to situations where the energy of a certain state i is not just due to the kinetic energy. In general, if a set of states are *a priori* equally probable, but there is an overall restriction on the average energy, the probability that a particular state will be selected is

$$p_i = const \times e^{-\beta u_i} \tag{44}$$

where we use the standard notation $\beta = \frac{1}{k_B T}$. The exponential factor in (44) is called the *Boltzmann factor* and the constant is given by the normalization condition that the probabilities add up to 1

$$const = \frac{1}{Z} = \frac{1}{\sum_i \exp(-\beta u_i)} \tag{45}$$

The sum Z over the Boltzmann factors is called the *partition function* and plays an important role in statistical physics.

The argument leading to (44, 45) is based on the identification of the logarithm of the number of ways that a certain thermodynamic state can be realized with the *entropy* (41). We will also need the *second law of thermodynamics*. In one formulation, this law states that the change in internal energy, dU, is given by

$$dU = TdS \tag{46}$$

in an infinitesimal process taking a system from one equilibrium state to another, if the process involves no work or transfer of matter. Equation (46) implies that

$$T = \left.\frac{\partial U}{\partial S}\right|_{N,no\ work} \tag{47}$$

Let us assume that we are primarily interested in a system $\{1\}$ that

- is in contact with a heat bath $\{2\}$ at temperature T.
- has thermodynamic states i with energy u_i that can be realized in w_i a priori equally probable ways.
- does not exchange work or matter with the reservoir.

The bath is much *larger* than system $\{1\}$, so that changes in to the energy of $\{1\}$ will have only a slight effect on the thermodynamic variables of the heat bath.

The number of ways that are compatible with energy $U_2 = U_{tot} - u_i$ is from (4.17)

$$W_2 = e^{S_2(U_2)/k_B} = e^{S_2(U_{tot}-u_i)/k_B} \tag{48}$$

where $S_2(U_2)$ is the *entropy* of the bath when its energy is U_2.

We want to find the probability distribution, $P(u_i)$, for system $\{1\}$. Since system $\{2\}$ is much larger than $\{1\}$, $u_i << U_{tot} = u_i + U_2$, where U_{tot} is the total energy of the two subsystems. We have

$$P(U_i) = \frac{w_i W_2(U_{tot} - u_i)}{\sum_i w_i(u_i) W_2(U_{tot} - u_i)}$$

We expand S_2 in a Taylor series

$$S_2(U_{tot} - u_i) = S_2(U_{tot}) - u_i \frac{\partial S_2}{\partial U} + \cdots \tag{49}$$

If the heat bath is big enough we can cut off the Taylor series after the first two terms and find using (47)

$$W_2(U_{tot} - u_i) - = W_2(U_{tot}) \exp\left[\frac{-u_i}{k_B T}\right] = const \times e^{-\beta u_i}$$

We conclude

$$P(u_i) = const \times w_i(u_i)e^{-\beta u_i} \tag{50}$$

where, normalizing the distribution, we find that the *const* is given by (45). The factor $e^{-\beta u_i}$ is the *Boltzmann factor.* When a system is in contact with a heat bath at a certain temperature, all possible microstates of the system are no longer equally likely. Instead, the Boltzmann factor acts as a weight factor biasing the distribution towards states with lower energy.

4.1.1 Isothermal atmosphere

To illustrate the use of Boltzmann factors, consider a gas in a gravitational field. We choose the potential energy to be zero at ground level. The potential energy at height z will be mgz with m the mass of a molecule. If the temperature of the atmosphere is constant, the number of molecules per unit volume at height h must be proportional to $\exp(-\beta mgz)$. With the density at ground level given as n_o the density at height z will be $n = n_o \exp(-\beta mgz)$. The isothermal assumption is *not* a realistic one, because the atmosphere is generally not in thermal equilibrium, being exposed to radiation from the sun and in turn radiating excess heat into space. We will discuss a better approximation, the *adiabatic atmosphere*, later in this course, in the context of *convection.*

4.1.2 Diatomic gas in an electric field

Our next example is a dilute gas made up of diatomic molecules that have a permanent electric dipole moment $\vec{\mu}$. The number of dipoles is N, the electric field is E, the volume is V and the temperature is T. The orientational energy of the dipole is (see figure 9)

$$U = -\vec{\mu} \cdot \mathbf{E} = -\mu E \cos\theta \tag{51}$$

We can define an element of solid angle $d\Omega = \sin\theta \; d\theta \; d\phi$ by specifying that the polar angle is between θ and $\theta + d\theta$, while the azimuthal angle is between ϕ and $\phi + d\phi$. All solid angles are *a priori* equally probable and the probability that the dipole is oriented in a given direction is

$$p(\theta,\phi)d\Omega = const \; \times \exp(-\beta U)d\Omega = A\exp(\beta\mu E\cos\theta)\sin\theta d\theta d\phi$$

The constant A is determined by the normalization condition

$$A\int_o^{2\pi} d\phi \int_o^{\pi} \sin\theta d\theta \exp(\beta\mu E\cos\theta) = 1$$

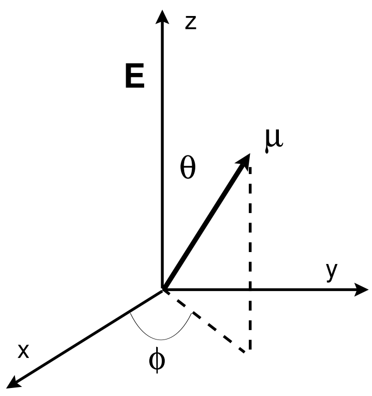


Figure 9: Dipole in an electric field

giving

$$A = \frac{\beta E \mu}{4\pi \sinh(\beta E \mu)}$$

The orientational probability distribution is then

$$p(\theta, \phi) d\Omega = \frac{\beta \mu E \exp(\beta \mu E \cos\theta)}{4\pi \sinh(\beta \mu E)} \sin\theta d\theta d\phi \tag{52}$$

4.2 Problems

Problem 4-1.

If the ratio of oxygen to nitrogen is 1/4 at sea level, what do you expect the ratio to be at an altitude of 3000 m (assume an isothermal atmosphere and $T = 0^0$C)? The mass of an oxygen molecule is 32 a.m.u., and the mass of a nitrogen molecule is 28 a.m.u., 1 a.m.u= 1.67×10^{-27} kg, and $k_B = 1.38 \times 10^{-23}$ J K^{-1}.

Problem 4-2

A certain gas consists of non-interacting molecules which are completely free to move in the $x-$ and $y-$directions, but the molecules are constrained in their motions in the $z-$direction by a harmonic oscillator potential. The kinetic plus potential energy of a single molecule of mass m and velocity v=(v_x, v_y, v_z)

and position $r = (x, y, z)$ is given by

$$Energy = K.E. + P.E. = \frac{m}{2}[{v_x}^2 + {v_y}^2 + {v_z}^2] + \frac{\kappa}{2}z^2 \qquad (53)$$

(a). If the gas is in equilibrium at temperature T, what can one say about the following average values:
$\langle v_x \rangle, \langle v_y \rangle, \langle v_z \rangle, \langle {v_x}^2 \rangle, \langle {v_y}^2 \rangle, \langle {v_z}^2 \rangle, \langle v^2 \rangle, \langle z \rangle, \langle z^2 \rangle$
(b). What is the heat capacity of one mole of this gas?

Problem 4.3

A zipper contains a large number N of links. The energy associated with opening a link is $\epsilon > 0$, $N\epsilon >> k_B T$. The zipper can only unzip from one end while the other end remains closed. Find the mean and the variance of the number of open links.

Problem 4.4

Calculate the root mean square fluctuation of the kinetic energy of a monatomic ideal gas about its mean value $\frac{3}{2}NK_BT$.

5 Molecular collisions.

We now discuss the effect of collisions between the molecules. Consider a gas of $n = \frac{N}{V}$ particles per unit volume. We will at first make the idealization that the molecules are hard spheres of radius r. The closest approach of two molecules is then a diameter $d = 2r$. The sphere of radius d surrounding a given molecule is called its *sphere of influence*. We assume that the molecules form an ideal gas with a Maxwellian velocity distribution (25). Suppose $\mathbf{v}_1$ and $\mathbf{v}_2$ are the velocities of two molecules. Their *relative velocity* is then $\mathbf{v}_r = \mathbf{v}_1 - \mathbf{v}_2$ and their center of mass velocity is $\mathbf{v}_{cm} = \frac{1}{2}(\mathbf{v}_1 + \mathbf{v}_2)$. Two molecules will collide in a time interval dt if the center of the second molecule happens to be located within a cylinder of height $v_r dt$ and radius d as shown in figure 10. The probability ϕdt that a given molecule will collide with any molecule in the gas during a time interval dt is then

$$\langle \phi \rangle dt = n\pi d^2 \langle v_r \rangle dt \qquad (54)$$

where ϕ is the *collision frequency* and $\langle v_r \rangle$ the average relative speed. It remains to find the probability distribution for relative velocities and to calculate $\langle v_r \rangle$. We have for the kinetic energy of a pair of particles

$$U_{12} = \frac{1}{2}m{v_1}^2 + \frac{1}{2}m{v_2}^2 = \frac{1}{2}M{v_{cm}}^2 + \frac{1}{2}\mu{v_r}^2 \tag{55}$$

where $M = 2m$ is the pair mass and $\mu = \frac{1}{2}m$ is the reduced mass. All relative velocities are a priori equally probable, and we find in analogy with (25) for the distribution of relative velocities

$$p(\mathbf{v}_r) = (\frac{\mu}{2\pi k_B T})^{\frac{3}{2}} \exp(-\frac{\mu{v_r}^2}{2k_B T}) \tag{56}$$

This distribution differs from the Maxwellian velocity distribution only in that the reduced mass replaces the particle mass in the formula. Thus from (32) we have for the mean relative speed

$$\langle v_r \rangle = \sqrt{\frac{8k_B T}{\pi\mu}} = \sqrt{2}\langle v \rangle \tag{57}$$

The factor πd^2 in (54) is the equatorial cross-section of the sphere of influence. In practice the hard sphere assumption is too restrictive. We can, however, replace πd^2 by an effective *collision cross section* σ. We thus find for the collision frequency

$$\langle \phi \rangle = \sqrt{2}n\sigma\langle v \rangle = 4n\sigma\sqrt{\frac{k_B T}{\pi m}} \tag{58}$$

A concept which is closely related to the collision frequency is the *mean free path* $l = \langle \lambda \rangle$, which is the average distance traveled between collisions

$$l = \langle \lambda \rangle = \left\langle \frac{v}{\phi} \right\rangle = \frac{1}{n\sigma\sqrt{2}} \tag{59}$$

Note that the speed factors out in (59); the mean free path is the same for all the particles in a Maxwellian gas. In particular, the mean free path in an ideal gas of a given density is independent of the temperature. This sometimes makes l the most convenient parameter to describe the collision properties of a gas.

5.1 Knudsen gas

It is useful to establish the order of magnitude of the mean free path under different conditions. Let us first find the mean free path of molecules of

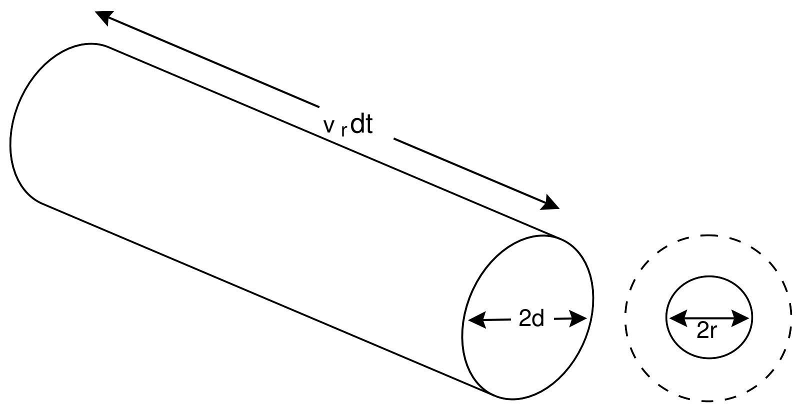


Figure 10: Collision tube.

diameter 4 Å in an ideal gas at atmospheric pressure and $T = 0^o$C. The number of particles per unit volume is $n = \frac{P}{k_B T}$, giving with $P = 1.013 \times 10^5$ N m^{-2}

$$l = \frac{k_B T}{P\pi d^2\sqrt{2}} = \frac{1.38 \times 10^{-23} \times 273.16}{101.3 \times 10^3 \times 16\pi \times 10^{-20}\sqrt{2}} = 5.210^{-8} \text{ m} = 520 \text{ Å} \quad (60)$$

If we now redo the calculation for a pressure $P = 10^{-9}$ atm we find that the mean free path will be by 10^9 times larger or 52 m. A *Knudsen gas* is a gas which is at such low pressure that the mean free path is longer than the dimensions of ordinary scientific equipment used to contain it.

5.2 Random walk

For the discussion that follows we need to work out the probability $p(r)$ that a particle can travel a distance r without undergoing any collisions. The probability that a collision occurs between r and $r + dr$ is proportional to dr. Let the proportionality constant be α. The probability $p(r + dr)$ that no collisions occurs in an interval of length $r+dr$ is thus $p(r+dr) = p(r)(1-\alpha dr)$ or

$$dp = -\alpha p dr \quad (61)$$

Since $p(0) = 1$ we find that $p(r)$ satisfies the *Poisson distribution* $p(r) = \exp(-\alpha r)$. The mean distance between collisions is

$$l = \int_0^1 r dp = -\int_0^\infty r\frac{dp}{dr}dr = \alpha\int_0^\infty r\exp(-\alpha r)dr = \frac{1}{\alpha}$$

giving

$$p(r) = exp(-r/l) \tag{62}$$

The picture of molecular motion which we have constructed is one in which each molecule travels in a straight line between collisions. We will make the approximation that after each collision the molecule starts in a random direction. This type of motion is called a *random walk*. Before considering the general 3-dimensional random walk problem let us consider the following simpler problem:

A particle moves with equal probability to the right or to the left in steps of unit length along a straight line. What is the probability that after t time steps the particle has moved a distance s?

If n_r steps are taken to the right and n_l steps are taken to the left the net distance traveled is $s = n_r - n_l$ if we let $s > 0$ correspond to motion to the right. For the desired outcome we must therefore have

$$n_r = \frac{1}{2}(t+s) \quad n_l = \frac{1}{2}(t-s)$$

¿From (16) the probability of this outcome is

$$p = \frac{t!2^{-t}}{[\frac{1}{2}(t-s)]![\frac{1}{2}(t+s)!]} \tag{63}$$

Example: An old exam question based on this formula is the following: A drunk leaves a beer parlor and takes random steps 1m long in one dimension. The entrance to another pub is 5 m from the first pub. What is the probability that the drunk has passed the second pub after 8 steps?

> *Solution:* The sought-after outcome can be reached in two ways: (i) by 8 steps in the direction of the second pub, or (ii) by seven steps towards the second pub and one step back. The total probability is the sum of the probabilities for these two outcomes
>
> $$p = \frac{8!2^{-8}}{8!0!} + \frac{8!2^{-8}}{7!1!} = \frac{9}{256}$$

Example: Assume that a particle with mean free path l can travel either to the left or to the right with speed v. After each collision the particle moves with equal probability to the left or to the right. What is the probability per unit length that the particle has traveled a distance x in time t? What is the root mean square (r.m.s.) distance traveled in 1 second for a particle with speed 1000 m/s and mean free path 1 mm?

Solution: The mean time interval per collision is l/v while the distance x on the average corresponds to $s = x/l$ steps. Problems in kinetic theory normally involve situations in which enough time has elapsed for the particle to have taken many steps, e.g. in the problem before us the particle will on the average have undergone one million collisions. We can then make use of the central limit theorem of chapter 3. The mean distance traveled per step is zero while the mean square distance is

$$\langle\lambda^2\rangle = \in_0^1 r^2 dp = -\int_0^\infty r^2 \frac{dp}{dr} dr = \frac{1}{l}\int_0^\infty r^2 \exp(-\frac{r}{l}) dr = 2l^2 N = 2lvt$$

The mean distance traveled after $N = tv/l$ steps is still zero but the variance is thus $2lvt$, and the r.m.s. distance is $\sqrt{2lvt}$. With the values provided, this distance is $\sqrt{2}$ meter. The probability distribution of distances traveled is from (13)

$$p(x,t) = \frac{1}{\sqrt{4\pi lvt}} \exp(\frac{-x^2}{4lvt}) \tag{64}$$

We will use this result later when we discuss diffusion.

5.3 Problems

Problem 5-1.

A small amount of a gas in which each molecule has mass m_1 is released into a gas in which there are n molecules per unit volume. The mass per molecule in the second gas is m_2. The temperature is T and the cross section for collisions of molecules of the two kinds is σ. Derive formulas for the mean relative velocity of molecules of the two kinds, and for the mean free path of molecules of the first kind. (Hint: (56) is still valid, but the reduced mass will no longer be $\frac{1}{2}m$).

6 Problems related to diffusion.

6.1 Molecules hitting a surface

In many applications one needs a formula for the mean number of particles that collide with a given surface per unit time. We assume that the spatial distribution of particles is uniform. Consider an element of volume immediately above the surface. One half of the particles in this volume have a velocity component in the direction of the surface. Let $\langle v_z \rangle$ be the average value of this velocity component for the particles traveling in the "right" direction. The number of particles hitting the surface per unit area is then $\frac{1}{2}\langle |v_z| \rangle n$. We put $v_z = v \cos\theta$. Averaging over solid angles for which $0 < \theta \leq \pi/2$ we find

$$\langle v_z \rangle = \langle v \rangle \frac{\int_0^{\frac{\pi}{2}} \cos\theta \sin\theta d\theta}{\int_0^{\frac{\pi}{2}} \sin\theta d\theta} = \frac{1}{2}\langle v \rangle$$

Thus, the number of particles hitting a surface per unit area and time is

$$j_N = \frac{n\langle v \rangle}{4} \tag{65}$$

which is the required formula. We next give some examples of problems making use of this result.

In order to illustrate the usefulness of (65) in surface adsorption problem consider the following example. A clean surface is suddenly exposed to oxygen at 25^{o}C. Assume that every molecule striking the surface is adsorbed, and that the effective diameter of the oxygen molecule is 4 Å. The problem is to estimate the time interval before the surface is covered with a layer of oxygen if the pressure P is **(a)** 10^{-6} Torr, **(b)** 10^{-10} Torr.

> *Solution:* The mean number of molecules striking the surface per unit area is given by (65) where
>
> $$n = \frac{P}{k_B T}$$
>
> $$\langle v \rangle = \sqrt{\frac{8 k_B T}{\pi m}}$$
>
> If the atoms in the oxygen layer are arranged in a triangular lattice as in figure 11, we see that the number ν of molecules per

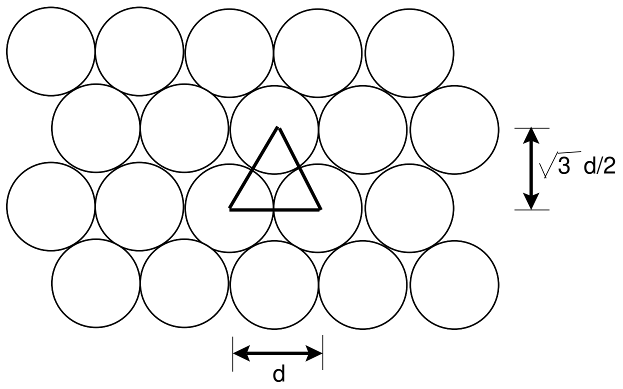


Figure 11: The triangular lattice.

unit surface area is $\frac{1}{2}$ molecule/(area of triangle), or

$$\nu = \frac{2}{\sqrt{3}d^2} \tag{66}$$

Collecting terms we find

$$t = \frac{\nu}{j_N} = \frac{2}{\sqrt{3}d^2} \times \frac{4k_BT}{P} \times \sqrt{\frac{\pi m}{8k_BT}} \tag{67}$$

Substituting $k_B = 1.38 \times 10^{-23}$ J K^{-1}, $T = 298$ K, $m = 32 \times 1.67 \times 10^{-27}$ kg, $d = 4 \times 10^{-10}$ m, 1 Torr$=$ 133 Nm^{-2} we find for the time it takes to cover the surface in the two cases **(a)** 2s, **(b)** 5.5 hours.

Note: The oxygen monolayer need not form in the triangular structure of the figure. The assumption that all molecules will stick if they hit the surface may not be realistic. Also, during the process some molecules will evaporate etc. In this type of problem there may therefore be many “correct” answers “in the same ball park”.

As a further example consider a gas at temperature T inside a container. What is the average kinetic energy of molecules colliding with the container walls?

Solution: We assume a Boltzmann velocity distribution. The average kinetic energy of the molecules is from (34) given by

$\frac{3}{2}k_BT$. Since all the molecules from time to time hit the walls, one might expect that the average energy of the molecules striking the walls is the same. In a more careful analysis however, one would note that the faster molecules hit the walls more often and this produces a bias in the average.

Let us consider a wall at $z = 0$, with the gas located below the wall ($z < 0$). In the time interval dt, molecules with velocity $v_z > 0$ will hit the wall if they start within a distance $v_z dt$ of the wall. The probability that a molecule has velocity component v_z is proportional to the Boltzmann factor

$$\exp\left(-\frac{m}{2k_BT}{v_z}^2\right)$$

We assume that the gas is uniformly distributed spatially. The average value of ${v_z}^2$ of the molecules hitting the wall will then be

$$\left({v_z}^2\right)_{av} = \frac{\int\limits_0^\infty {v_z}^2 v_z dv_z \exp\left(-\frac{m{v_z}^2}{2k_BT}\right)}{\int\limits_0^\infty v_z dv_z \exp\left(-\frac{m{v_z}^2}{2k_BT}\right)}$$

If we introduce the new variable $y = \frac{m}{2k_BT}{v_z}^2$, $dy = \frac{m}{k_BT}v_z dv_z$, we get

$$\left({v_z}^2\right)_{av} = \frac{2k_BT}{m}\frac{\int\limits_0^\infty ye^{-y}dy}{\int\limits_0^\infty e^{-y}dy} = \frac{2k_BT}{m} \tag{68}$$

The average value of ${v_x}^2$ and ${v_y}^2$ will be the same for molecules hitting the wall z= 0, as for the gas as a whole

$$\left(v_x^2\right)_{av} = \left(v_y^2\right)_{av} = \frac{k_BT}{m} \tag{69}$$

giving

$$(K.E.)_{av} = \frac{m}{2}\left[\left({v_x}^2\right)_{av} + \left({v_y}^2\right)_{av} + \left({v_z}^2\right)_{av}\right] = 2k_BT \tag{70}$$

The average kinetic energy of the molecules hitting the walls is thus different from the average kinetic energy of the molecules in the gas. The fact that it is often hard to avoid relying on biased samples is the source of many fallacies which occur when statistical arguments are used in ordinary life ("lies, damned lies and statistics").

As a further example consider a thin disk of radius r that is embedded in a monatomic gas of pressure P and temperature T_g. The disk is maintained at a temperature T_d. We assume that the gas molecules hit the disk with an average kinetic energy $2k_BT_g$ and leave with the average kinetic energy $2k_BT_d$. Also, assume that radiative losses are negligible and that there is no convection. We wish to estimate the rate at which heat is transferred between the gas and the disk.

Solution: The rate at which heat is transferred can be written $\dot{Q}$=(#of particles per unit area hitting the disk)× (area)× (the average energy transfer in each collision). We find

$$\dot{Q} = \frac{n\langle v\rangle}{4} \times 2\pi r^2 \times 2k_B(T_g - T_d) \tag{71}$$

with

$$\langle v\rangle = \sqrt{\frac{8k_BT_g}{\pi m}}$$

and

$$n = \frac{P}{k_BT_g}$$

6.2 Effusion

Consider a container of volume V which is evacuated. Suppose the container develops a small pinhole of area A. If the diameter of the pinhole is smaller than the mean free path of the air molecules we say that air *effuses* into the container. One can then assume that the rate at which molecules escape through the hole is equal to the rate at which the molecules would have hit the missing segment of the wall. If the condition is not satisfied we cannot assume that the molecules of the gas move independently, and we must solve a hydrodynamic flow problem. Assume air has molecular weight m, that the outside pressure is P and the temperature inside and outside is T. We wish to
(a) derive an expression for the pressure inside the container as a function of time, assuming effusion;
(b) estimate how long it takes for the pressure to rise to $\frac{1}{2}P$.

Solution: Let the density of air outside be n_0, while the inside density is $n_i = \frac{N}{V}$ where N is the number of molecules inside the container volume V. Air can effuse both in and out of the container through the hole of area A. We have from (65)

$$\dot{n}_i = \frac{\dot{N}}{V} = \frac{(n_0 - n_i)\langle v\rangle A}{4V} \tag{72}$$

The solution to this differential equation is

$$n_i = n_0 \left[1 - \exp\left(-\frac{\langle v\rangle tA}{4V}\right)\right] \tag{73}$$

and the answer to **a:** is $p_i = n_i k_B T$. To solve **b:** we put $n_i = \frac{1}{2}n_o$. The time taken is thus

$$t = \frac{4V}{\langle v\rangle A} ln2 = \frac{4V \ln 2}{A\sqrt{\frac{8k_BT}{\pi m}}} \tag{74}$$

At ordinary pressures and temperatures the mean free path is $\sim 10^{-1}$ μm from (60). The hole must therefore be very small indeed for (74) to be valid.

6.3 Diffusion

We next turn to a discussion of *diffusion.* The diffusion constant D is important in problems in which there is a *concentration gradient.* We will show that a consequence of the random motion of the molecules in a gas is that a concentration gradient will give rise to a particle current

$$\mathbf{j}_N = -D\nabla n(\mathbf{r}) \tag{75}$$

Equation (75) is known as *Fick's first law.* Similarly, for a time-dependent process at constant temperature we have

$$\frac{\partial n}{\partial t} = D\nabla^2 n \tag{76}$$

which is known as *Fick's second law.* We wish to use the kinetic theory of gases to express the diffusion constant D in terms of the mean velocity $\langle v\rangle$ and mean free path λ between collisions as

$$D = \frac{\langle v\rangle l}{3} \tag{77}$$

The SI unit for D is $\mathrm{m}^2\ \mathrm{s}^{-1}$. To derive (77) let us consider the case where the density of particles $n(z)$ varies with the height z in a sample while the density is constant in the x-y plane. Let us monitor the flow of particles through an imaginary surface dS in the x-y plane at height $z = 0$ (figure 12).

We estimate the flow by considering where the particles could have come from after their last collision. To get from there through the imaginary surface it must travel in the right direction. Furthermore, if a particle goes through dS from below it makes a positive contribution to the current; if it comes from above the contribution is negative. The number of particles undergoing collisions in a volume element dV with polar coordinates r, θ, ϕ per unit time is, with l the mean free path

$$\frac{\langle v \rangle n(z)}{l} dV = \frac{\langle v \rangle n(z)}{l} r^2 dr \sin\theta d\theta d\phi$$

The probability that such a particle can make it to dS without further collisions is $e^{-r/l}$. The probability that it is traveling in the right direction is

$$\frac{\mid \cos\theta \mid dS}{4\pi r^2}$$

If θ is between 0 and $\pi/2$ the particles are coming from above, and if θ is between $\pi/2$ and π the particle is coming from below. Collecting terms we find for the net flow

$$j_N dS = \int_0^\infty r^2 dr \left\{ \int_{\pi/2}^{\pi} d\theta - \int_0^{\pi/2} d\theta \right\} \sin\theta \int_0^{2\pi} d\phi \; n(z) \frac{\langle v \rangle \mid \cos\theta \mid}{l 4\pi r^2} dS e^{-r/l} \tag{78}$$

We assume that the density $n(z)$ varies slowly over distances comparable to the mean free path, and expand $n(z)$ in a Taylor series

$$n(z) = n(0) + z\frac{\partial n}{\partial z} + ... = n(0) + r\cos\theta\frac{\partial n}{\partial z} + ... \tag{79}$$

If we substitute (79) into (78) we note that the contribution proportional to $n(0)$ vanishes (if $n(z)$ is constant the net current should be zero). Using

$$\int_0^\infty r dr \exp(-\frac{r}{l}) = l^2 \tag{80}$$

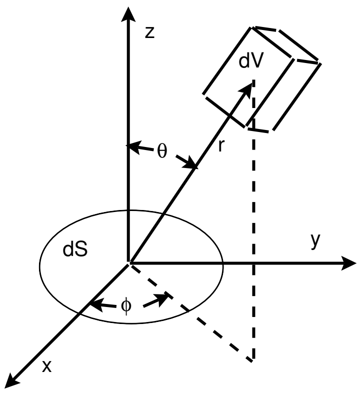


Figure 12: Diffusion through the xy-plane.

and

$$\int_0^\pi \cos^2\theta \sin\theta d\theta = \frac{2}{3} \tag{81}$$

the integrations in (78) can be carried out for the term proportional to the derivative of n. We find for the current density

$$j_z = -\frac{\langle v\rangle\ l}{3}\ \frac{\partial n}{\partial z} \tag{82}$$

or in vector notation

$$\mathbf{j} = -\frac{\langle v\rangle\ l}{3}\nabla n \tag{83}$$

Comparing with (75) we see that the diffusion constant D is given by (77).

In order to derive the time-dependent diffusion equation (76) we consider a volume element $dV = dxdydz$. As before, we let there be a flow in the z direction. If the flow into the volume element is larger than the flow out there will be a net increase in the number of particles. We have

$$dxdydz\frac{\partial n}{\partial t} = dxdy(j(z) - j(z+dz)) = dxdydz\frac{\partial}{\partial z}(D\frac{\partial n}{\partial z}) = dxdydzD\frac{\partial^2 n}{\partial z^2} \tag{84}$$

or

$$\frac{\partial n}{\partial t} = D\frac{\partial^2 n}{\partial z^2} \tag{85}$$

if we assume that the diffusion constant is independent of the density (which is approximately true if the density is slowly varying, but is not true in general). The three-dimensional generalization of (85) is (76).

Einstein relation

A quantity which is closely related to the diffusion constant is the *mobility* μ. Consider a particle which is subject to an external force F. This force will accelerate the particle, but from time to time this acceleration is interrupted by collisions with other particles. After each collision the particle will start out in a random direction, but will again accelerate in the direction of the force until it is interrupted by another collision and so on. The net result of this very irregular motion is a drift in the direction of the force and the average velocity of this drift is called the *drift velocity* v_d. The drift velocity will in general be proportional to the force F and the proportionality factor is called the mobility

$$v_d = \mu F \tag{86}$$

Let us go back to our example of the isothermal atmosphere where the equilibrium particle number density could be written

$$n = n_o \exp\left(\frac{-mgz}{k_B T}\right)$$

This expression for the density can be thought of as arising from two competing processes. There will be a downward current due to the *drift* in the gravitational field

$$j_d = -nmg\mu \tag{87}$$

This current is compensated by a *diffusion current* due to the density gradient

$$j_D = -D\frac{\partial n}{\partial z} = \frac{Dn_o mg}{k_B T}\exp\left(\frac{-mgz}{k_B T}\right) = \frac{Dnmg}{k_B T} \tag{88}$$

In equilibrium the two currents must cancel and we get the *Einstein relation*

$$D = \mu k_B T \tag{89}$$

Equation (89) is valid under quite general circumstances.

Solutions to the diffusion equation

We will in this course not have much to say about how to solve the one dimensional diffusion equation (85) or its three-dimensional analog (76). The details of that problem belongs to a course in partial differential equations. In chapter 5 we worked out the probability distribution for the position of a particle with speed v which undergoes many collisions in one dimension (64). We first show that (64) is a solution to the diffusion equation (85) if we make the identification $D = lv$. We have

$$\frac{\partial}{\partial x^2}\left(\frac{\exp\left[-\frac{x^2}{4Dt}\right]}{\sqrt{4\pi Dt}}\right) = \left(-\frac{1}{2Dt}+\frac{x^2}{4D^2t^2}\right)\left(\frac{\exp\left[-\frac{x^2}{4Dt}\right]}{\sqrt{4\pi Dt}}\right)$$

$$\frac{\partial}{\partial t}\left(\frac{\exp\left[-\frac{x^2}{4Dt}\right]}{\sqrt{4\pi Dt}}\right) = \left(-\frac{1}{2t}+\frac{x^2}{4Dt^2}\right)\left(\frac{\exp\left[-\frac{x^2}{4Dt}\right]}{\sqrt{4\pi Dt}}\right) \tag{90}$$

from which (85) follows directly. The three-dimensional probability distribution for a particle starting out at the origin and diffusing outwards is

$$p(\mathbf{r},t) = \frac{1}{(4\pi Dt)^{3/2}}\exp\left[-\frac{r^2}{4Dt}\right] \tag{91}$$

The reader may easily verify that (91) is a solution to the 3-dimensional diffusion equation by expressing $\mathbf{r}$ in Cartesian coordinates and working out the expressions analogous to (90). From (91) we see that the typical distance traveled by a particle will be of the order $\sqrt{Dt}$, which also could be expected on dimensional grounds since the diffusion constant has dimension $(\text{length})^2/$ time. Since the diffusion equation (76) is *linear* we can add solutions from several sources. Suppose we add N_i particles at the points $\mathbf{r}_i$ at time $t = 0$ the subsequent particle density will be

$$n(\mathbf{r},t) = \sum_i \frac{N_i}{(4\pi Dt)^{3/2}}\exp\left(\frac{-(\mathbf{r}-\mathbf{r}_i)^2}{4Dt}\right) \tag{92}$$

If there is a continuous distribution of initial sources the sums in (92) must be converted into integrals. Another commonly occuring problem is to find the steady state distribution after a long time has elapsed. In this case the diffusion equation (76) reduces to the *Laplace* equation

$$\nabla^2 n(\mathbf{r}) = 0 \tag{93}$$

Solutions to the Laplace equation are frequently required in electrostatics and many diffusion problems can be solved by making an analogy with a problem in electrostatics.

So far the only physical effect of diffusion is to smooth out concentration differences. In fact, diffusion, when occuring in conjunction with chemical reactions, can give rise to a rich variety of effects. Both the chemical reactions and diffusion processes supplying the chemicals to an active region occur at a certain rate. Competition between the two rates can give rise to effects such as the beautiful colored stripes of agate and many biological pattern formation processes. Similarly, if a melted mineral is allowed to cool and recrystallize, the size distribution of the crystallite grains will be governed by a competition between diffusion rates and the rate of cooling.

6.4 Problems

Problem 6-1.

Gas molecules effuse through a small leak from an oven which is held at constant temperature T. What is the average temperature of the escaping gas if the gas is monatomic with heat capacity $\frac{3}{2}k_B$ per molecule?

Problem 6-2.

Two flasks each having volume V are connected by a thin tube of length L and cross sectional area A. Initially one flask is full of pure water and the other contains "labeled water" (water in which the hydrogen is replaced by tritium) at concentration n_0. The diffusion constant of the labeled water is D. Tiny mixers maintain uniform concentration in each flask, but transport from one flask to the other through the connecting tube is accomplished only by diffusion.

Derive a formula for the concentration of labeled water in the two flasks as a function of time. State your assumptions.

Problem 6-3:

Tungsten atoms (atomic weight 183.7) are measured to leave a tungsten surface at the rate of 1.14×10^{-12} kg s^{-1} m^{-2} from a sample heated to 2000 K. Find the vapor pressure of tungsten at this temperature.

Problem 6-4:

10^{20} pollutant molecules with diffusion constant $D = 1.0 \times 10^{-10}$ m^2 s^{-1} are released at time $t = 0$ in the center of a large pool of water. Assume that the molecules diffuse away without any stirring taking place.

(a). Plot the concentration as a function of time a distance 1 cm away from the source.

(b). When does the maximum concentration occur 1 cm away from the source.

(c). How long will it take for the concentration everywhere to reach a "safe" level of 10^{20} m^{-3}?

(d). How long would it take for one half of the molecules to reach a distance 1cm from the source. (You will need to do a numerical integration of (91) to get your answer.)

Problem 6-5.

(a). Molecules with diffusion constant $1.0 \times 10^{-10}\mathrm{m}^2/s$ are released at a constant rate of 10^{10} molecules/s in the middle of a large pool and diffuse away. What is the steady state concentration 1 cm away from the source? (*Hint:* The electrostatic potential associated with a charge at the origin and no charges elsewhere is of the form $const/r$).

(b). How long after the release has started will it take for the concentration to reach 1/2 of its steady state value 1 cm away from the source?

Problem 6-6.

A gas of N_2 molecules at 300 K has a number density $N/V = 2.5 \times 10^{25}$ molecules per m^3. A small number of O_2 molecules is contained in the N_2 gas. The cross section for collisions between the O_2 and N_2 molecules is $\sigma \simeq 4 \times 10^{-19}$ m^2.

(a). Calculate the mean free path and the mean time between collisions for O_2 under these conditions.

(b). Estimate the diffusion constant of the O_2 molecules in the N_2 gas.

(c). If the O_2 molecules are initially localized at one end of a tube of length 1 m containing the N_2-O_2 mixture, approximately how long would it take them to diffuse to the other end of the tube? What type of measurement could be made to check the results of this calculation?

(d). A certain force is applied to the O_2 molecules and is observed to produce a "drift velocity" of magnitude 3 m s^{-1}. What is the magnitude of the force which produces this drift velocity?

Problem 6.7.

A narrow tube contains ether. Initially the tube is full but the level falls slowly by evaporation. Assume that the evaporation of ether is limited by diffusion. Near the liquid surface there is equilibrium concentration of ether vapor, while at the top of the tube the concentration is essentially zero. The vapor pressure of ether at room temperature is 5.33×10^4 Pa, the molecular weight is 74, the density of liquid ether is 0.7 g cm^{-3}, 1 atomic mass unit is 1.66057×10^{-27} kg.

(a). Find an equation for the depth h of the ether surface below the open end as a function of time, given the diffusion constant D for ether in air.

(b). Calculate the Diffusion constant D if the level falls by 1 cm in 30 minutes.

(c). How long will it take for the ether level to fall to 2 cm below the open end?

(d). Estimate the diameter of an ether molecule.

7 Thermal conductivity and viscosity.

In this section we wish to continue our discussion of *transport properties* of fluids. What we do is based on the picture that a gas or fluid is made up of individual particles, each of which do their own thing, except that from time to time they collide with other particles. We want to model materials properties such as thermal conductivity and viscosity using this picture. Suppose we set up a temperature gradient in a material. If the gradient is not too large the heat flow will be proportional to the temperature gradient and the proportionality constant is the thermal conductivity. The concept of viscosity dates back to Newton, who considered *laminar fluid flow.* While a fluid at rest offers no resistance to shear, there will be a shear stress when the fluid is flowing. In this type of flow the fluid is moving in parallel layers in such a way that we have a *velocity profile* (see figure 13) .

We assume that the shear stress P_{zy} is proportional to the velocity gradient. *Stress* is the force F per unit area A, so if the velocity field is changing in the z-direction

$$P_{zy} = \frac{F}{A} = -\eta \frac{\partial u_y}{\partial z} \tag{94}$$

The proportionality constant η is called the *viscosity*. A fluid which obeys 94 is called *Newtonian*. The three most important fluids air, water and petroleum are all Newtonian to a very good approximation. There are, however some fairly common fluids such as muds, paints, slurries, pastes and jellies that behave differently. We will come back to the non-Newtonian fluids later in this chapter. In the SI system the unit of viscosity is N s m^{-2}. A commonly used unit of viscosity is the *centipoise* which is 10^{-3} N s m^{-2}. Sometimes people prefer to use the *kinematic viscosity*, which is viscosity divided by the mass density $\nu = \frac{\eta}{\rho}$. The SI unit of kinematic viscosity is m^2 s^{-1} . The *centistoke* is defined as 10^{-6} m^2 s^{-1}, the reason for the common use of the nonstandard units is probably that ν for water is ≈ 1 centistoke, while μ is ≈ 1 centipoise.

In the case of heat conductivity there is a heat current j_Q which is proportional to the temperature gradient

$$j_Q = -\kappa \frac{\partial T}{\partial z} \tag{95}$$

In general, we can construct an approximate theory by starting with the assumption that there is a molecular property $a(z)$ which diffuses in the z direction. We assume that the property a is slowly varying over distances comparable to the mean free path so that we have

$$a(z) = a(0) + z\frac{\partial a}{\partial z} + ... \tag{96}$$

In the case of viscosity the molecular property of interest is the y-component (in the direction of the flow) of the average momentum per unit volume $a(z) = nm\langle u_y(z)\rangle$. The "current" is the shear stress P_{zy} which is the net flux in the z- direction of the y-component of momentum. We can estimate this shear stress by the same argument that we used in the previous section. In that case the molecular property which varied with z was the

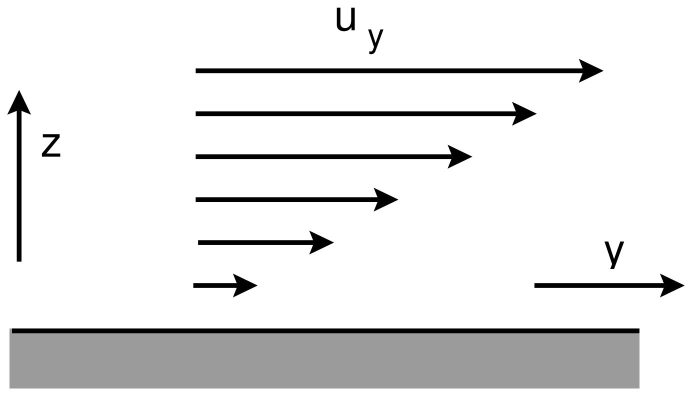


Figure 13: Velocity profile.

number density $n(z)$. Thus all we have to do is to replace $n(z)$ by $a(z)$ in (82) and we get using (94)

$$P_{zy} = -\frac{\langle v\rangle l}{3}\frac{\partial a}{\partial z} = -\frac{nm\langle v\rangle l}{3}\frac{\partial\langle u_y\rangle}{\partial z} = \eta\frac{\partial\langle u_y\rangle}{\partial z}$$

giving

$$\eta = \frac{nm\langle v\rangle l}{3} = \frac{m\langle v\rangle}{3\sqrt{2}\sigma} \tag{97}$$

where we have used (59) to relate the mean free path to the scattering cross section. Equation in (97) has the interesting consequence that the viscosity of a fluid should be independent of the density and proportional to the square root of the temperature (see(32)). Experimental results (see e.g. Flowers and Mendoza [23] for more details) on gases confirm these result at low to moderate pressures. At high pressures ($\geq$ 100 atm.) our theory is too simple minded.

We next turn to the thermal conductivity. The molecular property which is diffusing is the internal energy of the particles. A change ΔU in the internal energy per unit volume is associated with a change ΔT in the temperature. We have

$$\Delta U = nc_v\Delta T$$

where c_v is the specific heat per particle ($c_v = \frac{3}{2}k_B$ for a monatomic ideal gas). The heat current is thus

$$j_Q = -\frac{\langle v\rangle l}{3}nc_v\frac{\partial T}{\partial z} = -\kappa\frac{\partial T}{\partial z} \tag{98}$$

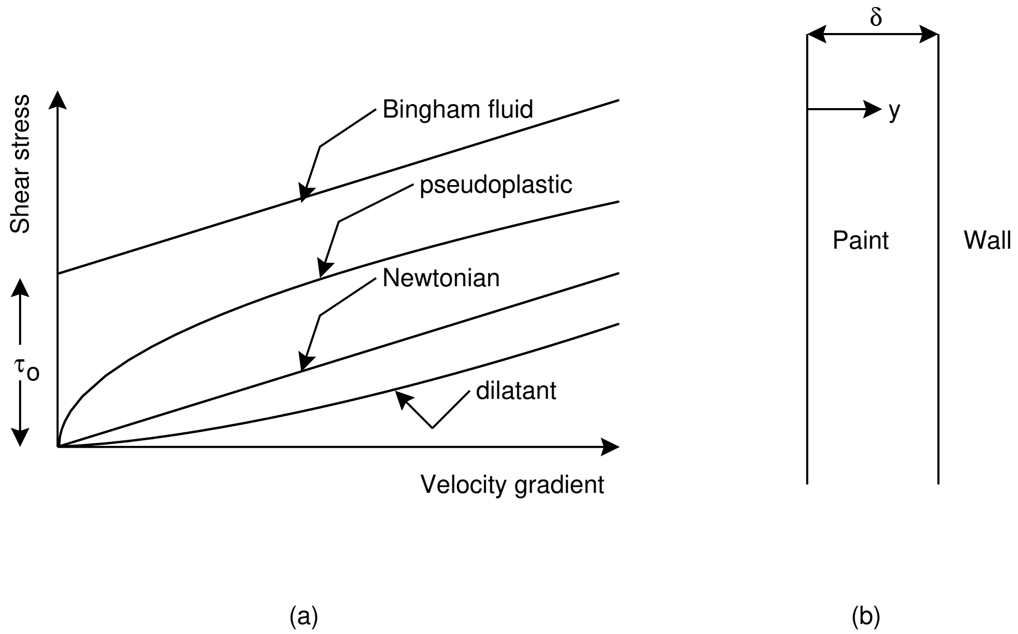


Figure 14: (a) Different types of Non-newtonian fluids. (b) Non-running paint.

Using (59) we find that

$$\kappa = \frac{\langle v \rangle l}{3} n c_v = \frac{\langle v \rangle c_v}{\sigma 3 \sqrt{2}} \tag{99}$$

Eq. (99) predicts that the thermal conductivity is independent of the pressure, and proportional to the square root of the temperature. Just as for the viscosity, this agrees with experiment on gases at moderate to low pressure, but there are corrections at high pressure ($\geq$ 100 atm). Finally from (77) and (59)

$$D = \frac{\langle v \rangle}{3 n \sigma \sqrt{2}} \tag{100}$$

i.e. the diffusion constant should be inversely proportional to the density, which seems indeed to be true experimentally (see e.g. [23]).

7.1 Non-Newtonian fluids

As mentioned earlier there are a number of fairly common fluids that do not satisfy (94). Examples are the so-called *Bingham fluids*, which do no flow unless a yield stress τ_0 is exceeded (see figure 14)

$$\begin{array}{ll} \text{If } |P_{yz}| > \tau_0 & P_{yz} = \tau_0 + \eta \frac{\partial u_y}{\partial z} \\ \text{If } |P_{yz}| < \tau_0 & \frac{\partial u_z}{\partial y} = 0 \end{array} \tag{101}$$

The above property of a Bingham fluid makes it useful as a paint. The threshold τ_0 keeps the paint from running, but it can still be spread if enough force is applied. As an example consider painting a vertical wall (figure 14(b)). If the thickness of the layer is δ the paint will not run if

$$P_{yz} = \rho g y < \tau_0$$

for all $y < \delta$. When $\delta = \frac{\tau_0}{\rho g}$ it will start running where the shear stress is the largest, i.e. near the wall.

In other non-Newtonian fluids one can model the stress-strain relationship by

$$|P_{yz}| = k \left| \frac{\partial u_y}{\partial z} \right|^n \tag{102}$$

If $n < 1$ the fluid is called pseudoplastic, if $n > 1$ it is called dilatant (see figure 14(a)).

7.2 Summary of results from kinetic theory

- *Boltzmann factor.* The probability that a system is in a state i with energy E_i at fixed V and N is

$$p_i = \frac{1}{Z} e^{-\beta E_i}; \quad Z = \sum_i e^{-\beta E_i}$$

- *Collisions.* The collision *frequency* and *mean free path* are given by

$$\langle \phi \rangle = \sqrt{2} n \sigma \langle v \rangle; \quad l = \frac{1}{n \sigma \sqrt{2}}$$

- *Diffusion* processes are governed by Fick's laws:

$$\mathbf{j}_N = -D \nabla n(\mathbf{r}); \quad \frac{\partial n}{\partial t} = D \nabla^2 n$$

In kinetic theory D is given by

$$D = \frac{\langle v \rangle l}{3}$$

- *Einstein relation.* The diffusion coefficient D is related to the *mobility* through

$$D = \mu k_B T$$

- *Maxwell velocity distribution* for particles in an ideal gas:

$$p(\mathbf{v}) = \left(\frac{m}{2\pi k_B T} \right)^{\frac{3}{2}} \exp \left(-\frac{mv^2}{2k_B T} \right)$$

- *Speed.* The *average* and *mean square* speeds are for an ideal gas

$$\langle v \rangle = \sqrt{\frac{8k_B T}{\pi m}}; \;\; \langle v^2 \rangle = \frac{1}{N} \int_0^\infty v^2 g(v) dv = \frac{3k_B T}{m}$$

- *Surfaces.* The number of particles hitting a surface per unit are and time is

$$j_N = \frac{n \langle v \rangle}{4}$$

- *Thermal conductivity*, denoted κ, is the proportionality constant connecting a *temperature gradient* and the resulting *heat current* per unit area

$$j_Q = -\kappa \nabla T$$

In kinetic theory

$$\kappa = \frac{\langle v \rangle l}{3} n c_V$$

where c_V is the specific heat per particle.

- *Viscosity*, denoted η, is the proportionality constant connecting a *velocity gradient* and the resulting *shear stress* in a Newtonian fluid. The *kinetic viscosity* is $\nu = \eta / \rho$.

$$P_{zy} = -\eta \frac{\partial u_y}{\partial z}$$

In kinetic theory

$$\eta = \frac{nm \langle v \rangle l}{3}$$

7.3 Problems

Problem 7.1.

The thermal conductivity of air at 0^0C is listed as 0.023 W m^{-1} K^{-1}, and is approximately independent of pressure.

(a) Estimate the collision cross section σ of air molecules.

(b) There is a gap of 1 mm. between two 1 m^2 window panes. The inside pane has a temperature 10^0C and the outside pane -10^0C. Calculate rate of heat loss through the window due to thermal conduction.

Problem 7.2.

A vial of height 3 cm is filled with an aqueous suspension of water particles. The particles are spherical with radius 100 Å, and have a density of 1200 kg m^{-3} (for comparison the density of water is 1000 kg m^{-3}). The sample temperature is $T = 300$ K. The viscosity of water is $\eta = 10^{-3}$ Pa s. The mobility of a spherical object in laminar flow is given by *Stokes' law*

$$\mu = \frac{velocity}{force} = \frac{1}{6\pi\eta r} \tag{103}$$

(a) Verify quantitatively that the equilibrium distribution of particles is nearly uniform over the height of the vial.

(b) The vial is placed in a centrifuge and subject to an acceleration equal to 10^4 times that of gravity, acting along the height of the vial (assume that the acceleration is nearly uniform over the height of the vial). Once the new equilibrium is reached, roughly how many times more concentrated are the particles near the outer end of the vial than they were before centrifuging?

(c). Roughly how long must the sample be spun in the centrifuge before equilibrium will be reached?

(d). If the vial stands undisturbed after removal from the centrifuge, roughly how long does it take for the particles to diffuse back to their original equilibrium distribution?

Part III
HYDRODYNAMICS

At this stage we abandon our attempt at a microscopic discussion of the properties of fluids. Instead, we will assume that fluid quantities such as the viscosity and density are known, and concentrate on the resulting macroscopic properties.

8 Laminar flow through a pipe.

We will begin our discussion of *hydrodynamics* by discussing laminar flow and show how viscosity governs the flow properties of slowly flowing fluids.

8.1 Poiseuille flow

Consider, as an example, a liquid which is flowing through a narrow horizontal, cylindrical pipe of radius r and length L. The flow is due to a pressure difference ΔP between the ends of the pipe. We will assume that the fluid is flowing in a streamline fashion with velocity parallel to the axis of the pipe. The viscosity of the fluid is η and we assume that the velocity is zero at the walls of the pipe. In order to find the *flow rate*, $\dot{V}$, (units volume/time) through the pipe (see figure 15(a)).

consider an imaginary cylinder of height L and radius $x < r$ which is concentric with the pipe. In the steady state the force $\Delta P \pi x^2$ generated by the pressure drop will balance the drag $2\pi r \eta L \; du/dx$ due to viscous forces. The velocity profile $u(x)$ satisfies the boundary condition $u(r) = 0$. We find

$$\Delta P \pi x^2 = -\eta \frac{du}{dx} 2\pi x L$$

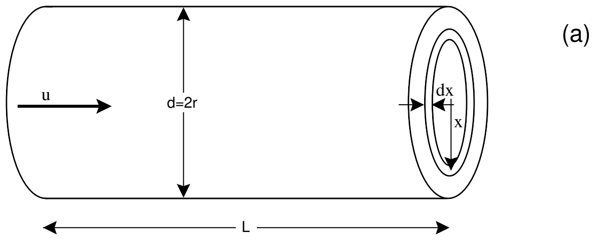
or

$$-x dx = \frac{2\eta L}{\Delta P} du$$

This equation can be integrated to give

$$u(x) = \frac{\Delta P}{4\eta L}(r^2 - x^2) \tag{104}$$

The velocity profile predicted by (104) is shown in figure 15(b).



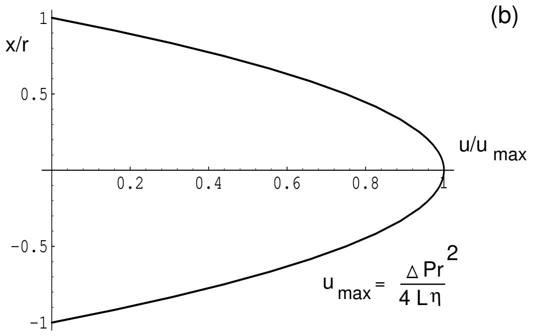


Figure 15: (a) Flow through a pipe. (b) Velocity profile.

To get the volume flow rate $\dot{V}$ consider a cylindrical shell of thickness dx. The amount of fluid flowing per unit time through the shell is

$$d\dot{V} = 2\pi x u(x) dx = \frac{\Delta P \pi}{2\eta L}(r^2 - x^2) x dx$$

Integrating we find for the total flow rate

$$\dot{V} = \int_0^r \frac{\Delta P \pi}{2\eta L}(r^2 - x^2) x dx = \frac{\Delta P \pi r^4}{8\eta L} \qquad (105)$$

which is *Poiseuille's formula.*

8.2 Pipe flow for a non-Newtonian fluid

The previous result can easily be generalized to the case of non-Newtonian fluids. If the flow is laminar the flow velocity will not have any component in a direction perpendicular to the pipe axis. The pressure will then be uniform in a cross sectional plane. Force balance then gives with $\tau(x) = P_{xz}$ the shear stress

$$\pi x^2 \frac{\Delta P}{L} = 2\pi x \tau(x) \tag{106}$$

For a *power law fluid* (102) we find from (106)

$$\tau = \frac{x\Delta P}{2L} = k\left(\frac{-du}{dx}\right)^n$$

or

$$-\frac{du}{dx} = \left(\frac{x\Delta P}{2kL}\right)^{1/n} \tag{107}$$

With the boundary condition $u(x) = 0$ for $x = r$ we can integrate (107) to get

$$u(x) = \frac{n}{n+1}\left(\frac{\Delta P}{2kL}\right)^{1/n}(r^{1+n^{-1}} - x^{1+n^{-1}}) \tag{108}$$

We substitute this result in the equation for the flow rate

$$\dot{V} = \int_0^r 2\pi x u(x)$$

to get after some algebra

$$\dot{V} = \frac{\pi r^3}{(3+\frac{1}{n})\eta L}\left(\frac{r\Delta P}{2Lk}\right)^{1/n} \tag{109}$$

which reduces to (105) in the special case $n = 1$. A *dilatant* fluid ($n > 1$) acts as if there is little or no viscosity for small pressure differences, but the resistance rapidly increases if the pressure is increased ("easy does it!"). On the other hand, a *pseudoplastic* ($n < 1$) flows easier the harder one pushes. Most people have had this experience of pouring ketchup: "first there is nothing, then there is nothing and then there is everything". Blood is a pseudoplastic; this has the important consequence that it can flow with remarkable ease in very narrow capillary tubes.

Pseudoplastics are *slippery*. If you gently test a small puddle with your foot, it will show resistance, but if you follow up by putting down your full weight, it will yield and you may fall.

We can find the laminar flow rate of a *Bingham fluid* in a pipe of circular cross section. There will be a critical pressure difference to make the fluid flow at all given by

$$\pi r^2 \Delta P_{crit} = 2\pi r L \tau_0 \tag{110}$$

Even if $\Delta P > \Delta P_{crit}$ the shear stress will be too small to overcome the yield threshold near the center of the pipe. The material in the center will then move like a *plug*, as when toothpaste is squeezed out of its tube. We define the dimensionless pressure

$$p = \frac{\Delta P}{\Delta P_{crit}} \tag{111}$$

and we obtain $x_0 = r/p$ for the radius of the plug. From (101) we find the differential equation for the velocity profile

$$-\eta \frac{du}{dx} + \tau_0 = \frac{x \Delta P}{2L} \tag{112}$$

valid for $r > x > x_0$. We can integrate (112) to obtain

$$u = \frac{1}{\eta}\left[\frac{\Delta P}{4L}(r^2 - x^2) - \tau_0(r - x)\right] \tag{113}$$

for $r > x > x_0$. For $x < x_0$ we have

$$u = u_0 = \frac{1}{\eta}\left[\frac{\Delta P}{4L}(r^2 - x_0^2) - \tau_0(r - x_0)\right] = \frac{\Delta P r^2}{4\eta L}(1 - \frac{1}{p})^2 \tag{114}$$

We can integrate to find the flow rate

$$\dot{V} = \int_{x_0}^{r} 2\pi x u(x) dx + \pi x_0^2 u_0$$

and find after some algebra

$$\dot{V} = \frac{\Delta P \pi r^4}{8\eta L}[1 - \frac{4}{3p} + \frac{1}{3p^4}] \tag{115}$$

A Newtonian fluid corresponds to the limit $p \to \infty$ In this limit (115) reduces to (105). A plot of the ratio

$$R = \frac{\dot{V}}{\dot{V}_{Poiseuille}} \tag{116}$$

vs. the reduced pressure is shown in figure 16.

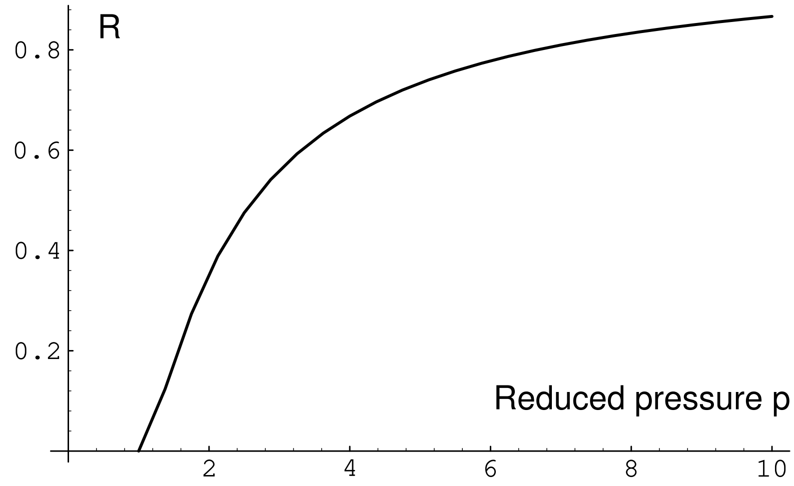


Figure 16: Ratio R defined by (116) vs. reduced pressure p for a Bingham plastic.

8.3 Problems

Problem 8.1.

Two flat solid plates of area 0.5 m^2 are constrained to be parallel to each other in the horizontal (x-y) plane. The plates are separated by a gap of $d = 0.001$ m of air. Suppose that the top plate is moved parallel to the stationary bottom one at a velocity $v = 0.1$ m s^{-1} along the x-axis. The viscosity of the air is 2.0×10^{-5}Pa s^{-1}.

(a). What is the shear force on the bottom plate?

(b). How is the force changed if the air pressure is

(i) increased from one to two atmospheres at constant temperature?

(ii) reduced to a very small value?

Problem 8-2:

A 2 kg solid cylinder is placed inside a vertical pipe with an inside diameter of 52 mm (see Figure 17). The length of the cylinder is 200 mm and the diameter is 50 mm. The cylinder is held concentric inside the pipe by a layer of oil with viscosity of

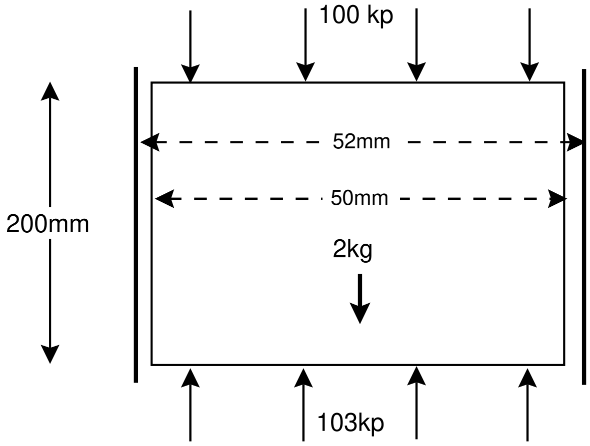


Figure 17: Cylinder of Problem 8-2.

9.2×10^{-2} Pa s. The pressure below the cylinder is maintained at 103 kPa and above the cylinder at 100 kPa. Determine the steady state speed with which the cylinder falls. Neglect the effect of the pressure difference on the flow of oil.

Problem 8-3.

Two square parallel plates with sides are kept apart with a clearance c. A pressure difference ΔP is maintained over the distance L between two opposite edges. This gives rise to flow in the plane of the plates in the direction of the pressure gradient. The clearance is filled with a fluid with viscosity η. Assume laminar flow. Derive a formula analogous to (105) for the total flow rate $\dot{V}$. Assume that $c << L$ so that end effects can be neglected.

9 Dimensional analysis. Reynolds number.

Poiseuille flow represents one of a small number of situations where we can find a closed form for the solution of a flow problem. As we shall see, the laminar type of flow is not the only type that can take place, and it is not even the most common type of flow. Since analytical methods are often not practical, we will often have to resort to empirical formulas. *Dimensional analysis* represents a way of reasoning which helps in constructing empirical formulas and which allows us to isolate the most important parameters. It is useful to familiarize ourselves with the method before we get involved with the details of hydrodynamics. In to order to illustrate the method let us apply it to the laminar flow problem of the previous section.

We note first that in laminar flow the fluid moves with constant speed, there is no acceleration. When constructing an empirical formula for the flow we therefore do not expect the mass density ρ to play a significant role. On what do we expect the flow rate to depend? Three obvious suspects are:

- *The pressure drop per unit length*, $(\Delta P/L)$. One expects that the harder we push, the more fluid will get through.
- *The radius r of the pipe.* Intuitively, the bigger the radius the more stuff will flow through.
- *The viscosity η of the fluid.* The viscosity is responsible for the drag force which makes it necessary to maintain the pressure difference.

We now jump to the conclusion that these three factors are the only ones that matter, and we attempt to construct a formula for the flow rate on the form

$$\dot{V} = const \left(\frac{\Delta P}{L}\right)^{\alpha} r^{\beta} \eta^{\gamma} \qquad (117)$$

To be a valid formula (117) has to be dimensionally consistent. This implies restrictions on the allowed values of the exponents α, β and γ. We use the notation that square brackets [] stands for "dimension of" and we reduce our units to the three basic ones of the SI system: kilogram (kg), meter (m) and second (s). The unit for force, the Newton, is

$$[\mathrm{N}] = \mathrm{kg\ m\ s^{-2}}$$

The pressure unit is the Pascal

$$[\mathrm{Pa}] = \mathrm{Nm^{-2}} = \mathrm{kg\ m^{-1}s^{-2}}$$

We have for the flow rate

$$[\dot{\mathrm{V}}] = \mathrm{m^3 s^{-1}}$$

and $[\frac{\Delta P}{L}]$ =Pa m^{-1} =kg m^{-2}s^{-2}, $[r]$ =m, $[\eta]$ = Pa s = kg m^{-1}s^{-1}. Dimensional consistency of (117) in the case of mass, length and time requires

$$\begin{array}{lrl} mass \rightarrow & 0 & = \alpha + \gamma \\ length \rightarrow & 3 & = -2\alpha + \beta - \gamma \\ time \rightarrow & -1 & = -2\alpha - \gamma \end{array} \tag{118}$$

The solution to (118) is $\alpha = 1$, $\gamma = -1$, $\beta = 4$. Substitution into (117) gives

$$\dot{V} = const\ \frac{\Delta P r^4}{L\eta} \tag{119}$$

This result is consistent with Poiseuille's formula (105) if we put $const = \pi/8$.

As a further example, let us consider the problem of finding the *drag force* F on an object with linear dimension r at low velocities where there is laminar flow. Let us assume that the fluid is a *power law fluid* for which the viscous shear stress is given by (102)

$$|P_{yz}| = k\left|\frac{\partial u_y}{\partial z}\right|^n$$

Let us assume that the drag force can be written

$$F = const\ r^\alpha u^\beta k^\gamma$$

We have $[F]$ =kg m s^{-2}, $[r]$ =m, $[u]$ =m s^{-1}, $[k]$ =kg s^{n-2}m^{-1}. Dimensional consistency then requires

$$\begin{array}{lll} mass & \rightarrow 1 & = \gamma \\ length & \rightarrow 1 & = \alpha + \beta - \gamma \\ time & \rightarrow -2 & = -\beta + (n-2)\gamma \end{array}$$

with solution $\gamma = 1$, $\beta = n$, $\alpha = 2 - n$, giving

$$F = const\ k r^{2-n} u^n$$

If the fluid is *Newtonian*, $n = 1$, $k = \eta$ and we find

$$F = const\ \eta r u \tag{120}$$

There is no way that the constant can be determined by dimensional arguments. However, for the special case of a spherical particle (120) is consistent with *Stokes' law* (103) if $const = 6\pi$.

Not all flow is laminar. Experience tells us that if the flow speed is very high the flow will be *turbulent.* In the case of flow through a pipe, one mechanism by which laminar flow can become unstable is if the pipe is rough. The fluid will then be accelerated around any irregularity. Whether or not the fluid can return to steady laminar flow will then depend on a competition between *viscous forces*, which tend to stabilize the flow, and *inertia* which causes eddies. In the case of our pipe flow problem, we assume that there is a critical velocity u_c beyond which the character of the flow changes from being laminar to being turbulent. Boldened by our success in "deriving" (119) by dimensional analysis we postulate that the critical velocity depends on the inertia, viscosity η, mass density ρ and the diameter d of the pipe [3].

$$u_c = const\ \ \eta^x \rho^y d^z$$

from $[u_c]$ =m s^{-1}, $[\eta]$ =kg m^{-1}s^{-1}, $[\rho]$ =kg m^{-3}, $[d]$ =m we find

$$\begin{array}{lll} length & \to 1 & = -x - 3y + z \\ mass & \to 0 & = x + y \\ time & \to -1 & = -x \end{array} \tag{121}$$

giving $x = 1$, $y = -1$, $z = -1$ or

$$u_c = const\ \ \frac{\eta}{\rho d} \tag{122}$$

We define the *Reynolds number* as

$$Re = \frac{u\rho d}{\eta} \tag{123}$$

and note that it a dimensionless quantity. The Reynolds number occurs in many different situation other than flow in pipes. In general, u has the interpretation of flow velocity, ρ of mass density, d of the typical dimension of object which the fluid flows in or around, and η is the viscosity. The nature of the flow (laminar vs. turbulent) depends on the Reynolds number. In the case of pipes one typically finds that if the Reynolds number is less than 2000 the Poiseuille formula is a good approximation. To what does a Reynolds number of 2000 correspond?

Suppose we have water flowing in a pipe of diameter 10^{-2}m. The viscosity of water is approximately 10^{-3}Pa s, the density is $\rho = 10^3$kg m^{-3}. The

[3] We could equally well have used the radius here, but it is more conventional to use the diameter to describe the dimension of a pipe

critical velocity is then about 0.2 m s^{-1}. If instead we have a storm sewer of diameter 1 m, the critical velocity is 2 mm s^{-1}, which is not very fast indeed. Air at room temperature has a viscosity of slightly less than 2×10^{-5}Pa s, at atmospheric pressure the density is 1.29 kg m^{-3}, so the critical velocity for air is higher than for water, as it also will be for viscous fluids such as oil. Note the role of the length dimension in (123). In medicine one deals with tiny objects such as viruses and cells which swim in the blood, the movement of such objects will typically involve laminar flow. On the other hand, the flows of air and water in meteorology and oceanography are almost never laminar.

The transition from laminar to turbulent flow in a pipe is our first encounter with a *hydrodynamic instability.* This particular transition is analogous to a *discontinuous phase transition* in equilibrium thermodynamics, such as the change from a liquid to a gas. In the gas-liquid transition the density changes discontinuously, and there is typically *hysteresis*: e.g. vapor can be maintained for long periods in a supercooled state below the transition without condensing. Similarly, it is possible with care to maintain laminar flow at Reynolds numbers which are significantly higher than the critical number.

9.1 Problems

Problem 9-1:

A cylindrical pipe is designed to carry a flow rate $\dot{V}$ (volume/time) of a fluid with viscosity η and density ρ .

(a). What is the minimum diameter of the pipe if the flow is to be laminar and the critical Reynolds number is Re.

(b). What is then the pressure difference between two points a distance L apart along the pipe.

(c). Give numerical answers if $Re = 2000$, $\eta = 10^{-3}$Pa s, $\dot{V} = 10^{-4}$m^3 s^{-1}, $L = 1$ m, $\rho = 1000$ kg m^{-3}.

Problem 9-2:

Show by dimensional analysis that for laminar flow in a pipe of a power-law fluid in which the viscous shear stress is given by

$$\tau = k\left(\frac{-du}{dx}\right)^n$$

the flow rate will be given by

$$\dot{V} = const.\, r^3 \left(\frac{r\Delta P}{Lk}\right)^{1/n}$$

10 Fluids at rest. Some problems of stability.

As we have mentioned earlier, what distinguishes a fluid from a solid is that a fluid in equilibrium cannot support any shear stress. If the system is out of equilibrium and there is a net flow, there will be viscous forces associated with changes in the velocity field, but if the fluid is at rest with regards to its surroundings there will be no viscous forces. This means that when a fluid is at rest relative to the surroundings there is only one kind of force which can be transmitted, namely the *hydrostatic pressure* which is *always normal to any surface of contact.* In many applications involving liquids it is a good approximation to assume that the density of a fluid is constant or that we are dealing with an *incompressible fluid.* If the mass density of such a fluid is ρ, and if the pressure at a free surface is P_s, then the pressure at depth z below the free surface is

$$P = P_s + \rho g z \tag{124}$$

If the fluid is compressible and the density depends on the depth we get instead

$$P = P_s + \int_0^z \rho(z) g dz \tag{125}$$

i.e. the pressure at depth z increases by an amount which is equal to the weight of the column of fluid above it. A closely related concept is that of *buoyancy.* Since the pressure inside a fluid in equilibrium depends on the depth there will be a net upwards thrust on any object submerged in it. It is easy to see that (124) or (125) leads to *Archimedes principle*, namely that the buoyancy is a force which is directed opposite to and is equal to the weight of the fluid which is displaced by a submerged object. The buoyancy force can be thought of as applied at the *center of buoyancy* B, which is the center of gravity of the displaced fluid. If an object is submerged in a fluid, and is at equilibrium, the force of gravity must cancel the force of buoyancy. We must also consider the *stability* of the equilibrium. Consider first the simple case of a balloon with a gondola submerged in air (figure 10(a) and (b)).

At a certain height the weight of the balloon + gondola will balance the force of buoyancy. Since the atmospheric density decreases with height, it is easy to see that if the balloon rises above its equilibrium height it will tend to sink, while if it sinks below the equilibrium height there will be a net upwards force. The equilibrium height is thus stable. We can next consider the stability against small torques. We see that if the center of gravity is below the center of buoyancy there will be a stabilizing torque if the balloon is rocked.

The situation is slightly more complicated if when dealing with a body which is floating on the surface of a liquid (e.g. boat or a log) . In this case an angular displacement of the body will shift the center of buoyancy from its previous position B (figure 10(c) and (d)) to its new position B'. We define the *metacenter* M as the projection of the line of action through B' on the line BG, where G is the center of gravity. If the metacenter is above the center of gravity the floating object is stable.

10.1 How does a log float?

We illustrate the points made above by an example. A uniform wooden cylinder (log) has a relative density of ρ For what ratios of diameter to height will it float **(a)** upright? **(b)** with the axis parallel to the water surface?

To solve this problem we must first determine the equilibrium position. We next consider an orientation which is tilted by a small angle θ from the equilibrium orientation and locate the shift in the center of buoyancy. This calculation is then used to determine the metacenter. If the metacenter is above the center of gravity, there is a restoring torque, and the equilibrium orientation is stable against small perturbations. If the metacenter is below the center of gravity, the floating object will topple.

(a). *Upright cylinder.*

In figure19(a) we plot the submerged part of the cylinder in a coordinate system centered on the cylinder. The submerged volume is $\pi r^2 \rho h$, the height of the center of buoyancy is $z_B = \frac{1}{2}\rho h$ and we choose our coordinate system so that $x_B = 0$. The y-coordinate of the center of buoyancy will depend on the tilt angle θ. The volume of the thin slice in figure 19(a) of thickness dy is

$$2\sqrt{r^2 - y^2}(y \tan\theta + \rho h)dy$$

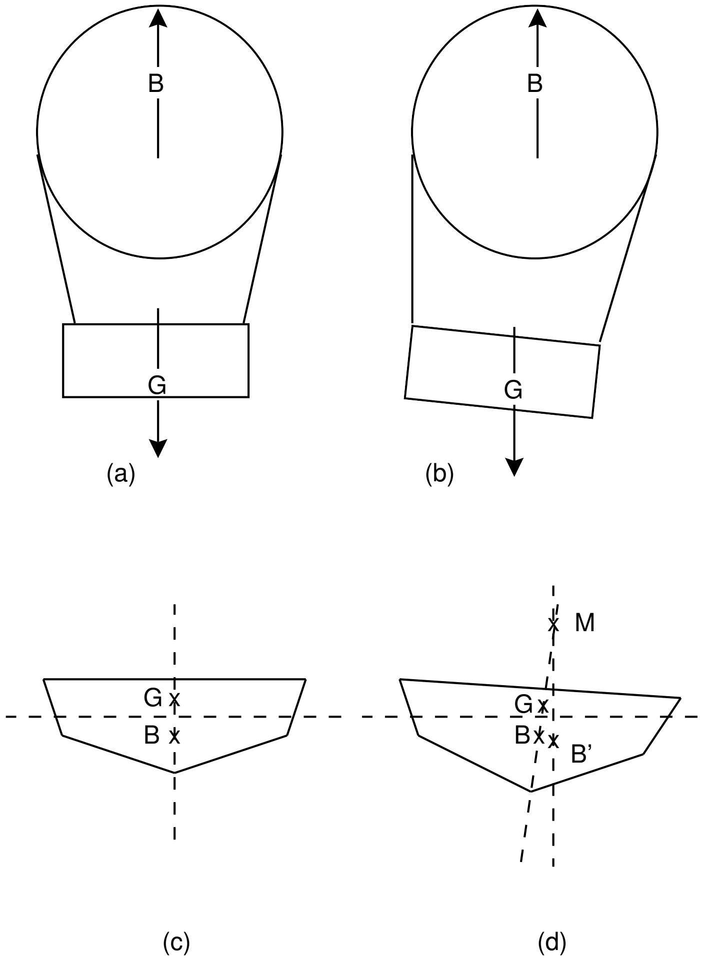


Figure 18: (a) Balloon floating in air. (b) Tilted balloon. (c) A floating ship. (d) Ship tilted from equilibrium.

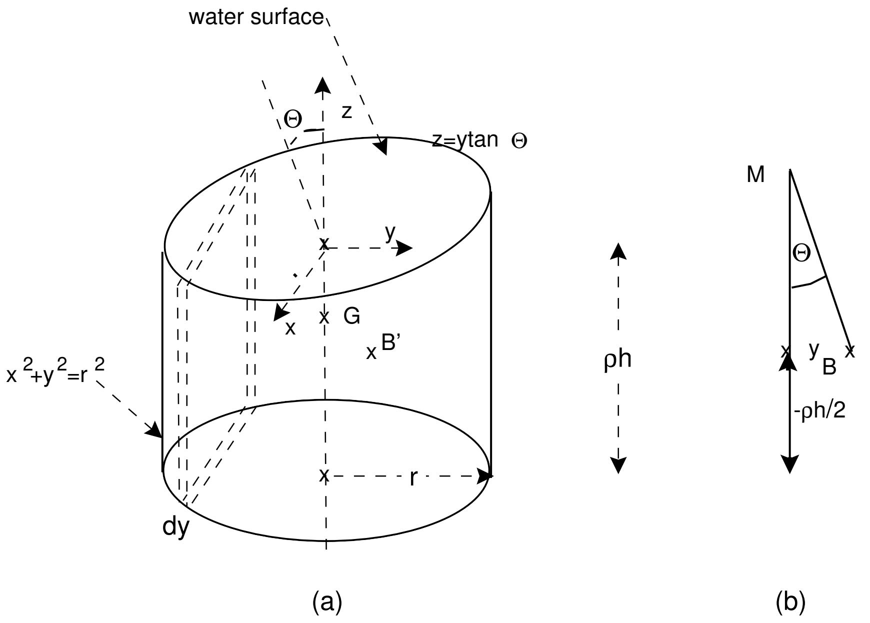


Figure 19: (a) Submerged part of upright log. (b) Metacenter and center of buoyancy.

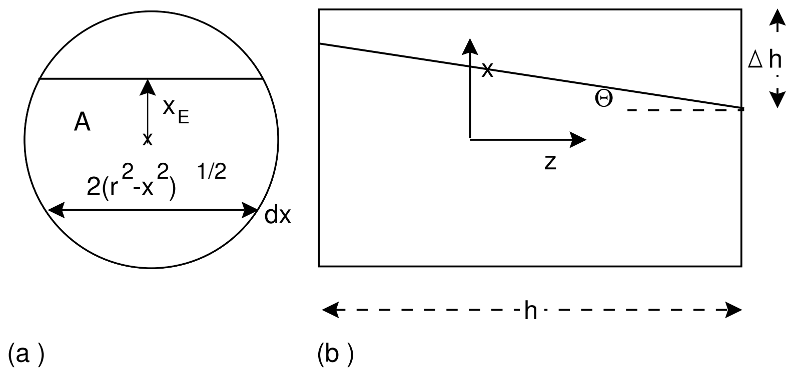


Figure 20: Submerged part of flat log.(a) Cross section of log. (b)Side view.

The y coordinate of the center of buoyancy is thus

$$y_B = \frac{2}{\pi r^2 \rho h} \int_{-r}^{r} y\sqrt{r^2 - y^2}(y \tan\theta + \rho h)dy$$

Notice that the term independent of θ in the integrand averages to zero and introducing a new variable ϕ given by $y = r\cos\phi$, we have $dy = -rsin\phi d\phi$ and obtain

$$y_B = \frac{2r^2\tan\theta}{\pi\rho h} \int_{o}^{\pi} d\phi \cos^2\phi \sin^2\phi = \frac{r^2\tan\theta}{2\pi\rho h} \int_{o}^{\pi} d\phi \sin^2 2\phi = \frac{r^2\tan\theta}{4\rho h} \quad (126)$$

The center of gravity of the cylinder is at $z_G = \frac{h}{2}$, $y_G = 0$, $x_G = 0$. From figure 19(b) the position of the metacenter on the cylindrical axis can be found as

$$z_M = \frac{\rho h}{2} + \frac{y_B}{\tan\theta} = \frac{1}{4}(2\rho h + \frac{r^2}{\rho h}) \quad (127)$$

The upright position is stable if $z_M > \frac{h}{2}$ or with $d = 2r$ if

$$8(1-\rho)\rho < \frac{d^2}{h^2} \quad (128)$$

The condition (128) is plotted in figure 21(c).

(b). *The log lies flat.*

We first locate the equilibrium height x_E when the cylinder floats with its axis parallel to the water surface (see figure20(a)). We find for the area A of a circle below x_E :

$$A = 2\int_{-r}^{x_E} dx\sqrt{r^2 - x^2}$$

This integral can be evaluated to yield with $x = \sin\phi$

$$A = 2r^2 \int_{-\pi/2}^{\sin^{-1}(x_E/r)} \cos^2\phi$$

$$= r^2 \int_{-\pi/2}^{\sin^{-1}(x_E/r)} (1 + \cos 2\phi)d\phi = \left[\phi + \frac{1}{2}\sin 2\phi\right]_{-\pi/2}^{\sin^{-1}(x_E/r)}$$

Finally using $\sin 2\phi = 2sin\phi\cos\phi$ we find using the fact that at the equilibrium height we must also have $A = \rho\pi r^2$

$$A = \rho\pi r^2 = x_E\sqrt{r^2 - x_E{}^2} + r^2\sin^{-1}\frac{x_E}{r} + \frac{\pi r^2}{2} \tag{129}$$

This equation must be solved numerically. A straightforward method is to guess a value of x_E/r, substitute it into (129) and solve for ρ. The result is shown in figure 21(a).

We have chosen a coordinate system with origin at the center of the cylinder so that the center of gravity is at $x_g = y_g = z_g = 0$. The center of buoyancy is given by the centroid of the area A of fig 20(a)

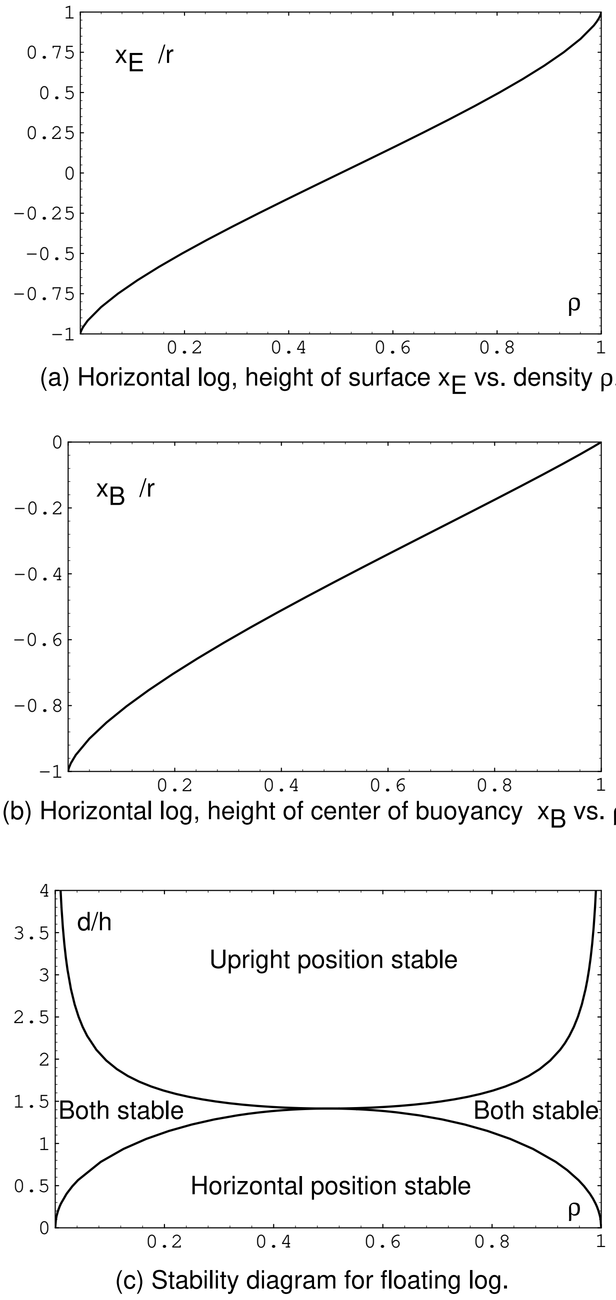
$$x_B = \frac{2}{\rho\pi r^2}\int_{-r}^{x_E} dx x(r^2 - x^2)^{\frac{1}{2}} = \frac{-2(r^2 - x_E^2)^{\frac{3}{2}}}{3\rho\pi r^2} \tag{130}$$

A plot of x_B vs. ρ is given in figure 21(b). If the cylinder is tilted by a small angle this height will stay approximately the same. The centroid of a wedge of height h is at height $\frac{h}{3}$. The z-component of the buoyancy is thus at

$$z_B = -(\frac{1}{2} - \frac{1}{3})\frac{h\Delta h}{2}2\sqrt{r^2 - x_E^2}\frac{1}{\rho\pi r^2} \tag{131}$$

The position with a horizontal axis is stable if the metacenter is above the center of gravity or

$$x_B - \frac{z_B h}{\Delta h} > 0 \tag{132}$$



(a) Horizontal log, height of surface x_E vs. density ρ.

(b) Horizontal log, height of center of buoyancy x_B vs. ρ.

(c) Stability diagram for floating log.

Figure 21:

Using (130) we find that the horizontal position is stable if

$$\frac{d}{h} < \sqrt{\frac{2}{1 - x_E^2/r^2}} \tag{133}$$

The stability situation is summarized in figure 21c. Note that for $\rho \neq 0.5$ there is a region of d/h where both the upright and horizontal positions are stable. This situation is analogous to what happens in pipe flow, when the Reynolds number is close to its critical value, and both laminar and turbulent flow may be stable against small perturbations.

10.2 Problems

Problem 10-1:

A hydrometer is a convenient instrument for measuring the density of a liquid from a determination of the depth at which the hydrometer floats. It is desired to construct a hydrometer for which the calibration marks for $1.00 gm\ cm^{-3}$ and $1.20 gm\ cm^{-3}$ are $10 cm$ apart on a stem of $1.00 cm^2$ cross-section (see figure 22(a)).

(a). What is the volume of the bulb below the 1.2 mark?
(b.) What is the total mass of the hydrometer?
(c.) How far from the 1.2 mark is the 1.1 mark?

Problem 10-2:

A open pop can has diameter $7 cm$, height $12 cm$. It is pushed upside down into water and as a result the trapped air inside the can is isothermally compressed (see figure 22(b)) . At a depth $25 m$ below the surface the force required to push the can down is zero.

(a). What is the mass of the can?
(b). Is the position of the can at depth $25 m$ stable?

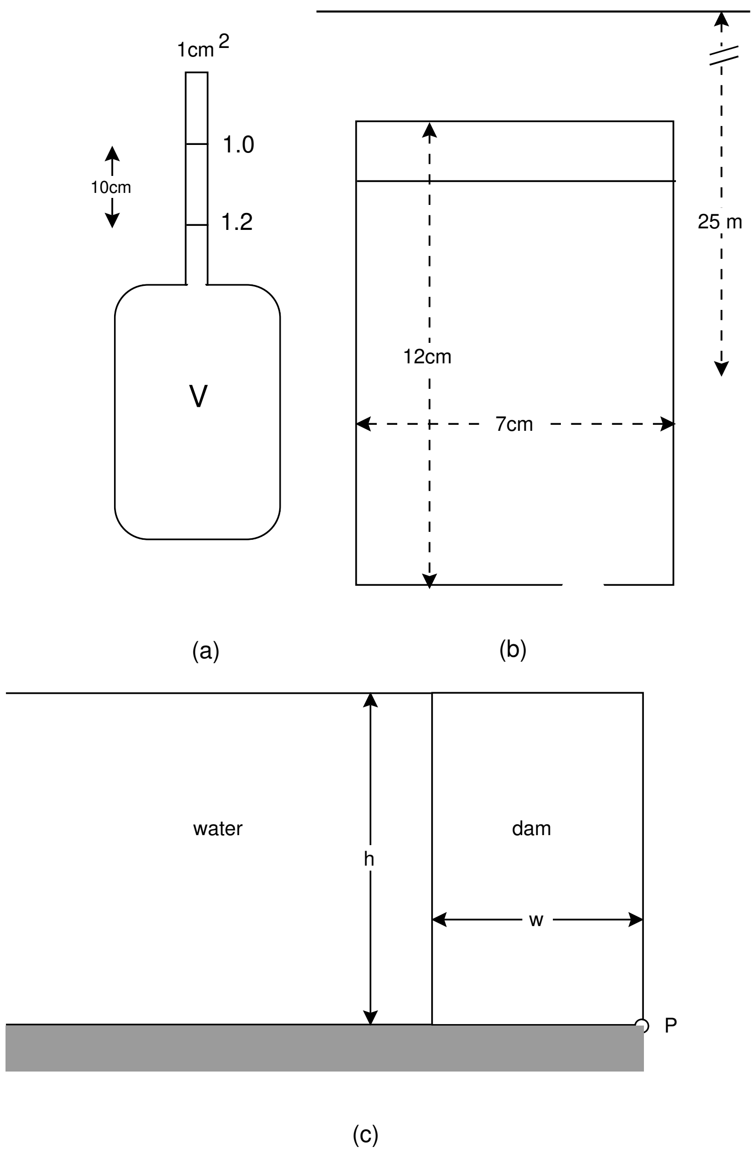


Figure 22: (a) Hydrometer. (b) Submerged pop-can. (c) Dam.

Problem 10-3:

The cross section of a dam has a rectangular shape as shown in figure 22(c). There is water seepage under the dam producing a lift pressure which is equal to the water pressure at depth h. The density of water and of the material in the dam is, respectively is $1000kg\ m^{-3}$ and $2000kg\ m^{-3}$. What is the minimum width to height ratio w/h to prevent the dam from toppling about the point P in the figure.

Problem 10-4:

A vertical mine shaft extends 3000 m down into the earth. The pressure at the top is $100kpa$.

(a). What is the air pressure at the bottom of the shaft? Assume that the temperature is constant and equal to $-10\ {}^0C$, and that the acceleration of gravity $g_0 = 9.8\ ms^{-2} = const.$.

(b). The temperature increases linearly with depth from -10^oC at the to 30^oC at the bottom. What is now the pressure at the bottom of the mine shaft.

(c). If the density of the earth is uniform, the acceleration g of gravity will decrease with depth z as $g = g_0(1 - \frac{z}{R})$, where $R = 6400\ km$ is the radius of the earth. Show that the effect of changes in gravity will be small compared to the effect of the temperature variation.

11 Ideal fluids. Euler and Bernoulli equation.

At this stage we need to define some concepts and discuss some useful idealizations. We will mostly be concerned with *steady flow*, which is flow which does not change in time. A slightly different concept is that of *uniform flow.* Uniformity means that the velocities at any time do not depend on position. Steady flow can be either uniform or nonuniform. If the density of the fluid doesn't vary spatially, or in time, the fluid is *incompressible.* Incompressible flow in a long straight pipe will be uniform irrespective of whether it is steady or not. We have already encountered the concept of *laminar flow* where the movement of a volume element of fluid follows a set of nearly parallel *stream lines* (figure 23). A cylindrical surface of stream lines is called a *stream tube.* Fluid matter can only enter and leave a stream tube through its ends.

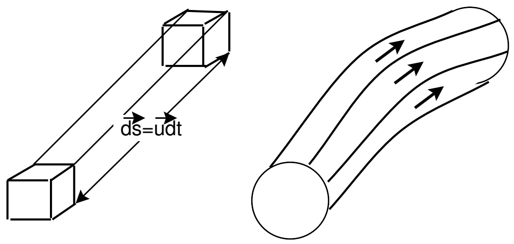


Figure 23: (a) Motion of volume element. (b) Stream tube.

In the case of laminar flow the pattern of streamlines is stabilized by viscous forces. *Ideal fluids* also flow along streamlines. However, we make the additional assumption that viscous forces can be neglected. The concept of ideal fluids is somewhat contradictory since from (123) the Reynolds number will be infinite if the viscosity is zero. There are, however, a number of situations in which most of the turbulence and viscous dissipation is confined to a *boundary layer* near the walls of the fluid. If this boundary layer is thin the ideal fluid approximation may be useful, particularly if one is only interested in a crude estimate of the parameters involved. Finally, *one-dimensional flow* is characterized by parameters which only depend on one spatial coordinate.

We next consider the acceleration of a volume element of an ideal fluid in one-dimensional flow and the consequences of conservation laws for mass and energy. Consider a volume element containing a certain mass of fluid which moves with the flow along a streamline. Let $u(s)$ be the velocity a distance s down the streamline. The velocity of this element may depend on time because the flow is non-steady (time dependent) or because the velocity changes along the streamline (non-uniform flow). The change in u if the time is incremented by dt is

$$du = \frac{\partial u}{\partial s} ds + \frac{\partial u}{\partial t} dt \tag{134}$$

where $ds = udt$. We have for the acceleration

$$a = \frac{du}{dt} = u\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \tag{135}$$

Equation (135) can be generalized to describe the derivative of any quantity $f(s,t)$ which is associated with the volume element of matter flowing along the stream line. We have

$$df = \left(u\frac{\partial f}{\partial s} + \frac{\partial f}{\partial t} \right) dt$$

Formally, we define the *convective derivative* as

$$\frac{d}{dt} = u\frac{\partial}{\partial s} + \frac{\partial}{\partial t} \tag{136}$$

We can extend our description to flow in three dimensions. Let

$$\mathbf{u} = u_x\hat{\mathbf{x}} + u_y\hat{\mathbf{y}} + u_z\hat{\mathbf{z}}$$

and

$$\mathbf{s} = x\hat{\mathbf{x}} + y\hat{\mathbf{y}} + z\hat{\mathbf{z}}$$

where $\hat{\mathbf{x}}$,$\hat{\mathbf{y}}$,$\hat{\mathbf{z}}$ are unit vectors in three Cartesian directions. We have

$$du_x = (u_x\frac{\partial u_x}{\partial x} + u_y\frac{\partial u_x}{\partial y} + u_z\frac{\partial u_x}{\partial z} + \frac{\partial u_x}{\partial t})dt = (\mathbf{u}\cdot\nabla + \frac{\partial}{\partial t})u_x dt$$

giving for the acceleration

$$\mathbf{a} = \frac{d\mathbf{u}}{dt} = (\mathbf{u}\cdot\nabla)\mathbf{u} + \frac{\partial \mathbf{u}}{\partial t} \tag{137}$$

Similarly, the generalization to three dimensions of the convective derivative of a scalar quantity such as e.g. the temperature is

$$\frac{dT}{dt} = (\mathbf{u}\cdot\nabla)T + \frac{\partial T}{\partial t}$$

Let us return to the case of one-dimensional flow and assume that a volume element Δds inside a stream tube of thickness ds, cross sectional area Δ, is subject to an external force per unit mass $\mathbf{f}$ and that the pressure drops by an amount $-dP$ from one end of the tube to the other. Newton's second law gives

$$f\rho\Delta ds - dP\Delta = \rho\Delta ds\frac{du}{dt} = \rho\Delta ds\left(u\frac{\partial u}{\partial s} + \frac{\partial u}{\partial t} \right)$$

or if we are dealing with steady flow $\frac{\partial u}{\partial t} = 0$, $\frac{\partial u}{\partial s} = \frac{du}{ds}$

$$f = \frac{dP}{\rho ds} + u\frac{du}{ds} \tag{138}$$

It is easy to generalize (9-6) to three dimensional flow. In vector notation

$$\mathbf{f} = \frac{1}{\rho}\nabla P + (\mathbf{u} \cdot \nabla)\mathbf{u} \tag{139}$$

Equation (139) is called the *Euler equation.* (If the time-dependent term, and the effect of viscous forces are included, the result is called the *Navier Stokes* equation. Numerical or analytical solutions to the Navier Stokes equation are beyond the scope of this course). An important special case of the Euler equation has $\rho = const$ and f the force of gravity acting in the negative z-direction. The component of $\mathbf{f}$ along the stream tube is - $g\frac{dz}{ds}$ and we get

$$\frac{d}{ds}(\frac{P}{\rho} + \frac{u^2}{2} + gz) = 0$$

or

$$\frac{P}{\rho} + \frac{u^2}{2} + gz = \text{constant} \tag{140}$$

This equation is commonly called the *Bernoulli equation.* It can be thought of as a statement of *conservation of energy.*

We stress that (140) is only an approximation. It may be worth repeating the assumptions that have been made:

1. The flow is steady.
2. The fluid is incompressible.
3. We have derived (140) for flow along a streamline; the value of the constant may be different along a different streamline.
4. Energy dissipation due to viscous forces and eddies are neglected.

11.1 Darcy's law

In order to investigate the range of validity of the different approximations that we have employed let us study the simple example of the draining of a water tank through a hose. Let the cross sectional area of the tank be $A = 0.5$ m^2 and the initial height of the water level be $h = 1$ m. The tank is drained by a hose of cross sectional area $a = 1$ cm^2, which ends $z = 1$ m below the bottom of the tank. Let us first assume that water is an ideal fluid and ask:

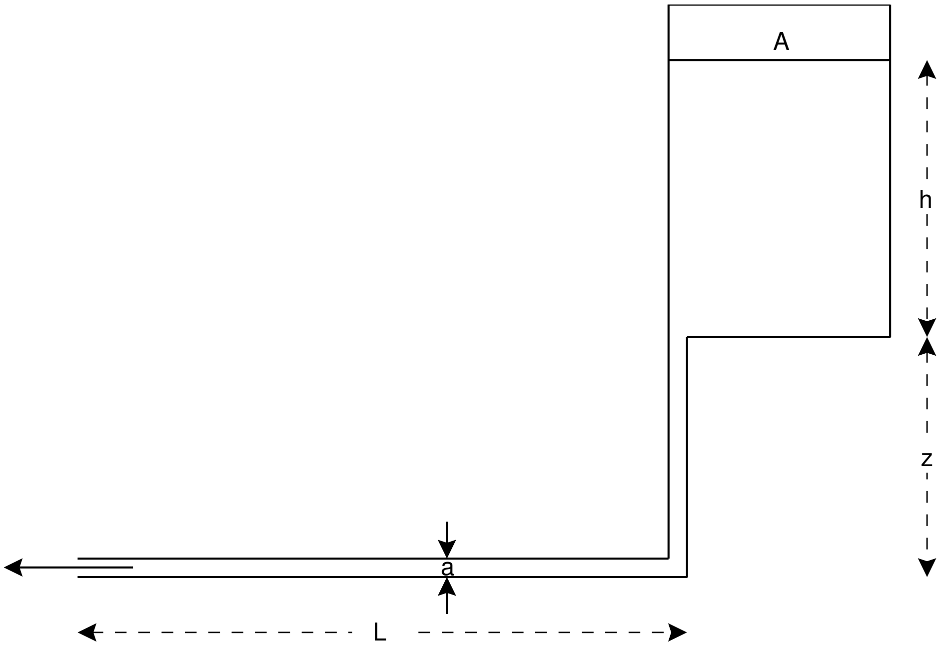


Figure 24:

(a) What is the initial flow rate?
(b) How long does it take to empty the tank?

Solution: The pressure P at the bottom of the tank is $P_{atm} + \rho g h$, where P_{atm} is the atmospheric pressure and ρ is the mass density of water. The flow velocity inside the water tank is negligibly small. The pressure at the end of the hose where the water escapes is P_{atm}. Bernoulli's equation gives for u, the flow velocity in the hose

$$g(h+z) = \frac{u^2}{2}$$

or $u = \sqrt{2g(h+z)}$. Substituting numbers we find for the initial velocity

$$u_o = \sqrt{2 \times 9.8 \times 2} = 6.26 \text{ m s}^{-1}.$$

Therefore

(a) The initial flow rate is

$$a u_o = a\sqrt{2g(h+z)} = 6.26 \times 10^{-4} \text{ m}^3 \text{ s}^{-1}$$

.

(b) We have $A\dot{h} =$ flow rate, or

$$\dot{h} = -\frac{a}{A}\sqrt{2g(h+z)}$$

This equation can be integrated to yield

$$t = -\int_h^0 \frac{dh}{\frac{a}{A}\sqrt{2g(h+z)}} = \frac{A}{a}\sqrt{\frac{2}{g}}(\sqrt{h+z} - \sqrt{z}) \qquad (141)$$

Substituting numbers we find

$$t = .5 \times 10^4 .45 \times (\sqrt{2} - 1) = 932 \text{ s} = 15.5 \text{ min}$$

We must now ask ourselves: how realistic is the assumption that water is an ideal fluid? We note that for an ideal fluid the flow rate is independent of tube length L. If there are frictional losses it is obvious that the water will flow more slowly if the hose is long. What then is "long" $L = 1$ m? $L = 10$ m? $L = 100$ m?

One way to proceed is to modify the Bernoulli equation to take into account "frictional" losses due to viscosity or turbulence. Without such losses the equation can be written

$$\frac{u^2}{2g} = h + z \tag{142}$$

We refer to the right hand side, $h + z$ of 142 as the "head". A natural way to proceed is to say that losses cause the effective head to be reduced by a "frictional head" h_f . The ratio between the frictional head/length and the gravitational head has the dimension of inverse length

$$\left[\frac{h_f/l}{u^2/2g}\right] = m^{-1} \tag{143}$$

The only other length in the problem is the diameter d (or if you prefer the radius r) of the hose. It is reasonable to assume an empirical relation

$$\frac{h_f}{l} = f\frac{u^2}{2dg} \tag{144}$$

where f is a dimensionless friction factor (or fudge factor) which depends on the Reynolds number of the flow, and on properties of the hose such as its roughness. Equation (144) is commonly known as *Darcy's law*[4]. In order to see that (144) is not as arbitrary as it might seem at first sight we note that Poiseuille's formula (105) can be cast in the form (144). We have for Poiseuille flow

$$\dot{V} = au_{ave} = \frac{\Delta P \pi r^4}{8\eta l} \tag{145}$$

where u_{ave} is the average flow velocity and $\Delta P = \rho g h_f$ is the pressure drop in the pipe. We can rewrite (145)

$$\frac{h_f}{l} = \frac{32\eta}{\rho u_{ave} d}\frac{{u_{ave}}^2}{dg} = f\frac{{u_{ave}}^2}{2dg} \tag{146}$$

with

$$f = \frac{64\eta}{\rho u_{ave} d} = \frac{64}{Re} \tag{147}$$

[4]Our definition of the friction factor f agrees with [48], but differs with that of [38] by a factor of 4

where we have used the definition (123) for the Reynolds number. So, what is the Reynolds number predicted by the Bernoulli equation? We have $\eta = 10^{-3}$ Pa s for water,

$$d = 2\sqrt{\frac{a}{\pi}} = 1.12 \times 10^{-2}\mathrm{m}, u_{ave} = 6.26 \text{ m s}^{-1}, \rho = 10^3 \text{kg m}^{-3}$$

giving

$$Re = \frac{6.26 \times 10^3 \; 1.12 \times 10^{-2}}{10^{-3}} = 70000$$

We find for the friction head

$$h_f = \frac{32(L+z)}{70000} \frac{{u_{ave}}^2}{dg} \tag{148}$$

giving $h_f = 28$ cm if $L = 1$ m, $h_f = 1.8$ m if $L = 10$ m and $h_f = 16$ m if $L = 100$ m. In the two first cases the correction due to frictional losses will be relatively small, while in the case of the long hose ($L = 100$ m) the correction will be significant[5].

We must next question if the assumption of laminar flow is reasonable. In chapter9 we stated that flow in a pipe will be turbulent if the Reynolds number is larger than ~ 2000. Since we estimated the Reynolds number to be 70 000 *the assumption of laminar flow is clearly wrong.*

A number of engineers have studied flow in pipes made up of different materials and with controlled roughness of the inner walls. The result when f is plotted vs. the Reynolds number is commonly referred to as the Moody diagram (see figure 25)

We write the modified Bernoulli equation on the form

$$\frac{u^2}{2g} = h + z - \frac{f(L+z)}{d} \frac{u^2}{2g} \tag{149}$$

which can be rewritten

$$u^2 = \frac{2g(h+z)}{1 + \frac{f(L+z)}{d}} \tag{150}$$

The flow is thus

$$\dot{V} = au = a\sqrt{\frac{2g(h+z)}{1 + \frac{f(L+z)}{d}}} \tag{151}$$

[5] We stress that our discussion is only qualitative - we have neglected losses occurring at the intake to the hose, the kink after z and the nozzle where the water escapes.

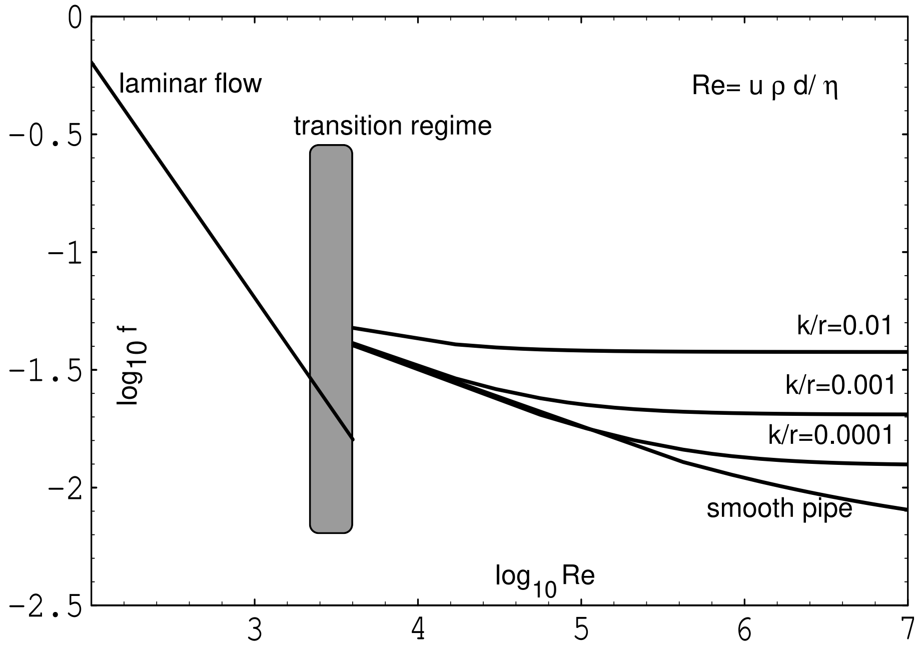


Figure 25: Moody diagram. Log-log plot of friction factor f vs. Reynolds number $\rho u_{ave} d/\eta$, k is a roughness parameter ($k \approx 1mm$ for concrete, $0.2-1mm$ for wood and $0.05mm$ for commercial steel pipe).

The numerator in (151) is the flow if there are no frictional losses. The denominator

$$\sqrt{1 + \frac{f(L+z)}{d}} \tag{152}$$

can be thought of as a correction factor. A typical value of f may be 0.03. If $L + z = 2$ m we have a correction factor ~ 3, when $L + z = 11$ m the correction factor is ~ 6 while for $L + z = 101$ m the factor is ~ 17. We conclude that the Bernoulli equation should be used with caution, ideal fluids are indeed an idealization, real fluids can be much more complicated.

11.2 Problems

Problem 11-1

Two water reservoirs having a difference in water surface level of 50 m are connected by a 1 km long pipe of 0.6 m diameter. Assuming the friction factor is $f = 0.01$ what is the rate of flow (in $\mathrm{m}^3\ \mathrm{s}^{-1}$) and shear stress at the wall of the pipe?

Problem 11.2.

An airplane of mass 8000 kg has wing area of 400 m^2. The speed of air flow is 100 m s^{-1} over the top of the wing and 70 m s^{-1} below the bottom of the wing. Assume the density of air varies with height as $\rho_0 \exp(-z/8000\ \mathrm{m})$ with $\rho_0 = 1.3$ kg m^{-3}.

(a) What is the lift force on the airplane at low altitude.

(b) How much load could be carried if the plane were to fly at an altitude of 5000 m?

Problem 11-3.

Suppose the velocity coordinates of a fluid in some region is given by

$$\mathbf{u} = [0.1y^2, 0.2xy, 0] \text{ m s}^{-1}$$

The temperature in the same region is given by

$$T = 270 + 0.5xt \text{ K}$$

(a) Show that the flow is irrotational in the region, i.e $\nabla \times \mathbf{u} = 0$.

(b) Find the convective derivative of T.

(c) The rate of change of the temperature at the point $\mathbf{r}$=[x,y,z] is

$$\frac{\partial T}{\partial t} = 0.5x \text{ K s}^{-1}$$

Explain what is the physical difference between the partial and convective derivative.

12 Water waves.

We are all familiar with waves on the ocean or on the surface of a lake. In the first section of this chapter we will consider ordinary water waves where the driving force is the wind. At very low speed a wind will cause a laminar shear flow. At moderate speeds of the order meters per second this flow becomes unstable and waves are generated. This effect is commonly called the Kelvin–Helmholtz instability. As it turns out, it is difficult to answer questions such as: what is the amplitude of the resulting wave? and which wavelength is selected? However, some features of water waves can be determined if we simply *assume* that we are dealing with a sinusoidal wave of wave number k. In particular we will find that with some simplifications we can determine the *dispersion relation*, which is the relation between the *angular frequency* ω and *wave vector* k. Once the dispersion relation is known, we can find the speed with which the wave propagates. In the second section of this chapter we consider solitary waves and discuss conditions for a hydraulic jump and discuss applications to tidal waves.

12.1 Gravity and capillary waves

Consider a channel of height h. We let y represent the vertical position with $y = 0$ the surface of water at rest. The wave propagates in the z-direction, with wave number k. The water which at rest was at height y, horizontal position z, will as a result of the wave motion be displaced by an amount $\boldsymbol{\psi}$ with components ψ_y, ψ_z. The amplitude of the wave will depend on the height y (waves are only prominent near the surface) but for a uniform channel we don't expect the amplitude of the wave to depend on z. We assume waves on the form

$$\begin{aligned}\psi_y(y,z,t) &= A_y(y)\cos(\omega t - kz)\\ \psi_z(y,z,t) &= A_z(y)\sin(\omega t - kz)\end{aligned} \tag{153}$$

Squaring both sides of the equations and using $\sin^2(x) + \cos^2(x) = 1$ we find

$$\left(\frac{\psi_y}{A_y}\right)^2 + \left(\frac{\psi_z}{A_z}\right)^2 = 1 \tag{154}$$

which is the equation for an ellipse. Therefore, the assumption (153) means that the particles near the surface of the water execute an elliptical path (figure 26(b)). The wave pattern during a complete cycle is described in figure 26(a).

The velocity field can be obtained by differentiating (153) with respect to time

$$\begin{aligned}u_y &= -\omega A_y \sin(\omega t - kz)\\ u_z &= \omega A_z \cos(\omega t - kz)\end{aligned} \tag{155}$$

We next wish to derive some formulas for the height-dependence of the amplitude. To do this we need to make some simplifying assumptions. The first is that we assume water to be *incompressible.* In the next chapter we discuss flow where density changes are important. The idealization of restricting ourselves to incompressible flow is fairly harmless in the present case where we consider flow with a free surface. Mathematically, the restriction of incompressible flow can be expressed by the condition that there is no net flow in or out of a volume element (figure 27(a)) $dV = dxdydz$ centered at $\mathbf{r} = \{x, y, z\}$

If the velocity field is $\mathbf{u} = \{u_x, u_y, u_z\}$ we can write for the inflow through the y, z surface at $x - \frac{dx}{2}$

$$u_x(x - \frac{dx}{2}, y, z)dydz$$

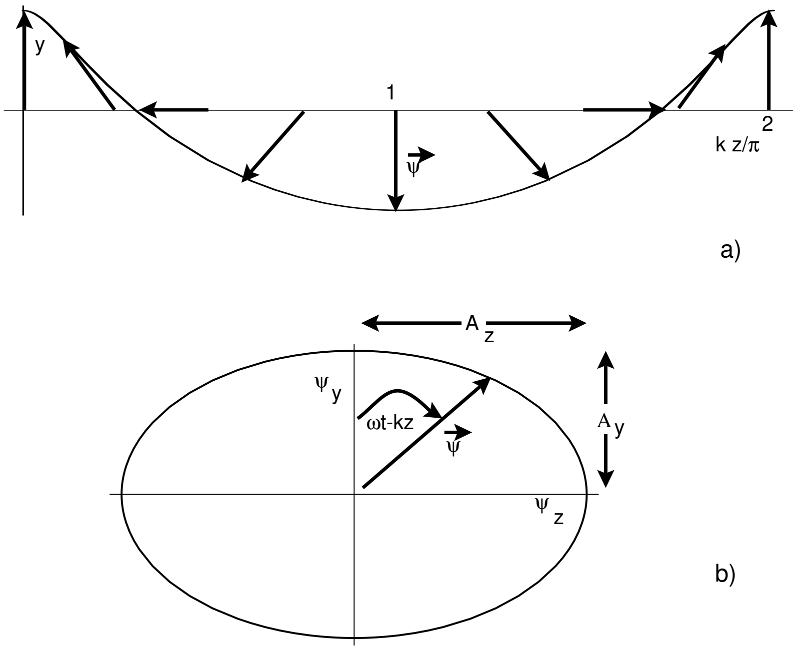


Figure 26: (a) Displacement pattern. (b) Path of a volume element of water.

similarly, the flow through the opposite end is

$$-u_x(x+\frac{dx}{2},y,z)dydz$$

Adding up the contributions through all six faces yields

$$0=\left[u_x(x-\frac{dx}{2},y,z)-u_x(x+\frac{dx}{2},y,z)\right]dydz$$

$$+\left[u_y(x,y-\frac{dy}{2},z)-u_y(x,y+\frac{dy}{2},z)\right]dxdz$$

$$+\left[u_z(x,y,z-\frac{dz}{2})-u_z(x,y,z+\frac{dz}{2})\right]dxdy$$

Using

$$u_x(x-\frac{dx}{2})=u_x(x)-\frac{\partial u_x}{\partial x}\frac{dx}{2}$$

and similar formulas for the other components, we find, if we put the flow out equal to the flow in

$$-\left[\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}\right]dxdydz=0$$

The condition of incompressible flow is thus

$$\nabla\cdot\mathbf{u}=\frac{\partial u_x}{\partial x}+\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=0 \tag{156}$$

In the special case of one dimensional flow in a shallow channel we find

$$0=\frac{\partial u_y}{\partial y}+\frac{\partial u_z}{\partial z}=-\omega\sin(\omega t-kz)\left(\frac{dA_y}{dy}-kA_z\right) \tag{157}$$

giving

$$\frac{dA_y}{dy}-kA_z=0 \tag{158}$$

We next neglect the effect of viscosity and assume that the flow is irrotational, i.e. that there are no eddies. If there are no eddies the circulation satisfies (see figure 27(b))

$$\oint_c \mathbf{u}\cdot\mathbf{ds}=0 \tag{159}$$

along any closed path.

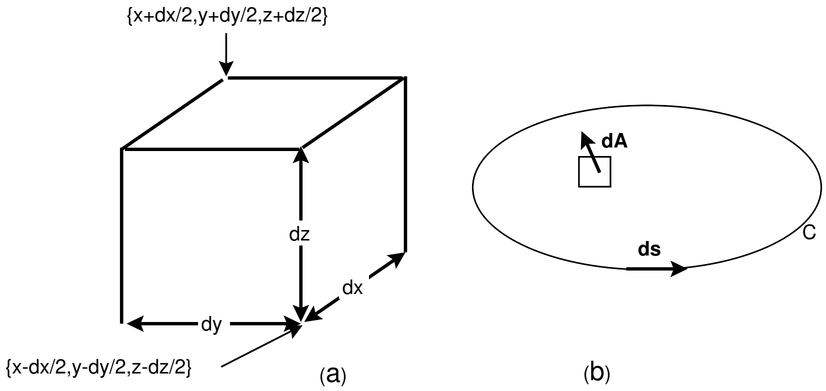


Figure 27: (a) Divergence theorem. (b) Stokes' theorem.

In order to produce eddies one must provide angular momentum to the system, something which requires a torque. If there is no viscosity there are no shear forces associated with the wind causing the waves. The assumption of irrotational flow is clearly an idealization; in practice it means that we are neglecting any mechanism which leads to damping of the waves. From Stokes' theorem (figure 27(b))

$$\oint_c \mathbf{u} \cdot \mathbf{ds} = \int_A (\nabla \times \mathbf{u}) \cdot \mathbf{dA} = 0 \tag{160}$$

In (160) $\mathbf{dA}$ is an element of area represented by a vector normal to the surface. If there is no circulation anywhere, (160) must hold independently of the contour c and

$$\nabla \times \mathbf{u} = 0 \tag{161}$$

everywhere. In our model problem,

flow in a shallow channel, everything is independent of the x-coordinate and (161) yields

$$0 = \frac{\partial u_y}{\partial z} - \frac{\partial u_z}{\partial y} = \omega \cos(\omega t - kz) \left[kA_y - \frac{dA_z}{dy} \right]$$

or

$$\frac{dA_z}{dy} - kA_y = 0 \tag{162}$$

We differentiate (158) once more with respect to y and substitute (162) to obtain

$$\frac{d^2A_y}{dy^2} - k^2A_y = 0 \tag{163}$$

The general solution to (163) is

$$A_y = ae^{ky} + be^{-ky} \tag{164}$$

where a and b are arbitrary constants. If we assume that the bottom of the channel is firm we must have $A_y(-h) = 0$. We also assume that the amplitude of the wave near the surface is $A_y(0) = A$. Substitution of the boundary conditions into (164) gives two equations with two unknowns a and b. Solving these equations we find after some algebra

$$\begin{aligned} A_y(y) &= A\frac{e^{k(h+y)} - e^{-k(h+y)}}{e^{kh} - e^{-kh}} = A\frac{\sinh[k(h+y)]}{\sinh(kh)} \\ A_z(y) &= A\frac{e^{k(h+y)} + e^{-k(h+y)}}{e^{kh} - e^{-kh}} = A\frac{\cosh[k(h+y)]}{\sinh(kh)} \end{aligned} \tag{165}$$

From (165) we see that the elliptical trajectories (154) always have the major axis parallel to the horizontal and that the ratio of the minor to the major axis is

$$\tanh[k(h+y)] < 1$$

Since A_y and A_z in (165) have the same sign, we note from (153) that when a wave is propagating in the positive z direction ψ_z lags 90^0 behind ψ_y and the motion along the elliptic orbit is *clockwise*.

If $kh >> 1$ we speak of *deep water waves* and the equations for the wave amplitudes simplify to

$$\begin{aligned} \psi_y &= Ae^{ky}\cos(\omega t - kz) \\ \psi_z &= Ae^{ky}\sin(\omega t - kz) \end{aligned} \tag{166}$$

In this case the particles in the wave will undergo a circular motion.

Another property of the wave which we would like to determine is the speed at which it travels. The ratio

$$u_\phi = \frac{\omega}{k}$$

is called the *phase velocity* of the wave. As we shall see, the phase velocity will depend on the wavelength. Such waves are called *dispersive*. In order to determine the phase velocity we must look at the energetics of the situation, suggesting the application of the Bernoulli equation. However, this equation is only valid for *steady flow*. On the other hand, the wave motion (153) is manifestly time dependent. This problem can be overcome by making a coordinate transformation to a frame of reference which is moving with velocity ω/k in the z-direction. In this frame of reference

$$z' = z - \frac{\omega t}{k} \tag{167}$$

and we find for the components of the velocity in the moving frame

$$\begin{aligned} u_y(y,z') &= \omega A_y(y)\sin(kz') \\ u_{z'}(y,z') &= \omega A_z(y)\cos(kz') - \tfrac{\omega}{k} \end{aligned} \tag{168}$$

In the moving frame the flow is thus steady and we can apply the Bernoulli equation. Let us consider a streamline at the water surface. The height of the surface can be found by substituting $y = 0$ into the first equation (165)

$$\psi_y(0) = A\cos(kz') \tag{169}$$

while we have from the second equation (165)

$$A_z(0) = A\coth(kh) \tag{170}$$

We find from (168)

$$u^2 = {u_y}^2 + {u_{z'}}^2 =$$

$$= \omega^2\left(A^2[sin^2(kz') + \coth^2(kh)\cos^2(kz')] - \frac{2A}{k}\coth(kh)\cos(kz') + \frac{1}{k^2}\right) \tag{171}$$

We now assume that the amplitudes $A_z(0)$, $A_y(0)$ are *small*, or more precisely small compared to $1/k$ (which is proportional to the wavelength $\lambda = 2\pi/k$). This allows us to neglect the terms proportional to A^2 in (171) and we have

$$u^2 \simeq \frac{\omega^2}{k^2} - \frac{2A}{k}\coth(kh)\cos(kz') \tag{172}$$

We must next consider the effect of *surface tension*. The surface tension comes about because there is an *energy cost* associated with the creation of a free surface

$$E_{surface} = \sigma \times area \tag{173}$$

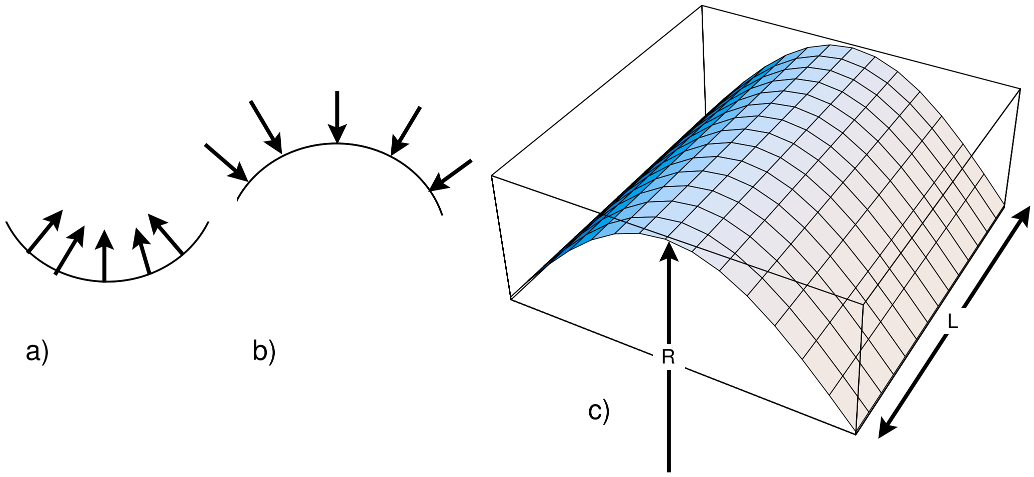


Figure 28: (a) Surface tension decreases pressure. (b) Surface tension increases pressure.(c) Radius of curvature.

Equation (173) defines the surface tension σ. The order of magnitude of σ can be obtained by considering the binding energy -ϵ per molecule in the fluid. Crudely speaking, the molecules in the surface layer only feel $\frac{1}{2}$ of their neighbors. This gives

$$\sigma \simeq \frac{1}{2}\epsilon n d \tag{174}$$

where n is the number of molecules per unit volume and d is the molecular diameter. Experimentally, for water at room temperature , $\sigma = 0.073$ Nm^{-1}.

Consider a part of a water wave which is curved upwards as in figure 12.1(a) The surface tension attempts to contract the surface and this gives rise to a negative contribution to the pressure just below the surface. If, on the other hand, the surface is curving downwards as in figure 12.1(b) the stress contracting the surface causes a positive contribution to the pressure.

To find the size of the contribution let us consider a half cylinder of radius R, of length L (figure 12.1(c). The volume of the half cylinder is $\frac{1}{2}\pi R^2 L$ while the area is $\pi R L$. If the radius is changed by an amount dR the surface energy changes by

$$dE = \pi L \sigma dR$$

The work associated with this is

$$dE = P_s dV = P_s d(\frac{\pi}{2}R^2 L) = P_s \pi R L dR$$

By comparison we find that the surface tension contributes a term

$$P_s = \frac{\sigma}{R} \tag{175}$$

We can write for the radius of curvature of a small amplitude wave

$$\frac{1}{R} = \frac{\partial^2 \psi_y}{\partial z^2}$$

allowing R to be positive if the surface curves upwards, negative if it curves downwards. If $\frac{\partial^2 \psi_y}{\partial z^2} > 0$ the surface curves

upwards, otherwise it curves downwards. The pressure which enters the Bernoulli equation is thus

$$P = P_{atm} - \sigma \frac{\partial^2 \psi_y}{\partial z^2} = P_{atm} + \sigma k^2 A \cos kz' \tag{176}$$

where P_{atm} is the atmospheric pressure. Adding terms we find that the Bernoulli equation for a streamline near the surface is

$$const = \frac{P}{\rho} + \frac{u^2}{2} + gA\cos(kz')$$

$$= \frac{P_{atm}}{\rho} + \frac{\sigma k^2}{\rho} A \cos kz' + \frac{\omega^2}{2k^2} - \frac{A\omega^2}{k} \coth(kh)\cos(kz') + gA\cos(kz') \tag{177}$$

This equation must hold for all values of z'. For this to be possible the terms proportional to $\cos(kz')$ must add to zero or

$$\omega^2 \coth(kh) = \frac{\sigma k^3}{\rho} + gk$$

from which we obtain the *dispersion relation*

$$\omega = \sqrt{(gk + \frac{\sigma k^3}{\rho})\tanh(kh)} \tag{178}$$

and we have for the *phase velocity*

$$u_\phi = \frac{\omega}{k} = \sqrt{(\frac{g}{k} + \frac{\sigma k}{\rho})\tanh(kh)} \tag{179}$$

The phase velocity is not the only velocity of interest, we must also consider the *group velocity*

$$u_g = \frac{d\omega}{dk} = \frac{1}{2}\left[g + \frac{3\sigma k^2}{\rho} + \left(g + \frac{\sigma k^2}{\rho}\right)\frac{kh}{\sinh(kh)\cosh(kh)}\right]\sqrt{\frac{\tanh(kh)}{gk + \frac{\sigma k^3}{\rho}}} \tag{180}$$

In most situations the waves which propagate along the water surface do not have a single frequency but will be made up of a *group* or superposition of waves with slightly different frequencies. Let us consider a superposition of two waves of frequencies ω_1 and ω_2 and wave numbers k_1 and k_2. Using the trigonometric identity

$$\cos a + \cos b = 2\cos\left[\frac{1}{2}(a+b)\right]\cos\left[\frac{1}{2}(a-b)\right] \tag{181}$$

we have

$$\cos(\omega_1 t - k_1 z) + \cos(\omega_2 t - k_2 z) = 2\cos\left[\frac{1}{2}\{(\omega_1 + \omega_2)t - (k_1 + k_2)z\}\right]$$
$$\times \cos\left[\frac{1}{2}\{(\omega_1 - \omega_2)t - (k_1 - k_2)z\}\right]$$

The resulting wave pattern is one in which beats modulate the carrier frequency $\frac{1}{2}(\omega_1 + \omega_2)$. The velocity of propagation of the beats is

$$\frac{\omega_1 - \omega_2}{k_1 - k_2} \simeq \frac{d\omega}{dk} = u_g \tag{182}$$

if ω_1, ω_2 are close to ω and k_1, k_2 are close to k. If we have a situation in which a calm surface is disturbed, such as when a stone is thrown into a pond, a group of waves is produced. This group is made up of individual waves of a range of frequencies. The component waves propagate with the phase velocity while the disturbance itself propagates with the group velocity. As long as the frequency range and the amplitude is not too large the group velocity will be given by (180).

Let us now consider some special cases:
If $kh >> 1$, $\tanh(kh) \simeq 1$, and we have deep water waves. For deep water waves

$$\omega = \sqrt{gk + \frac{\sigma k^3}{\rho}} \tag{183}$$

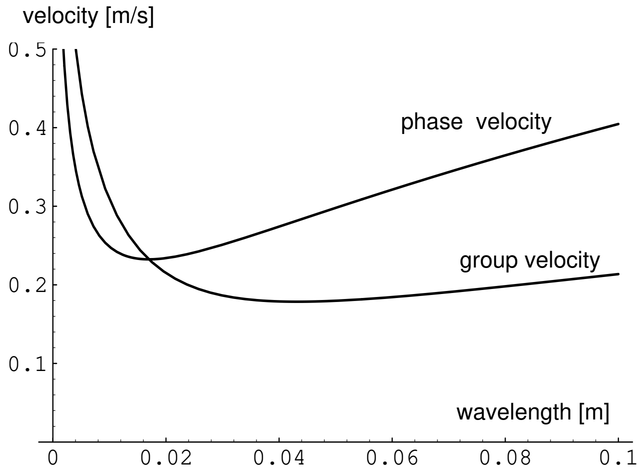


Figure 29: (a) Phase and group velocity for deep water waves.

There are two limiting cases, if

$$gk >> \frac{\sigma k^3}{\rho}$$

we have *gravity waves*. In the opposite limit

$$gk << \frac{\sigma k^3}{\rho}$$

we talk about ripples or *capillary waves*. Using $\sigma = 0.073$ N m^{-1}, $g = 9.81$ m s^{-2}, $\rho = 1000$ kg m^{-3} we find that

$$gk = \frac{\sigma k^3}{\rho}$$

when $\lambda = \frac{2\pi}{k} = 0.017$ m=1.7 cm. For gravity waves

$$\omega = \sqrt{gk} \tag{184}$$

The phase velocity of gravity waves is thus

$$u_\phi = \sqrt{\frac{g}{k}} \tag{185}$$

while for the group velocity

$$u_g = \frac{1}{2} u_\phi \tag{186}$$

For capillary waves

$$\omega = k^{3/2} \sqrt{\frac{\sigma}{\rho}} \tag{187}$$

The phase velocity is now

$$u_\phi = \sqrt{\frac{k\sigma}{\rho}} \tag{188}$$

and the group velocity

$$u_g = \frac{3}{2} u_\phi \tag{189}$$

¿From these equations we see that gravity waves move faster the longer the wavelength; also a group of waves will travel slower than the individual waves. This situation is reversed for ripples. In the latter case short wavelength waves move the fastest and the group will travel faster than the phase. In figure 29 we plot, for deep water waves, the behavior of the phase and group velocities in the transition region between gravity and capillary waves.

12.2 The hydraulic jump

In the limit that $kh << 1$ (shallow waves) we have

$$\tanh(kh) \simeq kh$$

and we see that for gravity waves in a shallow channel $\omega = k\sqrt{gh}$

$$u_g = u_\phi = \sqrt{gh} \tag{190}$$

We note that both the phase and group velocities will be slower in a shallow channel than in a deep channel. This has a consequence that waves tend to pile up as they reach a gently sloping beach and they eventually break (the surf is up!). Equation (190) implies that the speed of gravity waves in a shallow channel is independent of the wave vector i.e. is *dispersion-less*; the group and phase velocities are the same. This allows for the possibility of a solitary wave traveling along a river or canal without being dispersed. Striking example of this are tidal bores and tidal waves following earthquakes (tsunamis). In Canada, the most well known example of a tidal bore

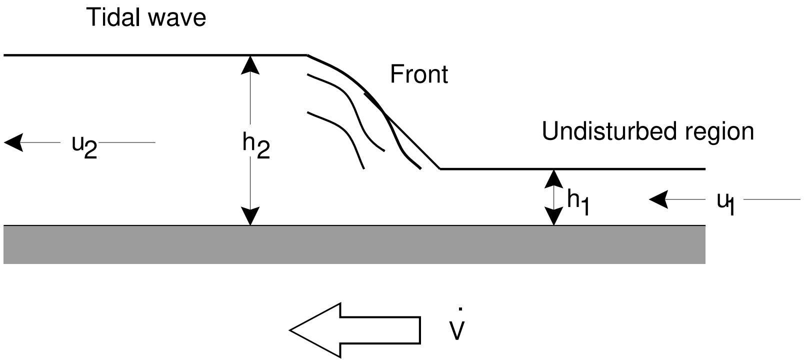


Figure 30: Hydraulic jump in moving frame

occurs on the Peticodiac river in New Brunswick, while there have been tsunamis causing considerable damage in the Alberni inlet on Vancouver Island. For a bore to occur one needs very high tides and a river with gently sloping bottom entering a funnel shaped estuary. A bore can be thought of as a hydraulic shock front moving upstream. Behind the front the water flows upstream for some time before the current reverses. Since the channel is deeper behind the bore the wave speed is faster behind the front than before it. This means that energy is fed into the front and prevents it from dissolving. The ratio of the speed of the front to the wave velocity in front of it is called the *Froude number*. As we shall see, it is a general condition for the formation of shocks that the Froude number is greater than one. If the Froude number is less than about 1.7 the bore will have an undular shape. For a larger Froude number the wave will break, and the larger the Froude number the more violent the front will be. Clearly tidal bores represent a complicated phenomenon which depends much on the details of the local conditions.

We will here consider the simple case of a hydraulic jump where an undisturbed shallow channel is invaded by an elevation wave or tidal wave. In order to have a manageable problem we neglect friction between the flowing water and the channel walls, and also assume a uniform velocity profile through the channel. Consider a frame of reference in which the front of the hydraulic jump is at rest (Figure 30). We will assume the situation is stationary. This assumption is not realistic, since after some time the height of the tidal wave will decrease, but for a sufficiently short time period we can take the amplitude to be constant. In the front frame, the undisturbed region is flowing towards it with velocity u_1, and we let u_2 be the flow velocity behind the front. Conservation of mass give for the volume flow per unit channel width

$$\dot{V} = h_1 u_1 = h_2 u_2 \tag{191}$$

¿From (124) we find for the net pressure force per unit width w of the channel

$$\frac{F}{w} = \int_0^h \rho g z dz = \frac{1}{2}\rho g h^2$$

The momentum balance equation can then be written as

$$F_1 - F_2 = (mass\ flow\ rate) \times (u_2 - u_1) \tag{192}$$

The mass flow rate per unit channel width is

$$j_M = \rho \dot{V}$$

We can rewrite (192)

$$\rho u_1^2 h_1 + \frac{1}{2}\rho g h_1^2 = \rho u_2^2 h_2 + \frac{1}{2}\rho g h_2^2 \tag{193}$$

Combining (193) and (191) one finds after a little algebra

$$0 = (h_1 - h_2)\left[h_1 + h_2 - \frac{2\dot{V}^2}{g h_1 h_2}\right] \tag{194}$$

Equation (194) has a trivial solution $h_1 = h_2$, and also a nontrivial solution in which a height difference $h_2 - h_1 \neq 0$ is maintained. Solving 194 for h_2 we find for the nontrivial solution

$$h_2 = -\frac{h_1}{2} + \sqrt{\frac{h_1^2}{4} + \frac{2\dot{V}^2}{g h_1}} \tag{195}$$

(when solving the quadratic equation for h_2 we discard one root which gives a negative value for h_2). The balance equations are perfectly symmetric with respect to h_1 and h_2; they permit both a positive and negative surge.

Only the positive surge $h_1 < h_2$ turns out to be stable. To see this consider the energy flow per unit width of the cannel

$$j_E = \int_0^h \left(\rho g z + \frac{1}{2}\rho u^2\right) u dz = \frac{j_M(gh + u^2)}{2} \tag{196}$$

We find from (193), (196) and (191)

$$(j_E)_1 - (j_E)_2 = g j_M (h_1^2 + h_2^2) \frac{h_2 - h_1}{4 h_1 h_2}$$

The turbulence and viscous shear generated at the jump makes the process *dissipative*; energy is not conserved. Rather we must have $(j_E)_1 - (j_E)_2 > 0$. As a consequence only the solution $h_2 > h_1$ is allowed. The unphysical solution $h_1 > h_2$ corresponds making a film of a tidal wave coming in and showing the movie backwards. Clearly something would appear to be "wrong".

12.3 Problems

Problem 12-1.

Breaking waves. The theory of section 12.1 is only valid for small amplitudes. If the height of the wave is increased it will eventually break. To see why this is bound to happen consider the amplitude equations (166) for deep waves. In these equations z and y represent the coordinates of a particle when the fluid is at rest. At the surface $y = 0$ the actual z coordinate of the particle will be

$$\zeta = z + A \sin(\omega t - kz)$$

where z is the position when the particle is at rest. The y-coordinate of the particle will be

$$\eta = A \cos(\omega t - kz)$$

If we plot the surface at some instant t (i.e. η versus ζ) we see that the wave will fold back on itself if $\frac{\partial \zeta}{\partial z}$ is negative (Figure 31). This happens if $Ak > 1$. Since the phase velocity is ω/k and the velocity amplitude is $A\omega$ a criterion for the amplitude for which a wave about to break is that the horizontal speed is equal to the phase velocity. Estimate the height of a deep water wave of wavelength 10 m which is just about to break.

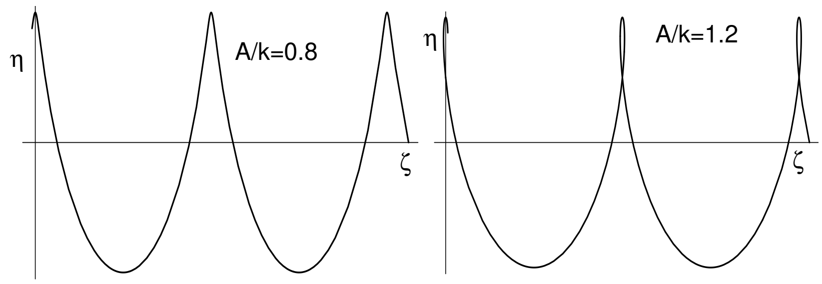


Figure 31:

Problem 12-2.

Shallow-draught canal boats pulled by horses are most easy to pull if they are made to travel on their own bow wave. Treat the bow wave as a shallow water wave. If the maximum speed at which the horses can pull is 4 m s^{-1}, what is the maximum canal depth at which this trick will work?

Problem 12-3.

A tidal wave of height $h_2 - h_1 = 5$ m is surging up a 5 m deep channel. How fast is it traveling? What is the Froude number of the front? What is the Froude number behind the tidal wave?

Problem 12-3.

A ship traveling with the wind overtakes waves with wavelength 20 m every 5 s. How fast is the ship going?

Problem 12-4.

A bay is open towards deep water at one end and closed at the other. The average depth is 50 m. The resonance frequency for long wavelength oscillations is 12.5 hrs. What is the effective length of the bay?

Problem 12-5.

A shallow lake is 100 km long. Sometimes one can observe an end to end oscillations of the surface called a *seiche* with period 5.5 hrs. What is the effective depth of the lake.

13 Compressible flow.

In the previous chapter we made the idealization that the fluid is incompressible, and we also neglected any possible change in the internal energy of the fluid causing changes in temperature along the flow. In this chapter we discuss *sound waves*, which are compression and dilation waves, and more extreme conditions such as shocks. To do this we must go beyond the Bernoulli equation which assumes incompressible flow and which doesn't take into account heat and work exchanged with the surroundings. For this purpose we must make contact with *thermodynamics.*

The first law of thermodynamics is a restatement of the principle of energy conservation which distinguishes between two forms of energy; heat and work.

$$\Delta Q = \Delta E + \Delta W \tag{197}$$

Here ΔQ is the *heat given to the system*, ΔE is the *change in the internal energy of the working substance* and ΔW is the *work done by the system.* Let us consider a mass element of fluid Δm which upstream has cross section A_1 and length l_1. The internal energy per unit mass is initially e_1. In order to get this mass element into our system we must do the work $P_1 A_1 l_1$. Since this is work done on the mass element the contribution to ΔW in (197) is negative. Eventually, after having done the work ΔW and having received the heat ΔQ, the mass elements escape downstream, performing the final work $P_2 A_2 l_2$. The height upstream is z_1, z_2 downstream. Conservation of energy then gives

$$\Delta Q - \Delta W = \Delta m[(e_2 - e_1) + \frac{1}{2}({u_2}^2 - {u_1}^2) + g(z_2 - z_1)] + P_2 A_2 l_2 - P_1 A_1 l_1$$

Dividing by Δm we get the balance equation

$$\frac{\Delta Q}{\Delta m} - \frac{\Delta W}{\Delta m} = e_2 - e_1 + \frac{1}{2}({u_2}^2 - {u_1}^2) + g(z_2 - z_1) + \frac{P_2}{\rho_2} - \frac{P_1}{\rho_1} \tag{198}$$

An important special case is a *free adiabatic* processes for which ΔQ and ΔW are zero. In the case of steady flow we have along a streamline, from (198)

$$\frac{P}{\rho} + \frac{u^2}{2} + gz + e = constant \tag{199}$$

It is convenient at this stage to introduce the *enthalpy*

$$H = E + PV$$

The enthalpy per unit mass is

$$h = e + \frac{P}{\rho}$$

The balance equation (199) is then

$$h + \frac{u^2}{2} + gz = constant \tag{200}$$

Note the similarity between (200) and the Bernoulli equation. The difference is that we now take into account the effect of conversion of kinetic energy into heat . We must then replace the pressure term in the Bernoulli equation by an enthalpy term. Consider an ideal gas working substance. Let C_P be the specific heat per mole at constant pressure and C_V the specific heat per mole at constant volume. You will have learned from an elementary thermodynamics course that

$$C_P = C_V + R$$

where R is the gas constant. The internal energy per mole is

$$E = C_V T$$

and the enthalpy per mole is

$$H = C_P T$$

In the case of air, the gravity term in (200) can usually be neglected and we find

$$c_P T + \frac{1}{2}u^2 \simeq constant \tag{201}$$

where c_p is the specific heat per unit mass.

13.1 Stagnation temperature

Consider a projectile or an airplane traveling through air at constant speed u. The temperature far away is T. Near the tip of the nose cone the air will be moving with approximately the same speed as the projectile while in a frame of reference moving with the projectile the speed far away is u (figure 32). Equation (201) then predicts that the *stagnation temperature* T_s just in front of the cone will be given by

$$c_p T + \frac{1}{2}u^2 = c_p T_s$$

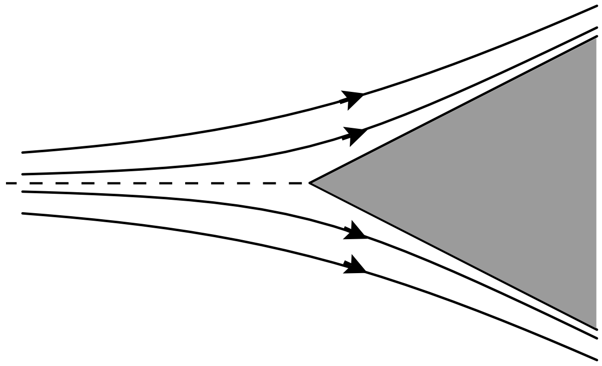


Figure 32:

or

$$T_s - T = \frac{u^2}{2c_p}$$

For air $c_p \simeq 1.0$ kJ kg^{-1} K^{-1}). We see that for an airplane traveling at 200 m s^{-1} the temperature rise will be about 20K. A bullet from a high speed rifle may travel at the speed of 1 km s^{-1}. The stagnation temperature will then be quite high, about 500 above the ambient temperature. Of course, we are dealing with a very idealized situation. The assumption of adiabatic flow will not always be that accurate - there may be a considerable amount of heat transport away from the cone.

13.2 Sound waves

Sound waves are small amplitude elastic waves that can propagate in a fluid or solid. In the case of fluids long wavelength sound waves are *longitudinal*, and the waves consist of an alternating pattern of rarefactions and compressions. For ordinary sound the changes in pressure tend to be very small. The *intensity* of a sound wave is often measured in decibels

$$I_{dB} = 20 \log_{10}(\frac{P_e}{P_{ref}})$$

The reference pressure is

$$P_{ref} = 2 \times 10^{-10} bar$$

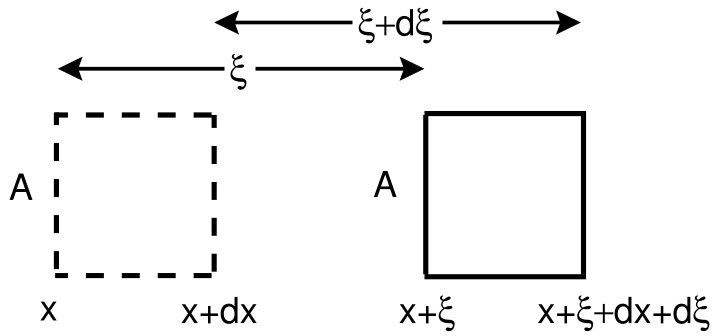


Figure 33:

and P_e is the root mean square pressure amplitude ($1/\sqrt{2}$ of the peak excess pressure). At the *pain level* of 120 dB the peak excess pressure will be $2\sqrt{2}\times 10^{-4}$ bar , which is small compared to the ambient pressure $\simeq$ 1 bar. The mechanism for a sound wave is that gas motion generates a change in the density, which causes a change in pressure. There will then be an unbalanced force which accelerates the gas and causes the cycle to repeat. We write for the pressure and density

$$\begin{aligned} P &= P_a + P_e \\ \rho &= \rho_a + \rho_e \end{aligned}$$

where the subscript a stands for average while e stands for excess.

The relationship between changes in density and pressure depends on the properties of the medium in which the sound waves propagate. We will assume that the processes are fast enough that they can be considered to be *adiabatic*, i.e. without any heat transport. Since the amplitudes P_e and ρ_e are small compared to the ambient conditions we assume that they are proportional to each other

$$P_e = \kappa\rho_e \tag{202}$$

with

$$\kappa = \left.\frac{\partial P}{\partial\rho}\right|_{adiabatic} \tag{203}$$

Let us consider a column of air of cross-section A (Figure 33).

When the air is at rest in equilibrium this column extends from x to $x+dx$. We assume that a sound wave is traveling in the x-direction and that at some instant the left end of the column is displaced an amount $\xi(x,t)$ which is small compared to the wavelength of the sound wave. Conservation of mass gives

$$A\rho_a dx = A(\rho_a + \rho_e)(d\xi + dx)$$

or

$$\rho_e dx = -(\rho_a + \rho_e)d\xi \simeq -\rho_a d\xi$$

if $\rho_e << \rho_a$, giving

$$\rho_e \simeq -\rho_a \frac{\partial \xi}{\partial x} \tag{204}$$

The column is subject to a net force

$$A[P(x+\xi,t) - P(x+\xi+dx+d\xi,t)] \approx -A\frac{\partial P_e(x,t)}{\partial x}dx$$

causing an acceleration

$$Adx\rho_a \frac{\partial^2 \xi}{\partial t^2} = -A\kappa dx \frac{\partial \rho_e}{\partial x} = A\kappa dx \frac{\partial^2 \xi}{\partial x^2}\rho_a \tag{205}$$

using (202) and (204). From (205) we get the *wave equation*

$$\frac{\partial^2 \xi}{\partial t^2} = \kappa \frac{\partial^2 \xi}{\partial x^2} \tag{206}$$

It is easy to see that (206) admits solutions in the form of traveling waves of the form $f(x \pm ct)$, where $c = \sqrt{\kappa}$, and it can be shown that the general solution to (206) can be written on the form

$$\xi(x,t) = f_1(x-ct) + f_2(x+ct)$$

We next wish to calculate the sound velocity assuming the waves travel in an ideal gas and that the sound vibrations are adiabatic. For an adiabatic process

$$PV^\gamma = constant$$

with $\gamma = C_P/C_V$ or

$$P = Const.\rho^\gamma$$

Differentiating we find

$$\left.\frac{\partial P}{\partial \rho}\right|_{adiabatic} = \frac{\gamma P}{\rho} = \frac{\gamma k_B T}{m} = \frac{\gamma RT}{\mu}$$

where m is the mass of a molecule and μ the molecular weight. Using (203) we finally get

$$c = \sqrt{\frac{\gamma k_B T}{m}} = \sqrt{\frac{\gamma R T}{\mu}} \tag{207}$$

It is interesting to compare the speed of sound (207) with typical molecular speeds. ¿From (35) we have for the rms speed

$$v_{rms} = \sqrt{\frac{3k_B T}{m}} = \sqrt{\frac{3}{\gamma}}c$$

Since $\gamma \simeq 1.4$ for air we see that the rms speed and the sound speed are quite comparable.

13.3 Shock waves

As we discussed earlier, sound waves are typically encountered as small amplitude waves. We have also met phenomena such as tidal waves where the amplitudes are large. As we shall see, a condition for maintaining a sharp shock front is that the wave velocity of the front is faster than the wave velocity ahead of it. In the case of tidal waves this came about because the wave velocity in a shallow channel increases with height and the water level is higher behind the bore. We will here consider a shock wave in air, caused by e.g. an explosion. The released heat causes the temperature to rise with the result that the sound velocity is larger behind the shock than in front of it.

We will assume that the conditions around the shock change slowly in time (however, as the volume traversed by the shock is increased the excess temperature and pressure will diminish). This slow time evolution takes place in the rest frame of the shock front. If you are in the laboratory frame and get hit by an oncoming shock wave things are not changing slowly at all. We will not here attempt to discuss the time evolution of a shock wave, instead we will only try to develop an equation of state which relates the conditions behind the shock to those ahead of it. We will wish to answer the question: *for a given shock speed what is the excess temperature and pressure behind the shock, given the conditions in the undisturbed region in front of the shock.* As we shall see this question can be answered from the conservation laws of the problem. We proceed in a manner which is quite similar to what we did in the case of the hydraulic jump, with one important difference. When discussing the hydraulic jump we could ignore the effect

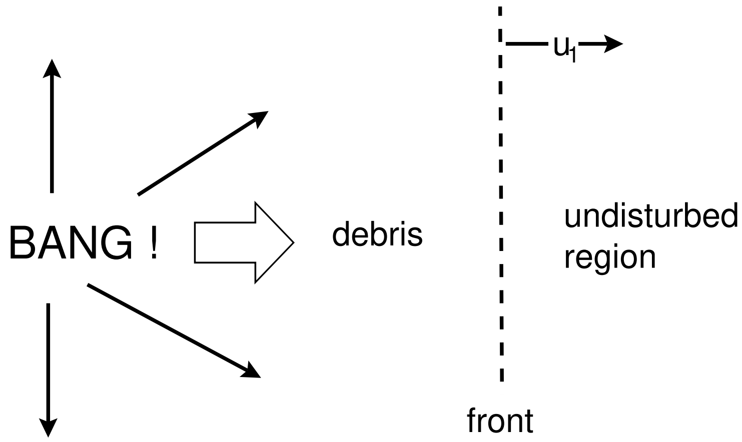


Figure 34:

of the heating of the water in the disturbed region behind the front. In the present case the effect of the dissipated heat is quite important and we need to consider the enthalpy balance in addition to the conservation laws for mass and momentum.

We use subscript 1 to describe the condition ahead of the shock while subscript 2 describes conditions behind the front. The speed u_1 of the air ahead of the shock is then equal to the shock front speed in the laboratory frame.

Conservation of mass gives rise to the condition

$$\rho_1 u_1 = \rho_2 u_2 \tag{208}$$

where ρ is the density, and u the speed. Conservation of momentum tells us (see also the discussion in connection with (192))that with P the pressure

$$(P_1 - P_2) \times area = mass\ flow\ rate \times (u_2 - u_1) = area \times \rho_2 u_2 \times (u_2 - u_1) \tag{209}$$

Dividing by the area and using (208) we find

$$P_1 - P_2 = \rho_2 {u_2}^2 - \rho_1 {u_1}^2 \tag{210}$$

Neglecting the effect of gravity we get from the energy balance (200)

$$h_1 + \frac{1}{2}{u_1}^2 = h_2 + \frac{1}{2}{u_2}^2 \tag{211}$$

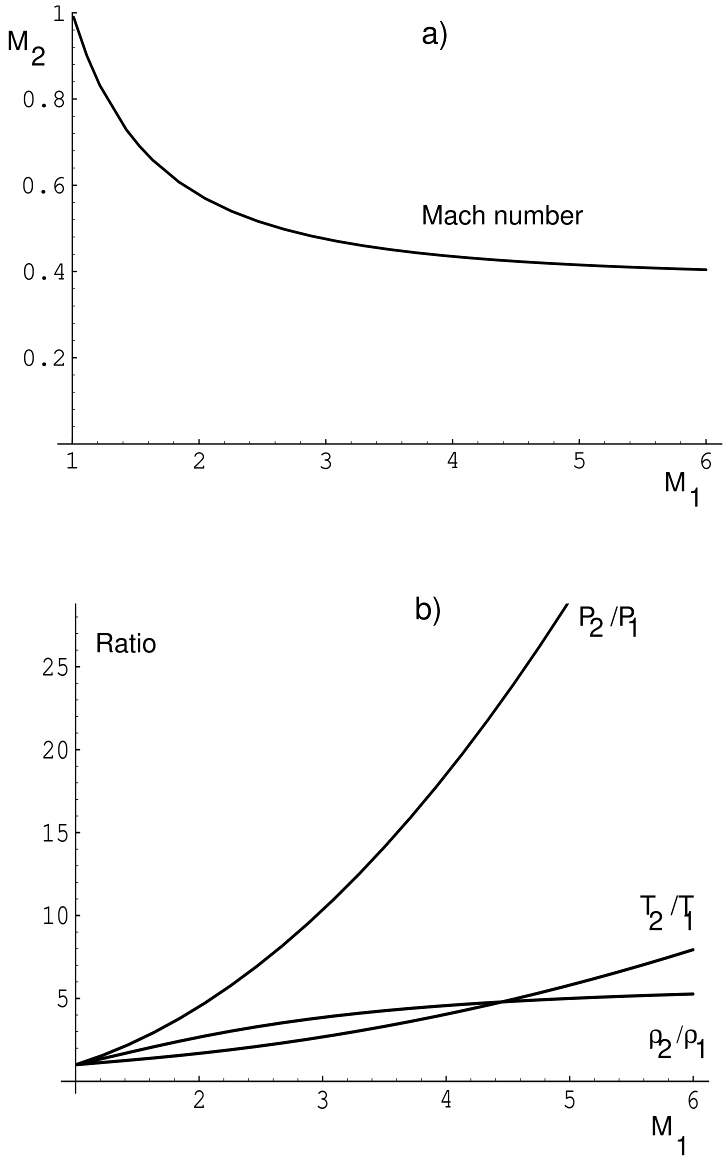


Figure 35: Ratios between physical quantities on the two sides of the shock front

We define the *Mach numbers*

$$M_1 = \frac{u_1}{c_1};\ M_2 = \frac{u_2}{c_2} \tag{212}$$

noting that the sound velocities c_1, c_2 may be different on the two sides of the front. We further assume that the working substance is an ideal gas for which the enthalpy per unit mass is $h = c_P T$. The sound velocity is given by

$$c = \sqrt{\gamma RT/\mu} = \sqrt{\gamma P/\rho} \tag{213}$$

Equation (211) can be rewritten in terms of the Mach numbers

$$c_P T_1 + {M_1}^2 \frac{\gamma R T_1}{2\mu} = c_P T_2 + {M_2}^2 \frac{\gamma R T_2}{2\mu}$$

or

$$\frac{T_2}{T_1} = \frac{c_P + \frac{{M_1}^2 \gamma R}{2\mu}}{c_P + \frac{{M_2}^2 \gamma R}{2\mu}}$$

Using $c_P = \gamma c_V$ and $R/\mu = c_P - c_V = \frac{\gamma-1}{\gamma} c_P$, we find

$$\frac{T_2}{T_1} = \frac{1 + \frac{1}{2}(\gamma - 1){M_1}^2}{1 + \frac{1}{2}(\gamma - 1){M_2}^2} \tag{214}$$

¿From the ideal gas law we also have

$$\frac{T_2}{T_1} = \frac{P_2 \rho_1}{P_1 \rho_2}$$

¿From (208) and (212) we find

$$\frac{T_2}{T_1} = \frac{P_2 u_2}{P_1 u_1} = \frac{P_2 M_2}{P_1 M_1} \sqrt{\frac{\gamma R T_2}{\gamma R T_1}}$$

giving

$$\frac{T_2}{T_1} = \left(\frac{P_2 M_2}{P_1 M_1} \right)^2 \tag{215}$$

¿From (210)

$$P_1 - P_2 = \rho_2 {M_2}^2 c_2^2 - \rho_1 M_1^2 c_1^2 \tag{216}$$

We use the expression (213) for the sound velocity and (216) becomes

$$\frac{P_2}{P_1} = \frac{1+\gamma {M_1}^2}{1+\gamma {M_2}^2} \tag{217}$$

Combining (214), (215) and (217), we finally get

$$\frac{1+\frac{1}{2}(\gamma-1){M_1}^2}{1+\frac{1}{2}(\gamma-1){M_2}^2} = \frac{(1+\gamma {M_1}^2)^2}{(1+\gamma {M_2}^2)^2}\frac{{M_2}^2}{{M_1}^2} \tag{218}$$

Equation (218) relates the Mach numbers on the two sides of the shock. It has the trivial solution $M_1 = M_2$ which corresponds to uniform flow without any shock. Numerical analysis also shows that there are also other solutions. If we hold one of the Mach numbers fixed, say M_2, there will always be one solution $M_1 < 1$ if $M_2 > 1$ and one with $M_1 > 1$ if $M_2 < 1$. We argued at the beginning of this section that for a front to be self sustained it must move faster than the sound velocity in the medium in front of it, i.e. only the solution $M_1 > 1$ is physically acceptable. This situation is completely analogous to what we had in the case of the hydraulic jump, where we only accepted the solution where the tidal wave moved faster than the wave velocity in the undisturbed channel. For each solution M_2 and with a knowledge of the temperature T_1, density ρ_1 and pressure P_1 in the undisturbed region in front of the shock we can use (215), (217) and the ideal gas law to calculate T_2, ρ_2, P_2. If one of these quantities, e.g. T_2 or P_2, is known from the properties of the explosion that caused the shock all the other quantities can then, in principle, be calculated.

The argument that only solutions with $M_1 > 1$ are physically acceptable can be put on a more formal basis by considering the second law of thermodynamics. Since the process is adiabatic the entropy of the system must increase as the shock passes by; we must have for the entropy per unit mass $s_2 > s_1$. It can be shown that this is only possible if $M_1 > 1$. A plot of the numerical properties of the shock parameters is given in figure 35.

13.4 Problems

Problem 13-1.

(a.) The bulk modulus of a material can be defined as

$$B = \rho\frac{\partial P}{\partial \rho}$$

Given that $B = 2.1 \times 10^9$ Pa, $\rho = 10^3$ kg m^{-3}. Find the sound velocity in water.

(b) Consider a liquid (like beer) in which gas bubbles are uniformly distributed throughout the fluid. Let the (adiabatic) bulk modulus of the pure fluid be B_l and the bulk modulus of the gas be B_g. The density of the pure fluid is ρ_l and the density of the gas is ρ_g. Find the sound velocity of the brew assuming it contains a volume fraction a of gas.

(c) Estimate the sound velocity of a mixture of water and 1 volume % air.

14 Physical similarity and modeling.

We will conclude our discussion of fluid mechanics by discussing dimensional analysis in more detail. The underlying idea is straight forward although the simple-mindedness of the arguments sometimes makes the practitioner somewhat uneasy. Each particular flow pattern arises because of the interplay of a number of forces and pseudo-forces due to inertia, pressure differences, viscosity, external forces such as gravity, surface tension, elasticity and stresses due to thermal expansion and buoyancy forces. Two systems are *physically similar* if the ratio between the relevant forces is the same in both systems. It is then possible to study a given system by building a scale model of the system in question. One can measure the properties of the model, and if it satisfies conditions of physical similarity one can rescale the measured properties and apply the results to the actual system of interest. Even if one is not actually building a model system, dimensional analysis often represents a convenient first line of attack on a complicated problem.

Let us assume that

$\alpha =$ thermal expansion coefficient
$g =$ acceleration of gravity
$\kappa =$ thermal diffusivity
$L =$ length parameter
$u =$ typical velocity
$\eta =$ viscosity
$\rho =$ mass density
$\sigma =$ thermal conductivity

Inertial pseudo-forces can be described as mass times acceleration. Often the acceleration is centripetal in origin and of the order u^2/L, while the mass is ρL^3. The resulting forces then scale as

$$Inertial\ force \sim \rho u^2 L^2 \tag{219}$$

Viscous stress (force per area) is given by the viscosity times the velocity gradient, which is typically proportional to u/L. The resulting force thus scales as

$$Viscous\ force \sim Lu\eta \tag{220}$$

The ratio between the inertial and viscous forces is the

$$Reynolds\ number = \frac{\rho u L}{\eta} \tag{221}$$

14.1 Flow past a submerged object:

Let L be a typical dimension of an object moving in a fluid with speed u. We assume that the flow around the object can be taken to be incompressible and assume a drag force law

$$F = Lu\eta f(Re) \tag{222}$$

where f is a function of the geometry of the object. In the case of a spherical object at low Reynolds number, the drag force is given by *Stokes' law*

$$F = 3\pi du\eta \tag{223}$$

where the diameter plays the role of length parameter. Therefore, for a spherical object we have $f(Re) = 3\pi$, in the limit $Re \to 0$. Note that we could equally well have used the inertial force expression (219) in constructing the drag law

$$F = \rho u^2 L^2 h(Re) \tag{224}$$

By comparing (222) and (224) we find that

$$f(Re) = Re\ h(Re)$$

Suppose we wish to estimate the drag force on a given submerged object of arbitrary shape by building a scale model to scale, say, 1:5. For tests with the model to represent a situation which is physically similar, the Reynolds number must be the same. If the fluid is the same we see from (221) that the flow velocity must be increased by a factor of 5. The force on the model and the real object would, from (222) then be the same.

In the case of a sphere the drag force is given by Stokes' law for very low Reynolds number $Re \leq 2$. In the range $20 < Re < 2000$ the drag coefficient $h(Re)$ is approximately proportional to the inverse square root of Re. For Reynolds numbers up to about 10^5 $h(Re)$ stays about constant. If the Reynolds number is increased coefficient there is a sudden drop in the drag coefficient and the behavior then depends not only on the Reynolds number but on the roughness of the sphere's surface [2]. This behavior is analogous to the dependence of the friction factor f on the roughness in the case of flow in pipes. The drag coefficient in the case of flow past a long cylinder behaves in a similar way to the drag coefficient for a sphere [1] [47].

If the velocity of flow approaches that of sound, the flow can no longer be taken to be incompressible, and physical similarity requires that both the Mach and the Reynolds number be the same. If the model is to be tested in the same medium as the object it simulates, the two must be of the same size. However, if the model is tested in a *pressurized* wind tunnel one can use a smaller model with the density of air $\times$ size constant. The sound velocity will be approximately independent of the density.

14.2 Convection

Buoyancy forces are important in convection problems. If a fluid is heated from below, thermal expansion may cause the density of the fluid to increase with height. The fluid is then unstable and convection will set in.

14.3 Convection in the atmosphere

Under most conditions the temperature of the atmosphere decreases with altitude. If this leads to a situation in which denser air lies atop of less dense air the atmosphere will become unstable against convection. The density is determined by the local temperature and pressure while the pressure will decrease with increasing altitude.

Let us begin by making a crude estimate of the critical temperature *lapse rate*

$$l = -\frac{\partial T}{\partial z} \tag{225}$$

at the on set of convection. Here z is the altitude. The mass density $\rho = MN/V$, where M is the mass of a molecule, given by

$$\rho = \frac{MP}{k_B T} \tag{226}$$

We have

$$\frac{dP}{dz} = -g\rho \tag{227}$$

Differentiating (226) and substituting (227) gives

$$\frac{d\rho}{dz} = \frac{M}{k_B T}\frac{dP}{dz} - \frac{MP}{k_B T^2}\frac{dT}{dz} = \frac{\rho}{T}\left(l - \frac{Mg}{k_B}\right)$$

With $M = 29$ a.m.u., 1 a.m.u. $1.66{\times}10^{-27}$kg, $k_B = 1.381 \times 10^{-23}$ J K^{-1}, we find that the critical lapse rate is 0.034 K m^{-1} or approximately 3 degrees per 100 m. This is an overestimate of the critical lapse rate.

To get a more realistic estimate consider an element of the gas which contains a mass dM. At equilibrium this mass will occupy a volume dV. Suppose that by a chance fluctuation the element is displaced by a small amount upwards. The element will then expand approximately adiabatically to the equilibrium volume at the pressure of the new location. If our volume element is heavier than the gas that it is replacing it will sink back to where it came from and the atmosphere is stable, if it is lighter it will rise. It follows that we have *marginal stability* for the *adiabatic lapse rate* given by (see Chapter 13) $c_p T + gz =$ const, where c_p is the specific heat at constant pressure per unit mass (approx. 3.5 k_B/M for air). The critical lapse rate is then 3.5 times smaller than calculated above or about 1 K per 100 m.

It is interesting to speculate why the observed lapse rate in the atmosphere is close to the adiabatic rate. If the atmosphere had been at equilibrium it would be isothermal. It is kept out of equilibrium by the incoming radiation from the sun. If the oceans and the ground are more efficient in absorbing sunlight than the atmosphere, a temperature gradient will be established. In the steady state the magnitude of the temperature gradient is determined by the ability of the atmosphere to transport away the excess heat. Heat conduction (Chapter 7) is an inefficient way of transporting heat and if this were the only mechanism the adiabatic lapse rate would be exceeded. On the other hand convection is a very efficient mechanism and once in place the rapid mixing may reduce the lapse rate below its critical value. This dichotomy gives rise to a self-organized pattern of changing weather near marginal stability. In order to understand the convection pattern in the atmosphere one must add in the effects of evaporation and condensation of water and also take into account Coriolis forces due to the earth's rotation.

14.4 Rayleigh Bénard convection

Consider next a liquid heated from below. Let us assume that the liquid is nearly incompressible, i.e. the only density variations are due to thermal expansion. The buoyancy force will be $\Delta\rho g L^3$ where

$$\Delta\rho = \alpha\rho(\Delta T)$$

and α is the bulk thermal expansion coefficient. We thus have

$$Buoyancy\ forces \simeq \alpha(\Delta T) g L^3 \rho$$

The viscous force is again given by (220), but for this formula to be useful we must estimate the convective velocity. Without convection the time dependence of the temperature will be governed by the diffusion equation

$$\frac{\partial T}{\partial t} = D\nabla^2 T \tag{228}$$

where D is the *thermal diffusivity.*

With convection we must replace the partial derivative with respect to time by the convective derivative

$$\frac{\partial T}{\partial t} + \mathbf{u}\cdot\nabla T = D\nabla^2 T \tag{229}$$

For steady state, $\frac{\partial T}{\partial t} = 0$ and if we assume that$\nabla T \sim (\Delta T)/L$, $\nabla^2 T \sim (\Delta T)/L^2$ we find

$$u \sim \frac{D}{L}$$

and we find that the viscous force is proportional to ηD. The *Rayleigh number* is defined as

$$Ra = \frac{F_{buoyant}}{F_{viscous}} = \frac{\alpha(\Delta T)\rho g L^3}{\eta D} \tag{230}$$

Without convection the heat flow is given by

$$\dot{Q} = \frac{AD(\Delta T)}{L} \tag{231}$$

The ratio between the actual heat flow and the heat flow without convection is called the *Nusselt number*, Nu. In the case of *Rayleigh-Bénard convection* due to heating from below we have physical similarity if the Rayleigh numbers of two different situations are the same or $Nu = Nu(Ra)$.

Convection is the driving mechanism for the great *ocean currents* which together with atmospheric convection are crucial in determining the earth's climate. In order to study convection in the oceans one consider density changes due to variations in salinity. Also, because of the great depth of the oceans one needs to worry about the compressibility of water. Finally the fact that the thermal expansion coefficient of water is not monotonic, leading to a maximum of the density of water at 4 ^{o}C must be taken into account. However, modeling of ocean currents represent a very important problem on which a great deal of effort is spent. The main reason for this interest is worry about the effects of global warming.

14.5 Wave resistance of a ship

We have seen in chapter 12 that gravity forces are often the dominant ones in the case of water waves. We have typically

$$Gravitational\ forces \sim \rho g L^3 \tag{232}$$

The square root of the ratio between the inertial and gravitational forces is defined as the *Froude number*

$$Fr = \sqrt{\frac{u^2}{Lg}} \tag{233}$$

This definition is compatible with the definition of the Froude number in section 12.2 where we discussed the hydraulic jump. If we let the characteristic length L be the depth of undisturbed channel, the Froude number (233) is just the ratio of the velocity of the bore to the velocity $\sqrt{Lg}$ of a gravity wave in the shallow channel, which was how we defined the Froude number in section 12.2.

The wave resistance of a ship is a function of the Froude number, although there are also other factors that contribute to the drag, such as skin resistance and eddy production, which depend on the Reynolds number . If we want to estimate the drag force due to the waves produced in the wake by making a model, we select a speed for the model so that the Froude numbers are the same and we assume the drag force F due to generation of the waves in the wake

$$F_{waves} = \rho u^2 L^2 f(Fr) \tag{234}$$

where f is some function that depends on the geometry. Suppose we build a scale model 1/25. For the Froude numbers to be the same the model should then be tested at 1/5 of the operating speed of the ship. From (234) the drag on the model due to waves is then $1/(25)^3$ that of the actual ship.

These examples should give some of the flavor of arguments based on physical similarity. The arguments are not very rigorous and it is obvious they have to be used with some caution. However, when combined with experience these type of arguments are very useful, particularly when the physical situation is to complicated to yield to direct calculations.

14.6 Problems

Problem 14-1.

A 1 mm deep layer of water has temperature ≈ 20 C. The thermal expansion coefficient of water is 2×10^{-4} K^{-1} at this temperature, the thermal conductivity is $\kappa = 0.0014$ cal cm^{-1} s K^{-1}, while the viscosity is 10^{-3} Pa s. The critical Rayleigh number for convection with a free surface is approximately 1100. One calorie is 4.1840 J.

(a) What is the critical temperature difference between top and bottom for onset of convection?

(b) The compressibility $\frac{-1}{V}\frac{\partial V}{\partial P}$ of water is about 4.5×10^{10} Pa^{-1}. Is the approximation of neglecting the density change due to changes in pressure justified?

(c) Redo the calculations under (a) and (b) for the case of a 2 m deep swimming pool.

Part IV
Review

15 Old exams and other problems.

1.

(a). For the dipolar gas in subsection 4.1.2 show that the average dipole moment parallel to the field is

$$\langle \mu \cos\theta \rangle = \mu \left(\coth(\beta\mu E) - \frac{1}{\beta\mu E} \right) \tag{235}$$

(b). Show that the average *Polarization* (dipole moment per unit volume) for small fields is

$$P = \frac{N\beta\mu^2 E}{3V} \tag{236}$$

(Hint: for small arguments the hyperbolic cotangent has the Taylor expansion
$\coth(x) = \frac{1}{x} + \frac{x}{3} + ...$)

---

2. A fluid of viscosity η exhibits laminar flow in a cylindrical pipe of radius r and length L. For a pressure drop ΔP, it may be shown that the flow velocity $u(x)$ at a distance x from the cylinder axis is given by

$$u(x) = \frac{\Delta P}{4\eta L}(r^2 - x^2)$$

(a). State the assumptions which yields this equation.

(b). The viscosity of H_2O at 20 °C in c.g.s. units is approximately 0.01 poise. What does this correspond to in SI units?

(c). Calculate the total flow rate (volume per second) if the flowing fluid is water at 20 ^{o} C, $\Delta P = 0.1$ atm $\simeq 10^4$ N m^{-2}, $L = 1$ m, $r = 0.1$ cm.

(d). Calculate the shear stress exerted on the pipe wall and the total force exerted by the flowing water on the pipe under the conditions in (c).

---

3. Consider a dispersion of spherical particles of radius r, density ρ, in a fluid with viscosity η and density $\rho_1 < \rho$. The particles feel a gravitational force mg (in the z-direction) and a buoyancy force. If r is sufficiently small the drag on particles moving with velocity v will be given by Stokes' law $F = 6\pi r v \eta$.

(a). If the number n of particles per unit volume is n_o, at height $z = 0$, what is the equilibrium value of n at height h?

(b). Derive a formula for the particle diffusion constant by balancing the drift due to gravity and diffusion.

(c). If $\rho = 2 \times 10^3$ kg m^{-3}, $\rho_1 = 10^3$ kg m^{-3} and the viscosity of the fluid is 10^{-3} kg m$^{-1}s^{-1}$, and T=300 K, for what particle radius will the number of particles per unit volume at $h = 1$ m be $\frac{1}{2}n_0$? What is the diffusion constant under these conditions?

---

4. A sphere of radius r is moving inside a fluid with velocity v. The drag force is measured to be F. What must the velocity of a sphere of radius $2r$ be for the motion to be dynamically similar? What is then the drag force?

---

5. It was shown in class that the frequency ω and wave vector k are related by

$$\omega^2 = (gk + \frac{\sigma k^3}{\rho}) \tanh(kh)$$

Assume that the surface tension of water is $\sigma = 0.073$ N m^{-1}.

(a). Derive a formula for the behavior of the phase and group velocity as a function of the wavelength in the general case.

(b). Find the wavelength for which the phase velocity has a minimum in the case of deep water waves. Give a numerical answer.

(c). Show that for long wavelengths the effect of surface tension is negligible. Explain qualitatively why the amplitude of an ocean swell increases as it approaches a sloping beach.

(d). What is the group velocity of water waves of wavelength 100 m (i) in the middle of the ocean? (ii) in a channel of depth 5 m?

---

6. Two thin soap bubbles of radii $R_1 = 9.9$ cm , $R_2 = 10.1$ cm are connected by a tube of length $l = 10$ cm and diameter 1 mm as shown in Figure 36 Assume that the surface tension of the film increases the pressure inside the bubble by an amount $\frac{4\sigma}{R}$ where $\sigma = 0.075$ N m^{-1}. The outside pressure is 1 atm and the temperature is 300 K, the viscosity of air is 2×10^{-5} kg m^{-1} s^{-1}.

(a). What is the pressure difference between the two bubbles? Which bubble has the highest pressure?

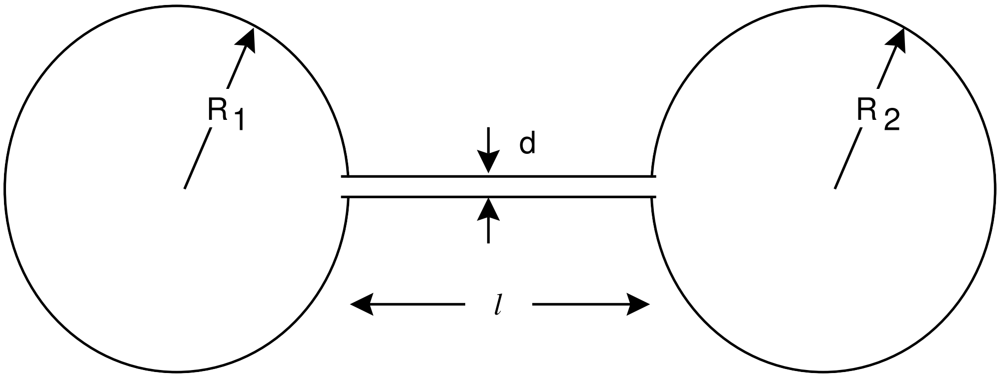


Figure 36: Soap bubbles of Problem 5.

(b). Assuming Poiseuille flow what is the initial flow rate in gm s $^{-1}$ or kg s $^{-1}$?

(c). Is the assumption of Poiseuille flow reasonable (assume the critical Reynolds number based on the diameter to be 2000)?

(d). What happens to the bubbles eventually (assume they don't evaporate!)?

Midterm examination October 1992

Closed book exam

---

1. Consider a suspension of spherical particles of radius r, density ρ, in a fluid with density $\rho_1 < \rho$. The particles experience a gravitational force and a buoyancy force. The temperature is T.

(a). If the number of particles per unit volume is $n = n_o$ at height $z = 0$, what is the equilibrium value of n at height $z = h$.

(b). If $\rho = 2 \times 10^3$ kg m^{-3}, $\rho_1 = 10^3$ kg m^{-3} and T=300 K for what particle radius will the number of particles per unit volume at $h = 1m$ be $\frac{1}{2}n_o$.

---

2. For an ideal gas the mean free path is

$$l = \frac{1}{\sqrt{2}n\sigma}$$

the diffusion constant is

$$D = \frac{\langle v \rangle l}{3}$$

the viscosity is

$$\eta = \frac{m\langle v \rangle}{3\sqrt{2}\sigma}$$

and the mean speed is

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}}$$

(a). If a gas is kept at constant temperature and the pressure is doubled, by what factors will the (i) mean free path (ii) viscosity (iii) diffusion constant change?

(b). Estimate scattering cross section σ and the diameter of the molecules of a gas with mass density 1.29 kg m^{-3} at 10^5 Pa and 300 K, if the viscosity is 2×10^{-5} Pa s.

The Boltzmann constant is 1.38×10^{-23} J K^{-1}.

Midterm Examination October 1993

Closed book exam. Allowed aid: Calculator

1:

(a)

A gas is made up of N_1 molecules of molecules of mass m_1 and N_2 molecules of mass m_2. A few of these molecules are adsorbed at the surface of the gas container with probability p_1 and p_2 respectively each time they hit the wall. Once the molecules are adsorbed the probability that they escape is negligible.

What will the ratio of concentrations of the two adsorbed species be on the surface?

(b)

Molecules are injected into a medium from a point source and diffuse away. The steady state concentration will be proportional to $1/r$ where r is the distance to the source. The diffusion constant is D and the rate at which molecules are injected is $\dot{N}$. Find the steady state concentration n(r) of the injected molecules.

2:

A gas of mass density 1.29 kg m^{-3} at $1. \times 10^5$Pa pressure and 300K has viscosity 2×10^{-5} Pa s. Find the diffusion constant.

Some formulas:

$$j_N = \frac{n\langle v\rangle}{4}; \quad k_B = 1.38 \times 10^{-23} \text{J K}^{-1}$$

For an ideal gas the mean free path and the diffusion constant are

$$l = \frac{1}{\sqrt{2} n\sigma}; \quad D = \frac{\langle v\rangle l}{3}$$

and the viscosity and mean speed are

$$\eta = \frac{m\langle v\rangle}{3\sqrt{2}\sigma}; \quad \langle v\rangle = \sqrt{\frac{8k_BT}{\pi m}}$$

Midterm Examination October 1994

Allowed aid: One double sided "cheat sheet", Calculator

Some formulas and useful formulas are listed on page 2 of the exam

1:

Liquid ether at T=300 K has an equilibrium vapor pressure of 5.33×10^4 Pa, the mass density of the liquid is $\rho = 700$ kg m^{-3}. The molecular weight of ether is 74. At equilibrium an equal number of molecules hit the liquid from above as leave the surface by evaporation.

(a)

How many ether molecules leave the surface by evaporation per s and m^2 of free surface?

(b) Assume that near the ether surface the ether vapor is in equilibrium with the liquid but that ether molecules 1 *cm* above the surface are blown away so that the ether concentration is effectively zero. Assume also that ether molecules diffuse from the liquid surface to a height 1 cm above. What is the steady state concentration of ether molecules 0.5 cm above the liquid surface?

(c) What is the net diffusion current (molecules per second and square meter) 0.5 cm above the liquid surface. The diffusion constant of ether in air is approximately 2×10^{-5} m^2 s^{-1}

2: A sealed container contains an ideal gas at T=300 K and 10^5 Pa pressure. The gas is made up of molecules of diameter 3 Å and molecular weight 30.

(a) Calculate the diffusion constant D and viscosity η of the gas.

(b) What happens to the diffusion constant and the viscosity if the temperature is lowered by 10%, but the volume of the container stays constant?

(c) At time t=0 a small amount of different molecules of the same size and mass as the original molecules of the gas are released from a point source inside the container and start diffusing. At what time after the release will the *diffusion current* 1 *mm* away from the source reach a maximum?

Some values of constants

1 a.m.u. $= 1.67 \times 10^{-27}$ kg $k_B = 1.38 \times 10^{-23}$ J K^{-1}

Some formulas:

$$l = \frac{1}{n\sigma\sqrt{2}}; \qquad D = \frac{\langle v \rangle l}{3}$$

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}}; \qquad PV = Nk_BT$$

$$\eta = \frac{nm\langle v \rangle l}{3}$$

$$n(\mathbf{r}, t) = \frac{N}{(4\pi Dt)^{3/2}} \exp\left[-\frac{r^2}{4Dt}\right]$$

$$j_N = \frac{n\langle v \rangle}{4}; \qquad j_N = -D\nabla n$$

Midterm Examination October 1995

Allowed aids: One double sided "cheat sheet", Calculator
Some constants and useful formulas are listed at the end of the exam

1:

(a)

A sphere of radius r, temperature T is embedded in a gas with molecules of mass m and specific heat per molecule $5/2k_B$. The pressure is P and the temperature of the gas is T_g. Assume that gas molecules hit the sphere with mean kinetic energy $3k_BT_g$ and then leave with mean energy $3k_BT$. Estimate the rate at which heat is transferred between the gas and the sphere assuming that collisions between gas molecules and the sphere is the only mechanism of heat exchange.

(b)

Assuming that initially (time $t = 0$) the temperature difference is ΔT. Derive an expression for the temperature of the sphere at later times t. Assume that the heat capacity of the sphere is C (units[J/K]).

(c)

How long will it take for the temperature difference to halve, assuming the temperature of the gas is kept constant? Also, assume that r=1 cm, $C = 15$ J K^{-1}, $P = 10^5$ Pa, T_g =300 K, $T = 275$ K, $m = 29$ a.m.u.

2: A container contains an ideal gas at $T = 300K$, pressure 10^5 Pa. The gas is made up of molecules of diameter 3 Å and molecular weight 32 a.m.u.

(a)

Calculate the diffusion constant D of the gas.

(b)

Estimate how long it will take for a typical molecule to diffuse 1 m.

(c)

What is the mean distance that a molecule has diffused in 1 s?

Some values of constants

1 $a.m.u. = 1.67 \times 10^{-27}$ kg

$k_B = 1.38 \times 10^{-23}$ J/K.

Some formulas:

$$l = \frac{1}{n\sigma\sqrt{2}}; \qquad D = \frac{\langle v \rangle l}{3}$$

$$\langle v \rangle = \sqrt{\frac{8k_BT}{\pi m}}; \qquad PV = Nk_BT$$

$$\eta = \frac{nm\langle v \rangle l}{3}$$

$$n(\mathbf{r}, t) = \frac{N}{(4\pi Dt)^{3/2}} \exp\left[-\frac{r^2}{4Dt}\right]$$

$$j_N = \frac{n\langle v \rangle}{4}; \qquad j_N = -D\nabla n$$

$$\int_0^\infty r^3 e^{-ar^2} dr = \frac{1}{2a^2}$$

Christmas examination December 1991

Allowed aids: "cheat sheet", calculator. Please note that some formulas and numerical values of some physical constants can be found at the end of the examination. Time 2 hours. ANSWER ALL 4 QUESTIONS.

————————————————————-

1. A wave with wavelength 100 m is traveling on the surface of a deep ocean

(a). What is the phase velocity?

(b). What is the group velocity?

(c). Estimate the phase and group velocity of a wave with the same wavelength in a $1m$ deep channel.

————————————————————-

2. Fine dust is dispersed uniformly in a room with a 3 m high ceiling. It takes about a minute for the dust to settle. Assume that the particles are made of an inert material of density $\rho = 2 \times 10^3$ kg m^{-3} and that the particles are spherical.

(a). Estimate the radius of the particles.

(b). Check that in equilibrium almost all dust will settle.

————————————————————-

3. A cylindrical drum of length L=10 cm, radius r=10 cm can rotate inside a fixed concentric casing with clearance c=1 mm. The space between the drum and the casing is filled with water. Assume laminar flow.

(a) What is the power required to rotate the cylinder with angular velocity $\omega = 1$ rad s^{-1}? Neglect frictional forces at the ends.

(b) What is the force of friction on the cylinder walls if the drum is pulled with velocity $v = 10$ cm s^{-1} parallel to the cylindrical axis which is assumed to be horizontal.

(c) The force pulling the cylinder is inactivated. If the mass of the cylinder is 0.1 kg how long will it take for the drum under the conditions in question (b) above to slow down to 1 cm s^{-1}.

————————————————————-

4. A valuable irregularly shaped object (VISO) of mass 100 kg is to be dropped from high altitude. You are required to estimate the velocity with which it is falling just before it hits the ground. The object is dense enough that the buoyancy of the air can be neglected.

(a). You have at your disposal a 1:10 scale model and a wind tunnel where you can measure the force on a suspended object and adjust and measure the wind speed. What should the force on the model be to achieve physical similarity?

(b). What is then the ratio between the terminal speed of the VISO and the air speed in the wind tunnel.

(c). The wind tunnel has broken down, but you are still required to estimate the terminal speed of the VISO. You have a friend with a water tank. You are able to pull the model with adjustable speed and to measure the drag force in the tank correcting for buoyancy. What force would you use and what is the ratio between the flow speed and the VISO terminal velocity in air?

——————————————————————-

SOME FORMULAS

$$F = u\eta r 6\pi$$

$$Re = \frac{u\rho d}{\eta}$$

$$\omega = \sqrt{(gk + \frac{\sigma k^3}{\rho})tanh(kh)}$$

$g = 9.8$ m s^{-2}, $\eta = 2.0 \times 10^{-5}$ Pa s for air and $\eta = 10^{-3}$ Pa s for water, $k_B = 1.38 \times 10^{-23}$ J K^{-1}, $\rho = 10^3$ kg m^{-3} for water, $\rho = 1.29$ kg m^{-3} for air.

Solution to 1991 Christmas Exam

Problem 1:

For deep water waves $\tanh(kh) \simeq 1$ and

$$\omega \simeq \sqrt{gk + \frac{\sigma k^3}{\rho}}$$

If the wavelength is as large as 100 m, the second term inside the square root can be neglected and

$$\omega \simeq \sqrt{gk}$$

(a) The phase velocity is

$$u_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k}} = \sqrt{\frac{g\lambda}{2\pi}} = \sqrt{\frac{9.8 \times 100}{2\pi}} = 12.49 \text{ m s}^{-1}$$

(b) The group velocity is

$$u_g = \frac{d\omega}{dk} = \frac{1}{2}\sqrt{\frac{g}{k}} = 6.24 \text{ m s}^{-1}$$

(c) For shallow water gravity waves $\tanh(kh) \simeq kh$ and $\omega = k\sqrt{gh}$ we have

$$u_\phi = u_g = \sqrt{gh} = \sqrt{1 \times 9.8} = 3.13 \text{ ms}^{-1}$$

Problem 2:

(a) If the Boltzmann factor

$$\exp(-\frac{mgh}{k_B T}) << 1$$

the steady state will be characterized by balance between the forces of gravity and friction $mg = 6\pi u\eta r$ We have

$$m = \frac{4\pi\rho r^3}{3}$$

giving

$$u = \frac{mg}{6\pi\eta r} = \frac{2\rho r^2 g}{9\eta}$$

We estimate the clearing time as

$$t = \frac{h}{u} = \frac{9\eta h}{2\rho r^2 g}$$

Solving for the radius we obtain

$$r = \sqrt{\frac{9\eta h}{2\rho t g}} = \sqrt{\frac{9 \times 2.10^{-5} \times 3}{2 \times 2.10^3 \times 60 \times 9.8}} = 1.5 \times 10^{-5} m = 15\mu\text{m}$$

(b) We need to check the Boltzmann factor. We have assuming $T \simeq 300K$

$$\frac{mgh}{k_B T} = \frac{4\pi\rho r^3 gh}{3k_B T} = \frac{4\pi \times (1.510^{-5})^3 \times 2.10^3 \times 9.8}{3 \times 1.38 \times 10^{-23} \times 300} h = 0.66 \times 10^{11} \frac{h}{meter}$$

For distances of the order of meters the Boltzmann factor will be extremely small. Even for distances as small as a particle radius the Boltzmann factor will be very small.

Problem 3:

(a) The velocity gradient is $\omega r/c$, the frictional force per unit area is thus $\omega r\eta/c$. The area of the cylinder is $2\pi rL$, and we get for the frictional torque

$$\frac{\omega r^3}{c}\eta 2\pi L$$

The required power is thus

$$\frac{\omega^2 r^3}{c}\eta 2\pi L = \frac{1^2 \times 0.1^3 \times 10^{-3} \times 2\pi \times 0.1}{10^{-3}} = 6.28 \times 10^{-4} \text{ W}$$

(b) The frictional force per unit area is now $v\eta/c$ giving

$$F = \frac{v}{c}2\pi\eta rL = \frac{0.1}{0.001}2\pi \times 10^{-3} \times 0.1 \times 0.1 = 6.28 \times 10^{-3} \text{ N}$$

(c) From Newton's second law we have for deceleration

$$m\dot{v} = -\frac{v}{c}2\pi\eta rL$$

with solution

$$v = v_o \exp\left\{-\frac{2\pi\eta rLt}{mc}\right\}$$

Solving for t we find

$$t = \frac{mc\ln\frac{v_o}{v}}{2\pi\eta rL} = \frac{0.1 \times 10^{-3}\ln 10}{2\pi \times 10^{-3} \times 0.1 \times 0.1} = 3.7 \text{ s}$$

Problem 4:

For physical similarity the Reynolds number has to be the same for the VISO and for the model. We let subscript v refer to the VISO and m to the model. Equating the Reynolds numbers gives

$$\frac{u_v\rho_v r_v}{\eta_v} = \frac{u_m\rho_m r_m}{\eta_m}$$

where r_v and r_m is the linear dimension of the VISO and the model respectively. It doesn't matter how this quantity is measured, as long as it is done the same way for both objects, We write the drag force in the form

$$F = u\eta rf(Re)$$

(we could equally well have written $F = u^2\rho r^2\tilde{f}(Re)$). The function $f(Re)$ (or $\tilde{f}(Re)$) is unknown, but the value would be the same for all flows with the same Reynolds number. Thus,

$$\frac{F_v}{F_m} = \frac{u_v\rho_v r_v}{u_m\rho_m r_m}$$

(a) If the experiment with the model is made in air $\rho_v = \rho_m$, $\eta_v = \eta_m$, and when the VISO has reached its terminal velocity $F_v = m_v g = 980$ N. Since $r_v = 10\ r_m$ we find $u_m = 10\ u_v$. We have $F_m = F_v = 980$ N.

(b) The terminal speed of the VISO is 1/10 of the speed in the wind tunnel when the drag force is $980N$.

(c) We have $\rho_v = 1.29$ kg m^{-3}, $\rho_m = 1000$ kg m^{-3}, $\eta_v = 2. \times 10^{-5}$ Pa s, $\eta_m = 10^{-3}$ Pa s. We have

$$u_m r_m \frac{1000}{10^{-3}} = u_v r_v \frac{1.29}{2\times 10^{-5}}$$

giving $u_m = 0.64u_v$. The ratio of the forces is

$$\frac{F_m}{F_v} = \frac{0.64\times 0.1\times 10^{-3}}{210^{-5}} = 3.2$$

Thus the speed of the model should be adjusted until the drag force is $3.2\times 980 = 3136$ N. The terminal velocity of the VISO will be $1/0.64 = 1.56$ times the speed of the model when the drag force on it is 3136 N.

Final exam December 1992

Allowed aids: 1 double sided "cheat sheet", calculator. Time 2 hours. Answer all 4 questions.

1: It was shown in class that, subject to some simplifying assumptions, deep water waves obey a dispersion relation in which the frequency ω is related to the wave vector k through

$$\omega = \sqrt{gk + \frac{\sigma k^3}{\rho}}$$

where g is the acceleration of gravity, σ is the surface tension and ρ is the mass density.

(a). Describe in words the approximations made in deriving the formula.

(b). Calculate the wave length for which the phase velocity is a minimum.

(c). What is the value of the group velocity when the phase velocity is a minimum? We have $\rho = 1000$ kg m^{-3} and $\sigma = 0.073$ N m^{-1} for water, and $g = 9.8$ m^2 s^{-1}.

2: Water is flowing steadily in a horizontal cylindrical pipe of length L inner radius r. To compensate for viscous and turbulent losses a pressure difference $\delta P = \rho g h_f$ is maintained between the ends with h_f the "frictional head". Assume that the flow rate is $\dot{V}$ and that h_f is given by Darcy's law

$$h_f = f\frac{Lu^2}{rg}$$

where the mean speed is $u = \frac{\dot{V}}{\pi r^2}$ and f is a dimensionless parameter.

(a). What is the force on the pipe?

(b). Derive a formula for the shear stress exerted on the inner surface of the pipe by the flowing water.

(c). Give a numerical answer for the net force in (a) assuming $\dot{V} = 10^{-4}$ m^3 s^{-1}, $f = 10^{-2}$, $r = 10^{-2}$ m, $L = 1$ m.

3: A container of volume V is filled with a gas at temperature T, pressure P, made up of molecules of mass m. According to kinetic theory the number of particles hitting the walls of the container per unit time and surface area is $j_N = n\langle v\rangle/4$ where n is the number of molecules per unit volume and

$$\langle v\rangle = \sqrt{\frac{8k_BT}{\pi m}}$$

is the mean speed. The mean free path of the molecules is

$$\langle\lambda\rangle = \frac{1}{n\sigma\sqrt{2}}$$

where σ is the collision cross section. Gas escapes into outside vacuum through a small circular hole. The area of the hole is A.

(a). Derive a formula for the rate of change of pressure with respect to time due to the escape of molecules.

(b). For the kinetic theory to be valid the distribution of particle velocities must not be effected by the hole. This requires that the radius of the hole should not be larger than the mean free path of the molecules. What is the range of pressures, given r, for which the formula derived in part (a) is valid?

(c). Give numerical estimate of the maximum area of the hole for the conditions under (b) to be valid assuming $P = 100$ k Pa, $\sigma = 20 \times 10^{-20}$ m^2, $T = 300$ K, $k_B = 1.38 \times 10^{-23}$ J K^{-1}.

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4: It is desired to measure the viscosity η of a fluid by dropping a steel ball of radius r in the fluid and observing the terminal velocity u assuming Stokes' law $F = 6\pi\eta r u$. The density of the fluid is ρ_f and that of the ball ρ_b.

(a). Derive a formula for the viscosity as a function of ρ_f, ρ_b, u, r.

(b). For turbulent drag corrections not to be important, the Reynolds number $Re = \rho_f ur/\eta$ must not exceed a critical value R. What restrictions does this put on the size of the ball?

(c). If the ball is too small the experiment won't work too well. Why not?

Final Exam, December 1992 Solution

1

a)

The most important assumptions are:

(i). Viscous and turbulent losses are neglected.

(ii). The fluid is incompressible.

(iii). The wave amplitude is small compared with the wavelength.

(iv). The channel is much deeper than the wavelength.

b)

The phase velocity

$$v_\phi = \frac{\omega}{k} = \sqrt{\frac{g}{k} + \frac{\sigma k}{\rho}} \tag{237}$$

has a minimum when

$$\frac{dv_\phi}{dk} = 0 = \frac{-\frac{g}{k^2} + \frac{\sigma}{\rho}}{2\sqrt{\frac{g}{k} + \frac{\sigma k}{\rho}}} \tag{238}$$

i.e. for

$$k = k_0 = \sqrt{\frac{g\rho}{\sigma}} \tag{239}$$

giving for the wavelength $\lambda = 2\pi\sqrt{\frac{\sigma}{g\rho}}$.
Substituting the values for g, σ, ρ we find $\lambda = 0.017$ m.
c)
The group velocity is given by

$$v_g = \frac{d\omega}{dk} = \frac{g + \frac{3\sigma k^2}{\rho}}{2\sqrt{gk + \frac{\sigma k^3}{\rho}}} = \sqrt{2\sqrt{\frac{g\sigma}{\rho}}} \tag{240}$$

Substituting numerical values for g, σ, ρ we find $v_g = 0.23$ m s^{-1}
2
a)
We have for the net force

$$F = \delta P \pi r^2 = \rho g \pi r^2 h_f = \rho \pi r f L \frac{u^2}{g} \tag{241}$$

Using

$$u = \frac{\dot{V}}{\pi r^2} \tag{242}$$

we find the desired result

$$F = \frac{\rho f L \dot{V}^2}{\pi r^3} \tag{243}$$

b)
The shear stress is

$$\tau = \frac{F}{2\pi r L} = \frac{\rho f \dot{V}^2}{2\pi^2 r^4} \tag{244}$$

c)

$$F = \frac{1000 \times 10^{-2} \times 1 \times 10^{-6}}{\pi 10^{-6}} = 3.2 \text{ N} \tag{245}$$

3
a)
Starting with

$$P = \frac{N k_B T}{V} \tag{246}$$

we find

$$\dot{P} = -\frac{k_B T N < v > A}{4V^2} = -\frac{PA < v >}{4V} \tag{247}$$

b)
If

$$\lambda = \frac{1}{n\sigma\sqrt{2}} = \frac{k_B T}{P\sigma\sqrt{2}} \tag{248}$$

we see that for $\lambda > r$ we must have

$$P < \frac{k_B T}{r\sigma\sqrt{2}} \tag{249}$$

c)
We have

$$\lambda = -\frac{k_B T}{P\sigma\sqrt{2}} = 1.4 \times 10^{-7} \text{ m} \tag{250}$$

If $r = \lambda$ we have

$$A = \pi r^2 = 6.710^{-14} \text{ m}^2 \tag{251}$$

4
a)
The force on the steel ball is

$$F = (\rho_b - \rho_f) g \frac{4\pi r^3}{3} = 6\pi\eta r u \tag{252}$$

giving for the viscosity

$$\eta = \frac{2g(\rho_b - \rho_f) r^2}{9u} \tag{253}$$

b)
Substitution of

$$u = (\rho_b - \rho_f) \frac{2r^2 g}{9\eta} \tag{254}$$

into

$$Re_{crit} = \frac{\rho_f u r}{\eta} \tag{255}$$

gives

$$r < (Re_{crit} \frac{9\eta^2}{2\rho_f(\rho_b - \rho_f) g})^{1/3} \tag{256}$$

c)
If the ball is too small the velocity will fluctuate due to Brownian motion and it will be hard to measure an average speed.

Final exam December 1993

Allowed aid: Calculator, 2 page "cheat sheet"
Answer 4 out of the 5 questions. Time 2 1/2 hours.

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1:

A virus of spherical shape and mass 10^{-18} kg, density $\rho = 1000$ kg m^{-3}, is floating in air at the temperature 300 K, pressure 100 k Pa. The mass of an air molecule is 4.84×10^{-26} kg. Boltzmann's constant is $k_B = 1.3810^{-23}$ J K^{-1}.

(a). What is the r.m.s. speed of a virus? Of an air molecule?

(b). What is the mean time between air-molecule virus collisions?

(c). Calculate the r.m.s. momentum of the virus and of an air molecule. Assume that the r.m.s. momentum transfer in each collision is approximately equal to the r.m.s. momentum of an air molecule. How many collisions will typically be needed for a virus to change its direction?

(d). How far will the virus typically travel between each time it changes direction?

———————————————————

2:

The cross section of a dam has the shape of an equilateral triangle with side l. There is water seepage under the dam producing a lift pressure which is equal to the water pressure at depth $h = l\sqrt{3}/2$ where ρ_w is the density of water. What is the minimum density ρ_d of the dam material for it to be stable against toppling about the point P in figure 37, when the water level reaches the apex of the triangle?

———————————————————

3:

A circular disk is rotating with angular velocity ω. There is a clearance c to a stationary disk below. The radius of the disk is r and the viscosity of the fluid between the disks is η. Assume laminar flow.

(a). What is the flow velocity at height $y < c$ above the stationary disk when the radial distance from the axis is x.

(c). How much heat must be transported away per unit time to prevent the temperature from rising.

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4:

The bulk thermal expansion coefficient of water at 20°C is 0.2×10^{-3} K^{-1}. The compressibility of water at that temperature is 4.5×10^{-10} Pa^{-1}.

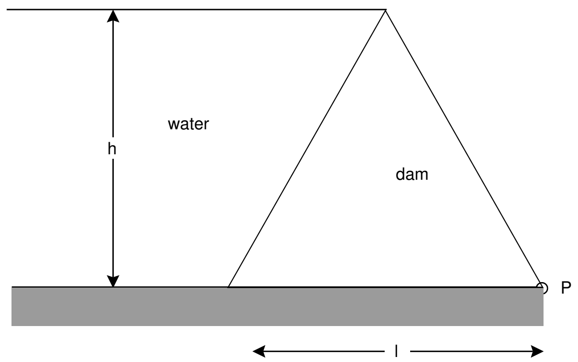


Figure 37: Figure to problem 2 on 1993 exam.

(a). What is the critical thermal lapse rate (temperature change per unit height) for which the density of water heated from below will increase with height.

(b.) The thermal conductivity is $K = 0.0014$ cal cm^{-1} s^{-1} K^{-1} and 1 cal $= 4.18$ J. The specific heat of water is 1 cal g^{-1} K^{-1} and the density of water approximately 1 g cm^{-3}. What is the heat flow (Watt m^{-2}) for the critical lapse rate.

(c.) If the critical lapse rate is exceeded convection may set in. Are there conditions where the lapse rate exceeds the rate calculated under (a) but there is no convection? If convection takes place will the heat flow increase or decrease assuming that the temperature difference is maintained?

5:

In a certain fluid the viscous stress is related to the velocity gradient by

$$\tau = k\left(-\frac{\partial u_x}{\partial y}\right)^n$$

(a). Use dimensional analysis to show that the laminar flow rate in a pipe can be written

$$\dot{V} = cr^3\left(\frac{r\Delta P}{Lk}\right)^{1/n}$$

where L is the length of the pipe r is the radius, ΔP is the pressure drop and c is a dimensionless constant.

(b). Use the result found under (a) to find a formula for the friction factor f in Darcy's law

$$\frac{h_f}{L} = f\frac{u^2}{rg}$$

where h_f is the frictional head.

Solution to final Exam December 93

1:
(a)

$$\frac{1}{2}MV_{rms} = \frac{3}{2}k_BT$$

For the virus:

$$V_{rms} = \sqrt{\frac{3 \times 1.381 \times 10^{-23} \times 300}{10^{-18}}} = 0.111 \text{ m s}^{-1}$$

For the air molecules:

$$v_{rms} = \sqrt{\frac{3 \times 300 \times 1.381 \times 10^{-23}}{4.84 \times 10^{-26}}} = 507 \text{ m s}^{-1}$$

(b)
The radius of the virus can be obtained from

$$\frac{4\pi\rho r^3}{3} = M$$

giving

$$r = \left(\frac{3 \times 10^{-18}}{4\pi \times 1000}\right)^{1/3} = 6.2 \times 10^{-8} \text{ m}$$

The number of air molecules per unit volume is with P the pressure

$$n = \frac{P}{k_BT}$$

The mean time τ between collisions can be obtained, either by arguing that

$$\frac{1}{\tau} = n\pi r^2\langle v\rangle$$

or by saying that

$$\frac{1}{4}n\langle v\rangle$$

air molecules hit per unit area of the virus. Since the area is $4\pi r^2$ the two approaches give the same result. If we use v_{rms} for $\langle v\rangle$ we get

$$\tau = \frac{k_B T}{P\pi r^2 v_{rms}} = \frac{300 \times 1.381 \times 10^{-23}}{10^5\pi(0.62 \times 10^{-8})^2 \times 507} = 6.76 \times 10^{-15} \text{ s}$$

If instead we use the more accurate formula

$$\langle v\rangle = \sqrt{\frac{8k_B T}{\pi m}}$$

we get

$$\tau = 1.08 \times 10^{-14} \text{ s} \tag{257}$$

(c)
The rms momentum of the virus is

$$P_{rms} = MV_{rms} = 1.11 \times 10^{-19} \text{ kg m s}^{-1}$$

while for the air molecules

$$p_{rms} = mv_{rms} = 2.45 \times 10^{-23} \text{ kgm s}^{-1}$$

The number of collisions typically needed is

$$N_c \approx \frac{P^2_{rms}}{p^2_{rms}} = 2 \times 10^7$$

(d)

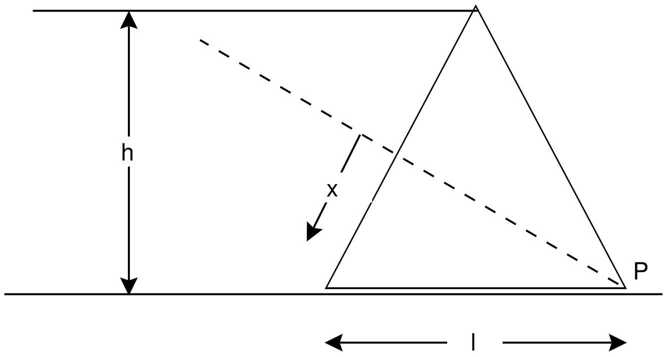
$$l \approx V_{rms}\tau N_c = 1.56 \times 10^{-8} \text{ m}$$

If we use the more accurate value 257 we obtain

$$l = 2.5 \times^{-8} \text{ m}$$

2:
The height of the apex is

$$h = \frac{\sqrt{3}l}{2}$$



The cross-sectional area of the dam is

$$\frac{hl}{2} = \frac{\sqrt{3}l^2}{4}$$

We need to calculate the torque about the point P (see figure). If counter-clockwise torques are considered to be positive we have for the torque per unit width due to the weight of the dam

$$T_d = \frac{\sqrt{3}l^3\rho_d}{8}$$

The pressure of the seeping water is

$$P = \frac{\sqrt{3}lg\rho_w}{2}$$

The upwards force per unit width is thus

$$F_w = \frac{\sqrt{3}l^2g\rho_w}{2}$$

The opposing torque due to seeping water is

$$T_w = -\frac{\sqrt{3}l^3\rho_w}{4}$$

The pressure on the side of the dam acts perpendicular to the surface. The torque due to the pressure on the side of the dam is

$$T_s = \rho_w g \int_{-l/2}^{l/2} dxx(x + \frac{l}{2})\frac{\sqrt{3}}{2} = \rho_w g \frac{\sqrt{3}l^3}{24}$$

The condition $T_d + T_w + T_s > 0$ gives

$$\frac{2\rho_w - \rho_d}{\rho_w} < \frac{1}{3}$$

If $\rho_d > \frac{5}{3}\rho_w$, the dam will be stable against toppling. We may also wish to check if the downwards forces are enough to overcome the upwards lift due to the seeping water. The downwards force due to the pressure on the side of the dam is (the downwards component of the pressure is $\frac{1}{2}$ the pressure)

$$F_s = \frac{\rho_w g}{2} \int_0^l dx \frac{x\sqrt{3}}{2} = \frac{\sqrt{3} l^2 g \rho_w}{8}$$

Giving for the net upwards force

$$F_w - F_s - F_d = \frac{\sqrt{3} l^2 g \rho_w}{2} - \frac{\sqrt{3} l^2 g \rho_d}{4} - \frac{\sqrt{3} l^2 g \rho_w}{8}$$

or $\rho_d > \frac{3}{2}\rho_w$. If this condition is not met the seeping water may lift the dam allowing the pressure along the side to move it sideways. However, $\rho_d > \frac{5}{3}\rho_w$ is the more severe restriction.

3:

(a).

The flow velocity at height $y < c$ above the stationary disk at radial distance x is

$$\frac{\omega y x}{c}$$

(b).

The shear stress at radial distance x is

$$\tau = \frac{\eta \omega x}{c}$$

The dissipated power is

$$\int_0^r dx \omega x \frac{\eta \omega x}{c} 2\pi x = \frac{\pi \omega^2 r^4 \eta}{2c}$$

4:

(a).

The rate of change of the density with height h is given by

$$\frac{d\rho(P,T)}{dh} = \frac{\partial \rho}{\partial P}\frac{\partial P}{\partial h} + \frac{\partial \rho}{\partial T}\frac{\partial T}{\partial h}$$

The compressibility is given by

$$\kappa = \frac{1}{\rho}\frac{\partial \rho}{\partial P}$$

The thermal expansion coefficient is

$$\alpha = \frac{1}{V}\frac{\partial V}{\partial T} = -\frac{1}{\rho}\frac{\partial \rho}{\partial T}$$

and we have

$$\frac{\partial P}{\partial h} = -\rho g$$

and

$$\frac{\partial T}{\partial h} = -l$$

where l is the thermal lapse rate. Collecting terms gives

$$\frac{d\rho}{dh} = \rho(-\kappa\rho g + \alpha l)$$

The critical lapse rate is thus

$$l = \frac{\kappa \rho g}{\alpha} = \frac{4.5 \times 10^{-10} \times 1000 \times 9.8}{0.2 \times 10{-}3} = 0.022 \text{ K m}^{-1}$$

(b).

The thermal conductivity in SI units is $K = 0.5852$ J m^{-1} s^{-1} K^{-1}, the heat flow is thus $Kl = 0.013$ W m^{-2}.

(c).

It is possible to have a fluid which is stable against convection even though the density increases with height, if the layer of fluid is thin enough. The condition for convection for an incompressible fluid is that the Rayleigh number must exceed a critical value.

If convection sets in, while the temperature difference is maintained, the heat flow will *increase*.

5:

(a.)

We assume that

$$\dot{V} = c r^{\alpha}\left(\frac{\Delta P}{L}\right)^{\beta} k^{\gamma}$$

¿From

$$k = \tau\left(-\frac{\partial u_x}{\partial L}\right)^{-n}$$

we find

$$[k] = [\tau][s^{-1}]^{-n} = [Pa][s]$$
$$[r] = [m]$$
$$\frac{\Delta P}{L} = [Pa][m]^{-1}$$
$$[\dot{V}] = [m]^3[s]^{-1}$$

and

Pressure $\rightarrow 0 = \beta + \delta$
Seconds $\rightarrow -1 = n\delta$
Meters $\rightarrow 3 = \alpha - \beta$
The solution is $\delta = -1/n$, $\beta = 1/n$, $\alpha = 3 + 1/n$.

$$\dot{V} = cr^3 \left(\frac{r\Delta P}{Lk}\right)^{1/n} \tag{258}$$

(b).
We have

$$\rho g h_f g = \Delta P; \;\; \dot{V} = \pi r^2 u$$

Substitution into (258) gives

$$\frac{h_f}{L} = \frac{u^2}{rg}\left(\frac{\pi}{c}\right)^n \frac{k}{\rho u^{2-n} r^n}$$

or

$$f = \left(\frac{\pi}{c}\right)^n \frac{k}{\rho u^{2-n} r^n}$$

Final exam. December 1994

Allowed aid: Calculator, 2 page "cheat sheet"
Answer 4 out of the 5 questions. Time 2 1/2 hours.

If you answer more than 4 questions you will only be given credit for the 4 best answers.

1:

A vial of height 3 cm is filled with an aqueous suspension. The particles in the suspension are spherical with radius 100 Å, and have a density of

2000 kg m^{-3} (twice that of water). The sample temperature is T=300 K. The viscosity of water is $\eta = 10^{-3}$ Pa s. The mobility of a spherical object in laminar flow is given by Stokes' law

$$\mu = \frac{velocity}{force} = \frac{1}{6\pi\eta r}$$

(a) Verify quantitatively that the equilibrium distribution of particles is nearly uniform over the height of the vial. Boltzmann's constant is $k_B = 1.3810^{-23}$J K^{-1}.

(b) The vial is placed in a centrifuge and subject to an acceleration equal to 10^4 times that of gravity, acting along the height of the vial (assume that the acceleration is nearly uniform over the height of the vial). Once the new equilibrium is reached, roughly how many times more concentrated are the particles near the outermost end of the vial (where the particles tend to concentrate) end of the vial than they were before centrifuging?

(c) Roughly how long must the sample be spun in the centrifuge before equilibrium will be reached?

2:

A cylinder of length L and radius R can rotate inside a fixed concentric casing with a clearance, c, which is much less than R. The space between the drum and the casing is filled with a lubricant with viscosity η. Assuming laminar flow

(a) What is the power required to rotate the cylinder with angular velocity ω? Neglect end effects.

(b) Calculate the power loss due to viscous forces at the ends assuming that the clearance at top and bottom is c and that the flat surfaces are parallel.

3:

Water is flowing steadily in a horizontal cylindrical pipe of length L inner diameter d. To compensate for viscous and turbulent losses a pressure difference $\delta P = \rho g h_f$ is maintained between the ends with h_f the "frictional head". Assume that the flow rate is $\dot{V}$ and that h_f is given by Darcy's law

$$h_f = f\frac{Lu^2}{2dg}$$

where the mean speed is $u = \frac{4\dot{V}}{\pi d^2}$ and f is a dimensionless parameter.

(a) What is the force exercised by the water on the pipe?

(b) Derive a formula for the shear stress exerted on the inner surface of the pipe by the flowing water.

(c) Give a numerical answer for the net force in (a) assuming $\dot{V} = 10^{-4}$ m^3 s$^-$, $f = 10^{-2}$, $d = 10^{-2}$ m, $L = 1$ m.

———————————————

4:

A hot air balloon contains a volume $V = 4\pi R^3/3$ of air at temperature $T_h > T_a$, where T_a is the ambient temperature. The pressure inside is approximately the same as the pressure P outside. The payload of the balloon (exclusive of the hot air) is M. The average mass of an air molecule is m.

a Find a formula for the minimum value of $V = V_{min}$ for the balloon to rise assuming the other constants are given.

b The drag force on a rising balloon is $F = \rho u^2 R^2 h$ where h is a constant, ρ is the density of the outside air and u is the speed with which the balloon rises. Find a formula for the steady state speed with which the balloon will rise if $V > V_{min}$.

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5:

Imagine that a hole has been drilled to the center of a planet of radius R. If the acceleration of gravity at the surface of the planet is g it will, at distance $r < R$ from the center, be $\frac{gr}{R}$. The pressure at the surface is P, and the atmosphere on the planet consists of molecules of mass m.

a Find the pressure $P(r)$, for $r < R$, assuming an isothermal ideal gas at temperature T.

b Give a numerical answer for the pressure at the center of the planet ($r = 0$) assuming $P = 10^5$ Pa, $m = 5 \times 10^{-26}$ kg, $g = 10$ m s^{-2}, $k_B = 1.38 \times 10^{-23}$J K^{-1}, $R = 6 \times 10^6$ m, $T = 300$ K.

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Final exam. December 1995

Allowed aids: One double sided "cheat sheet", Calculator.
Some constants and useful formulas are listed at the end of the exam.
Time 2 Hours. Answer 4 out of the 5 questions. If you answer all 5 questions you will be given credit for the 4 best answers.

1:

The density of the fluid in a body of water depends on the *pressure* P and the *temperature* T. Assume that with increasing depth z we have $T = T_0 + lz$, where l is the *lapse rate*, and that the *compressibility* $K = \frac{\partial \rho}{\rho \partial P}$, and *thermal expansion coefficient* $\alpha = -\frac{\partial \rho}{\rho \partial T}$ are constant, as is the acceleration of gravity g.

(a)

Under what conditions will the density ρ decrease with depth (leading to overturning of the water)?

(b)

Estimate numerically the critical lapse rate beyond which the density will decrease with depth.

2:

It is observed that raindrops with diameter approximately 10^{-3} m fall towards the ground with a terminal speed proportional to the diameter of the drop. The proportionality constant is found to be 2×10^3 s^{-1} if the diameter is given in meters.

(a)

What is the drag force on a raindrop of diameter 2 mm traveling at its terminal speed? What is the terminal speed?

(b)

Find the form of the drag coefficient $h(Re)$ if the drag force is of the form

$$F = \rho_{air} u^2 d^2 h(Re)$$

u is the velocity of the drop, d the diameter of the drop and the viscosity of air is $\eta = 2 \times 10^{-5}$ Pa s and the Reynolds number is defined as $Re = \rho_{air} du/\eta$.

3:

(a)

A shallow lake is 100 km long and 50 m deep. Sometimes one can observe an end-to-end oscillation of the surface called a seiche. What is the period of this oscillation?

(b)

Describe the physical significance of the Reynolds number, the Froude number and the Mach number. When are these numbers important in problems where one wishes to establish physical similarity between a system and a scale model of the system.

4:

It is desired to pump a fluid with density ρ through a horizontal pipe with a volume flow rate $\dot{V}$. The pipe has a circular cross section with diameter d . Assuming that the frictional head is given by Darcy's law

$$\frac{h_f}{l} = \frac{fu^2}{2dg}$$

where u is the average flow velocity and f can be taken to be constant.

(a)

What is the shear stress between the fluid and the walls of the pipe.

(b)

What is the power required to pump the fluid.

5:

(a)

A dam has a rectangular cross section with height h and width w. The density of the material in the dam is ρ_D, while the density of the water is ρ_W. The water behind the dam is filled to level x. There is water seepage under the dam producing a lift pressure equal to the water pressure at depth h. What is the minimum height to width ratio w/h to prevent the dam from toppling?

(b)

Assuming $x = 1.0\ h$, $w/h = 1$, what is the minimum density ratio ρ_D/ρ_W to prevent the dam from toppling?

Some formulas and values of constants

$\rho_{water} = 1000$ kg m^{-3}, $\rho_{air} = 1.29$ kg m^{-3}.

$g = 9.81$ m s^{-2}.

The thermal expansion coefficient of water is $\alpha = 2 \times 10^{-4}$ K^{-1}.

The compressibility of water is $K = 4.5 \times 10^{-10}$ Pa^{-1}.

The shallow water wave velocity is $v = \sqrt{gh}$, where h is the depth.

16 Suggested topics for term paper.

The essays are due on the last day of classes.

One type of essay which you may write is to review a topic, e.g. in the style of a serious newsmagazine such as "The Economist". Alternatively, you may use a more informal style and instead describe some calculations and experiments of your own. You may collaborate with a friend on researching a topic, but each student should write her/his own essay. Typically, when you do the background reading for one of these topics there will be some things you understand and some that you don't. When you write the report concentrate on the parts you understand. Try not to exceed 10 typed pages including figures.

1. **Patterned ground.** Research how geometrical patterns are created by convection cells in water logged soil that undergoes repeated freeze-thaw cycles. A possible starting point is the pictorial essay by Kranz et al. [31], see also [18, 25, 28] .

2. **The wake pattern from a boat**. Discuss what determines the pattern of wakes after a moving boat [16, 59].

3. **Why a fluid flows faster when the tube is pinched**. A Jearl Walker "Amateur Scientist" note [58] describes a very counter-intuitive experiment by Gatzek. You may wish to verify the results by experiments of your own.

4. **How and why bacteria swim, or life at low Reynolds number**. For a number of readable articles on the motion of bacteria see [6, 7, 9, 10, 11, 24, 45]. A book on hydrodynamics at low Reynolds number is [29]. Collective effects may also be important. For the rôle of convection, and how bacteria and algae can set up convection cells in order to get more nutrients see [30]. There is much more material in the references cited above than you can cover in a single term paper, so you should specialize on some aspect of the problem, but try not too rely too much on a single reference.

5. **Fluid uptake by a blotter.** For a simple demonstration of scaling behavior that you can reproduce yourself see [5, 42].

6. **The flight of beer bubbles**. You may start with the articles of Crawford [17] or Coghlan [14]. Shafer and Zare [49] claim that "The deformation, oscillation, wandering and ultimate breakup of a rising, rapidly growing bubble make beer bubble dynamics a rich phenomenon, worthy of study".

7. **Convective instabilities**. A possible starting point is the review by Normand et al. [40]. Much of this article contains advanced material which requires a strong math background. But, there is a long introductory section which contain many historical references which could form the basis of an interesting paper.

8. **The hydraulic jump in the kitchen sink.** A jet of water that falls on a horizontal plane will spread out radially in a thin layer. This layer will be surrounded by a circular hydraulic jump [26, 41, 60]. Discuss the factors that determine the position of the jump and its height. Supplement your theories with experimental observations of your own.

9. **Tidal bores.** A possible starting point is the Scientific American article by Lynch [32].

10. **Bathtub vortices.** Will the bathroom vortex spin in the opposite sense in the Northern and the Southern hemispheres. If so what will happen at the equator? For studies on these problems see [53, 8, 55, 15, 54, 33, 37, 50].

11. **How objects float**. The problem of finding the orientation of a floating object can be solved for non-cylindrical objects too. The results are sometimes counter-intuitive, see [20, 21].

12. **Wind energy conversion**. For a possible starting point when discussing modern windmills see [19].

13. **Motion of a falling paper**. Why is the motion of a falling sheet of paper so irregular and unpredictable? For a very idealized model see [52].

14. **Traffic flow**. This is a topic where a kind of "hydrodynamics" influences our daily lives. The book by Braun et al. [13] contains a series of readable sections on how to model traffic flow. Arnott and Small [3] give some interesting examples of how attempted remedies for traffic congestion may have the opposite of the intended effect. Migowsky et al. [39] describe cellular automata modeling of the effects of individual driving patterns and accidents affect overall flow in a freeway model. An important book is that of Prigogine and Herman [44]. As expected, considering the economic importance of the problem, the literature on traffic flow is enormous, and you should be able to find many useful references yourself.

17 Values of some constants

atomic mass unit,1 a.m.u.= 1.67×10^{-27} kg
Ångström, 1Å= 10^{-10} m
Avogadro's number, $N_A = 6.023 \times 10^{23}$
Boltzmann constant, k_B, $1.38 \times 10^{-23} JK^{-1}$
Calorie, cal, $4.184J$
Velocity of light, $c = 3 \times 10^8$ m s^{-1}

References

[1] E. Achenbach 1971, *Influence of surface roughness on the cross-flow around a circular cylinder*, J. Fluid Mech. **46** 321.

[2] E. Achenbach 1972, *Experiments on the flow past spheres at very high Reynolds numbers*, J. Fluid Mech. **54** 565.

[3] R. Arnott and K. Small 1994, *The economics of traffic congestion* American Scientist **82** 446-55.

[4] D.J. Barber and R. Loudon 1989, *An introduction to the properties of condensed matter*, Cambridge: University Press.

[5] C.P. Bean et al. 1991, *Fluid uptake by a blotter*, Amer. J. Phys. **59** 533.

[6] H.C. Berg 1975, *Bacterial behavior*, Nature **254** 389.

[7] H.C. Berg and E.M. Purcell 1977, *Physics of Chemoreception*, Biophysical Journal **20** 193.

[8] A.M. Binnie 1964, *Some experiments on the bath-tub vortex* J. Mech. Eng. Sci. **6** 256-7.

[9] R.P. Blakemore and R.B. Frankel 1981, *Magnetic Navigation in Bacteria*, Scientific American December Issue 42 .

[10] R. Blakemore 1975, *Magnetotactic Bacteria* Science **190**, 377.

[11] R.P. Blakemore, R.B. Frankel and A J Kalmijn, 1980, *South seeking magnetotactic bacteria in the southern hemisphere* Nature **286** 384.

[12] L. Boltzmann 1992, *A German professor's trip to el dorado*, Physics Today January issue p44.

[13] M. Braun, C. S. Coleman and D. A. Drew 1978, eds. *Differential equation models* Modules in applied mathematics Vol. 1 Springer Verlag.

[14] Coghlan A. 1991, *The flight of beer bubbles from first principles*, New Scientist, 9 Nov. issue p24.

[15] W. Cope 1983, *The bathtub vortex* Am. Sci. **71** 566.

[16] F.S. Crawford 1984, *Elementary derivation of the wake pattern of a boat*, American J. Physics **52** 782.

[17] F. S. Crawford 1990, *Hot water, fresh beer and salt*, Am. J. Phys. **58** 1033.

[18] J.G. Dash 1989, Science 246,1591 (1989) *Thermomolecular pressure in surface melting motivation for frost heave* , Science **246**, 1591.

[19] J.B. Dragt 1993, *Wind energy conversion* Europhysics News **24** 27.

[20] Paul Erdös, Gerard Schibler and Rou Herndon 1992, *Floating equilibrium of symmetrical objects and the breaking of symmetry Part 1 Prisms*, Amer. J. Phys **60** 335.

[21] Paul Erdös, Gerard Schibler and Rou Herndon 1992, *Floating equilibrium of symmetrical objects and the breaking of symmetry Part 2: The cube the octahedron and the tetrahedron*, Amer. J. Phys **60** 345.

[22] R.P. Feynman, R.B. Leighton and M.Sands 1964, *The Feynman lectures on physics*, Addison-Wesley.

[23] B.H. Flowers and E. Mendoza 1970, *Properties of matter*, John Wiley&Sons.

[24] R B Frankel, R.P Blakemore, FF Torres de Araujo, DMS Esquivel and J. Danon 1981, *Magnetotactic bacteria at the geomagnetic equator*, Science **212** 1269.

[25] K.J. Gleason et al.1986, *Geometrical aspects of sorted patterned ground in recurrently frozen soil*, Science, **232**, 216.

[26] R.P. Godwin 1993, *The hydraulic jump (shocks and viscous flow in the kitchen sink)*, Amer. J. Phys. **61** 829.

[27] E.S.R Gopal 1974, *Statistical mechanics and properties of matter*, John Wiley&Sons.

[28] B. Hallet 1990, *Self-organization in freezing soils:from microscopic ice lenses to patterned ground*, Can. J. Phys. **68** 842.

[29] John Happel and Howard Brenner 1973, *Low Reynolds number hydrodynamics-with special applications to particulate media*, Nordhoff second rev. ed.

[30] J.O. Kessler and N.A. Hill 1995, *Microbial consumption patterns* in *Spatio-temporal Patterns*, Ed. P.E. Cladis and P.Palffy-Muhoray, SFI studies of complexity, Addison Wesley.

[31] W.B. Kranz, K.J. Gleason and N. Caine 1988, *Patterned Ground*, Scientific American December issue, page 68.

[32] D.K. Lynch 1982, *Tidal bores*, Scientific American October issue page 146.

[33] D.L. Kelly, B.W. Martin and E.S. Taylor 1964, *A further note on the bathroom vortex*, J. Fluid Mech. **19** 539-42.

[34] C. Kittel and H. Krömer 1980, *Thermal physics*, W.H. Freeman and Company.

[35] M. J. Klein 1990, *The physics of J, Willard Gibbs in his time*, Physics Today Sept. issue p40.

[36] I. G. Main 1984, *Vibrations and waves in physics*, Cambridge University Press.

[37] A.W. Marris 1967, *Theory of the bathtub vortex*, J.Appl. Mech. **34** 11-15.

[38] B. S. Massey 1989 *Mechanics of fluids*, fifth ed., van Nostrand.

[39] S. Migowsky, T. Wanschura, P. Rujian, 1994, *Competition and cooperation on a toy autobahn model*, Zeitschrift fur Physik B-Condensed Matter. **95**, 407-414.

[40] C. Normand, Y. Pomeau and M. G. Velarde 1977, *Convective instability: A physicist's approach*, Reviews of Modern Physics **49** 581.

[41] R.G. Olsson and E.T Turkdogdan 1966, *Radial spread of a liquid stream on a horizontal plate*, Nature **211** 813.

[42] G.C. Peiris and K. Tennakone 1980, Amer. J. Physics **48** 415.

[43] D. Pneuli and C. Gutfinger 1992, *Fluid Mechanics*, Cambridge University Press.

[44] I. Prigogine and R. Herman 1971. *Kinetic Theory of Vehicular Traffic.* Elsevier, New York.

[45] E.M. Purcell 1977,*Life at low Reynolds number*, Amer. J. Phys. **45** 3.

[46] L. Reichl 1980, *A modern course in statistical physics*, University of Texas Press.

[47] A. Roshko 1961, *Experiments on the flow past a circular cylinder at very high Reynolds number*, J. Fluid Mech **10** 345.

[48] R.H. Sabersky, A.J. Acosta and E.G. Hauptmann 1989, *Fluid flow; a first course in fluid mechanics*, third edition McMillan .

[49] N. Shafer and R. Zare 1991, Physics Today October issue.

[50] Merwin Sibulkin 1983, *A note on the bathroom vortex and the earth's rotation*, Am. Sci. **71** 352-3.

[51] D. Tabor 1991, *Gases, liquids and solids and other states of matter*, third edition Cambridge University Press.

[52] Y. Tanabe and K. Kaneko 1994, *Behavior of a falling paper*, Phys. Rev. Lett. **73** 1372-5.

[53] A. H. Shapiro 1962, *Bath-tub vortices*, Nature **196** 1080-81.

[54] M. Sibulkin 1962, *A note on the bathroom vortex*, J.Fluid Mech. **14** 21-4.

[55] L.M. Trefethen, R.W. Bilger, P.T. Fink, R.E. Luxton and R.I. Tanner 1965, *The bath-tub vortex in th southern hemisphere*, Nature **207** 1084-5.

[56] Loup Verlet 1967, *Computer experiments on classical fluids. I Thermodynamical properties of Lennard-Jones Molecules*, Phys Rev **159** 98.

[57] Loup Verlet 1968, Computer experiments on classical fluids. II Equilibrium correlation functions Phys. Rev. **165** 201 .

[58] J. Walker 1984, *Why a fluid flows faster when the tube is pinched*, Scientific American "The Amateur Scientist" October issue.

[59] J. Walker 1988, Scientific American "The Amateur Scientist" February issue 124.

[60] E.J. Watson 1964, *The radial spread of a liquid jet over a horizontal plane*, J. Fluid Mech. **20** 481.

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