# The Department of Physics \& Astronomy <br> The University of British Columbia 

# PHYSICS 314 FLUIDS 

Summary of lectures by Birger Bergersen

Department of Physics and Astronomy
University of British Columbia

Telephone: (604) 822-2754
birger@physics.ubc.ca

Revised July 2001

## Contents

I INTRODUCTION ..... 4
1 Course organization ..... 4
2 Gases, liquids and solids. ..... 5
2.1 Problems ..... 10
3 Probabilities and averages. ..... 13
3.1 Outcome of coin tosses ..... 17
3.2 Some properties of Gaussians ..... 18
3.3 Problems ..... 19
II Kinetic theory ..... 20
4 Maxwell velocity distribution. ..... 22
4.1 Most probable distribution, Boltzmann factor ..... 24
4.1.1 Isothermal atmosphere ..... 28
4.1.2 Diatomic gas in an electric field ..... 28
4.2 Problems ..... 29
5 Molecular collisions. ..... 30
5.1 Knudsen gas ..... 31
5.2 Random walk ..... 32
5.3 Problems ..... 34
6 Problems related to diffusion. ..... 35
6.1 Molecules hitting a surface ..... 35
6.2 Effusion ..... 38
6.3 Diffusion ..... 39
6.4 Problems ..... 44
7 Thermal conductivity and viscosity. ..... 46
7.1 Non-Newtonian fluids ..... 50
7.2 Summary of results from kinetic theory ..... 50
7.3 Problems ..... 52
III HYDRODYNAMICS ..... 52
8 Laminar flow through a pipe. ..... 53
8.1 Poiseuille flow ..... 53
8.2 Pipe flow for a non-Newtonian fluid ..... 54
8.3 Problems ..... 57
9 Dimensional analysis. Reynolds number. ..... 58
9.1 Problems ..... 62
10 Fluids at rest. Some problems of stability. ..... 63
10.1 How does a log float? ..... 64
10.2 Problems ..... 70
11 Ideal fluids. Euler and Bernoulli equation. ..... 72
11.1 Darcy's law ..... 75
11.2 Problems ..... 81
12 Water waves. ..... 82
12.1 Gravity and capillary waves ..... 83
12.2 The hydraulic jump ..... 93
12.3 Problems ..... 96
13 Compressible flow. ..... 98
13.1 Stagnation temperature ..... 99
13.2 Sound waves ..... 100
13.3 Shock waves ..... 103
13.4 Problems ..... 107
14 Physical similarity and modeling. ..... 108
14.1 Flow past a submerged object: ..... 109
14.2 Convection ..... 110
14.3 Convection in the atmosphere ..... 110
14.4 Rayleigh Bénard convection ..... 112
14.5 Wave resistance of a ship ..... 113
14.6 Problems ..... 115
IV Review ..... 116
15 Old exams and other problems. ..... 116
16 Suggested topics for term paper. ..... 143
17 Values of some constants ..... 146
Bibliography ..... 146Index151

## Part I

## INTRODUCTION

## 1 Course organization

The purpose of the course is to present a survey of the properties of fluids, a term which describes both gases and liquids. We start by describing the different forms of matter and common types of interaction between atoms and molecules. We then move on to kinetic theory. Observable quantities, such as thermodynamic variables, are interpreted as averages over probability distributions. We present, for gases, the Maxwell-Boltzmann velocity distribution, and the more general idea of a Boltzmann factor. Results for diffusion, thermal conduction, and viscosity are derived from the concepts of collision cross section and mean free path. We then show how these ideas are modified when working with dense fluids. Generally, diffusion has the effect of smoothing out concentration inhomogeneities. However, diffusive processes can also lead to pattern formation with important consequences in geology and biology.

Most of the rest of the course is spent on an introduction to hydrodynamics. We start out with laminar flow, including a discussion of non-Newtonian fluids, continue with the Bernoulli equation, and move on to convection, and turbulence. We will also discuss waves in fluids, including capillary and gravity waves, tidal waves in a shallow channel, sound waves and shocks. In the case of flow problems one often finds that it is infeasible to work all the way from first principles, and there will be some discussion of modeling and the use of dimensional analysis. An attempt has been made to stick with SI units throughout the course, although I have not been completely successful in this respect. I have also included a rather long review section consisting of old exam problems. Some of these come with solutions, others don't!

It is expected that students taking this course will have had a course in thermal physics such as PHYS 313 (or an equivalent course in Chemistry or Geophysics). The way it is commonly expounded, the theory of fluids will present formidable mathematical obstacles. I try in this course to emphasize qualitative aspects avoiding excessive formalism. Nevertheless, use of the material in the standard second year mathematics courses (MATH 200, 221, 215) cannot be avoided. It would also be an advantage, but it is not required, for students to have had a second year mechanics course such as PHYS 216.

There will be 6 regular assignments, a midterm and a final exam. I
will also require a short term paper. The basic marking scheme is $20 \%$ assignments, $20 \%$ term paper, $20 \%$ midterm and $40 \%$ final.

I don't know of any available text that covers all the material in this course. For an elementary account of the properties of matter and kinetic theory I recommend the books by Flowers and Mendoza [23] and Tabor [51]. The student may also find the books by Gopal [27], Kittel and Krömer [34] and the more advanced book by Reichl [46] useful for the kinetic theory part. When it comes to hydrodynamics I found that chapters 40 and 41 in Volume II of the Feynman lectures [22] offer a fresh point of view. The book by Massey [38] contains much information and is easy to read, and Pnueli and Gutfinger [43] and Sabersky et al. [48] are also helpful. The text by Main [36] contains an excellent discussion of waves. Other more specialized references will be given when appropriate.

I have "borrowed" freely from old exams prepared by my colleagues David Balzarini, Meyer Bloom, Roland Cobb, Harold Davis, Dan Murray and Murray Neuman when selecting problems and examples. I am grateful to Michael Craddock for bringing errors and typos to my attention.

The course description in the UBC Calendar is
PHYS 314 (3) Fluids. Kinetic theory: Diffusion, viscosity and sound waves. Introduction to hydrodynamics: Laminar flow, capillary and gravity waves, convection and turbulence. Dimensional analysis. Pre-requisite: 1 of PHYS 203, PHYS 313.
Corequisite: MATH 215. [3-0-0]

## 2 Gases, liquids and solids.

The materials which we meet in our daily lives, air water, metals semiconductors, rocks, plastics, can all be classified as being either gases, liquids or solids. In this course we generally consider matter to be made up of molecules. These are not the fundamental building blocks of nature since molecules in turn are made up of atoms, which contain electrons, protons and neutrons. The latter are composites of quarks and gluons etc. The reason we work with molecules as the basic building block is that they tend to stay intact over a reasonable time under the conditions encountered in the study of materials. We shall be concerned with the forces between molecules in a fairly general way. The origin of the forces lies in the behavior of the electronic charge clouds around the nuclei. For this reason quantum mechanics is required to compute such forces, but once the forces are known
one can often to a good approximation use Newtonian mechanics to work out their effect. We will in this course concentrate on achieving a qualitative understanding of the properties of fluids, and quite idealized models are often adequate for this purpose. It is not only very difficult to obtain a quantitative understanding of detailed macroscopic properties from microscopic forces, but when the goal can be achieved, the calculations are often quite tedious.

A distinction between gases, liquids and solids can be made, because sharp transitions take place between them. For example, the substance water can freeze to form ice or boil to form a vapor, and the processes can be reversed through melting or condensation. Our climate is to a very large extent determined by the transitions between these phases.

The least ordered phase is the gas. For the present purposes a gas is a substance which fills the space available. Gases have relatively low density and high compressibility. A gas has no rigidity, i.e. there is no resistance to changes in shape if the change is carried out slowly. The viscosity, or resistance to shear, is low. The molecules are usually far apart compared to their diameter, and they can move over distances corresponding to many diameters without experiencing collisions. The spatial distribution of molecules is close to random. The gas phase occurs at low pressures and high temperatures. At ambient pressures all known substances condense if cooled to low temperatures. The kinetic theory of gases is a favorite tool to describe the equilibrium properties of gases. As we shall see, we can also use this theory to derive expressions for important quantities such as the, thermal conductivity, viscosity and diffusivity which describe how gases respond when disturbed from equilibrium.

When a gas is cooled at ambient pressures it will as a rule condense to form a liquid. This phase is characterized by a higher density than the gas phase. At atmospheric pressure a density difference of a factor of $10^{3}$ is common when a vapor condenses to form a liquid. If the pressure is lowered a liquid may evaporate or boil, but it is a self-bound system, which cannot expand to fill an arbitrarily large volume. Since the density is not strongly dependent on the pressure the compressibility is low. The molecules in a liquid are densely packed, one cannot describe a liquid as a collection of independent particles that only occasionally interact through collisions. Instead there is considerable amount of local molecular order caused by the dense packing. Liquids have no rigidity . A liquid can change its shape without any cost of bulk energy if the change is done slowly enough. If we generate flow with nonuniform flow velocity there will be frictional or
viscous forces which we characterize by the viscosity. Liquids generally have a higher viscosity than gases.

The transition from a gas to a liquid can take place suddenly by a phase transition. At the transition liquid and vapor coexist. For mechanical stability the coexisting phases must have the same pressure so at a given temperature the liquid has a certain vapor pressure. There will also be a latent heat associated with the change of phase. As the temperature is increased the vapor pressure will increase, the latent heat will decrease and the discontinuity in density will decrease. Finally, at the critical point the latent heat and the density discontinuity vanishes and there is a continuous change between a liquid and a gas. Thus, apart from the phase transition, which only occurs in a certain temperature range, the difference between a liquid and a gas is not fundamental; they are both fluids. A liquid is a high density fluid, while a gas is a low density fluid, and a vapor is a gas whose temperature is below the liquid gas critical point.

When a fluid is cooled it will eventually solidify. (An interesting exception is helium, which at low pressures stays fluid down to the lowest recorded temperatures, and instead undergoes a transition to a superfluid.) Solids tend to have densities and compressibilities which are similar to those of liquids, and the latent heat in a transition from a liquid to a solid is usually less than for the liquid gas transition. However, solids are rigid and do not change their shape easily under the action of small forces, while a liquid shows no resistance to shear if applied slowly enough.

Glasses are forms of matter intermediate between solids and liquids. They are hard and brittle, i.e. their mechanical properties are those of a solid. Atomically they are non crystalline supercooled liquids, and they are generally believed to be only metastable and should eventually crystallize. However, the time required may be extremely long.

For a pure substance, the pressure $P$ is in principle determined, if the number of particles $N$, per unit volume $V$, and the temperature $T$ are known. The functional form of $P$ is known as an equation of state and can be plotted as an equation of state surface (see figure 1). Ideally, one would like to be able to compute the equation of state, and to predict the location of interesting points such as the critical point where the vapor liquid coexistence line ends, and the triple point where the solid, liquid and vapor phases coexist. This will require a knowledge of the interaction between the molecules. If the molecules are ions the dominant forces are electrostatic in origin, and the interaction energy between two charges $q_{1}$ and $q_{2}$ a distance


Figure 1: Equation of state surface $P\left(\frac{V}{N}, T\right)$.


Figure 2: Lennard-Jones potential.
$r$ apart is given by the Coulomb potential

$$
\begin{equation*}
V(r)=\frac{q_{1} q_{2}}{4 \pi \epsilon_{o} r} \tag{1}
\end{equation*}
$$

The interaction between the atoms within a molecule is typically covalent. The covalent energy is quantum mechanical in origin and comes about because electrons occupy orbitals that are shared between the different atoms of a molecule. Covalent forces can be very strong and they are directional, i.e. they depend not only on the distance between atoms but on bond angles. Covalent forces are usually not important between molecules.

There are two main contributions to the potential energy $V(r)$ associated with two neutral molecules a distance $r$ apart. If the two molecules get so close together that the electronic clouds of the nuclei significantly overlap, there is a strong repulsion due to the Pauli exclusion principle. At larger separation the dominant contribution comes from van der Waals attraction, which is caused by fluctuating dipole forces, and the van der Waals potential energy is proportional to the inverse sixth power of the separation. A commonly used interaction energy for spherical molecules is the Lennard-Jones potential

$$
\begin{equation*}
V(r)=4 \epsilon\left[\left(\frac{\sigma}{r}\right)^{12}-\left(\frac{\sigma}{r}\right)^{6}\right] \tag{2}
\end{equation*}
$$

This potential is shown in figure 2. It is easy to verify that the potential has a minimum $-\epsilon$ at separation $r_{0}=2^{1 / 6} \sigma$. The interaction is repulsive for $r<r_{0}$ and is attractive for $r>r_{0}$. There is no particular reason to use the value 12 for the exponent associated with the repulsive term, and other forms are sometimes employed. The $1 / r^{6}$ behavior of the attractive part of the interaction is, however, realistic for neutral molecules. The force $\mathbf{F}$ between two spherical molecules can be expressed in terms of the gradient of the potential

$$
\begin{equation*}
\mathbf{F}(\mathbf{r})=-\hat{\mathbf{r}} \frac{d V(r)}{d r}=-\nabla V(\mathbf{r}) \tag{3}
\end{equation*}
$$

Here $\hat{\mathbf{r}}$ is a unit vector in the direction of $\mathbf{r}$.
Computer simulations using Molecular dynamics (see e.g. Verlet [56, 57] ) have demonstrated that the Lennard-Jones potential (2) can give a quite adequate description of equation of state related properties of "simple" substances such as the rare gas elements argon, krypton and xenon.

For molecules, more complicated potentials than (2) are required, but molecular dynamics simulations still often work quite well for equilibrium properties. When it comes to dynamical properties one faces the formidable obstacle that even a "state of the art" simulation describing millions of collisions per molecule will correspond to only a fraction of a microsecond in real time. For this reason we will not attempt a completely microscopic description of fluids in this course, but we will instead take a more phenomenological approach. We will not be interested in the equilibrium properties of materials as much as in how a system responds when disturbed from equilibrium. A major theme will be to relate material properties such as diffusivity and viscosity to macroscopic flow properties.

For an overview of condensed matter physics at an introductory level I recommend the text by Barber and Loudon [4].

### 2.1 Problems

Problem 2-1.

When solving problems it is frequently practical to work in reduced units in order to identify characteristic scales of energy, length and time. To illustrate this point consider two small magnets of mass $m$, arranged so that they repel each other, as shown in figure 3. The magnets are restrained from moving horizontally.


Figure 3: Figure to Problem 2.1

The interaction energy between two magnetic dipoles is proportional to the inverse third power of the distance. Write $c / z^{3}$ for the magnetic interaction energy, with $c$ a constant.
(a) Plot the magnetic, gravitational and total potential energy against height $z$. Use $(m g)^{3 / 4} c^{1 / 4}$ as unit of energy and $(c / m g)^{1 / 4}$ as unit of height.
(b) Find the equilibrium height of the top magnet.
(c) Calculate the frequency of small oscillations about the equilibrium position.

## Problem 2-2.

The two columns to the left in table 1 show data for the density of water in the range 0 to $10^{\circ} \mathrm{C}$, while the two columns on the right present more detailed data in the range 0 to $0^{\circ} \mathrm{C}$.
(a) Fit the two data sets to the expression

$$
\rho(T)=a+b T+c T^{2}
$$

where $T$ is the temperature in ${ }^{\circ} \mathrm{C}$ and $a, b, c$ are constants.
(b) Make separate plots for the two temperature regions and show the data points and the two fits.

You should find that both fits appear to work well in their respective regions. However, the overall fit will not agree well with data in the smaller region. Similarly, the fit to the smaller region doesn't give a very good overall fit.


Figure 4: Figure to Problem 2.3

## Problem 2-3.

A sealed cylindrical vessel (see figure 4) is in contact with a heat bath at constant temperature $T$. A friction-less airtight piston of weight $m g$ divides the container into two volumes $V_{1}$ and $V_{2}=V-V_{1}$. There are $N$ ideal gas atoms in both the top and in the bottom partitions.
(a). Find the equilibrium height of the piston.
(b). The ideal gas in the two partitions is replaced by steam. At a certain temperature the bottom partition is found to contain a puddle of water coexisting with its vapor. Which of the following statements may be true at equilibrium:

1. The top partition contains water in coexistence with its steam.
2. The top partition contains only steam.
3. The top partition contains only liquid water.

Table 1: Data for the density of water

| $T\left[{ }^{\circ} \mathrm{C}\right]$ | $\rho\left[\mathrm{kg} \mathrm{m}^{-3}\right]$ | $T\left[{ }^{\circ} \mathrm{C}\right]$ | $\rho\left[\mathrm{kg} \mathrm{m}^{-3}\right]$ |
| ---: | :--- | ---: | :--- |
| 0 | 999.841 | 0 | 999.841 |
| 10 | 999.700 | 1 | 999.902 |
| 20 | 998.203 | 2 | 999.941 |
| 30 | 995.646 | 3 | 999.965 |
| 40 | 992.21 | 4 | 999.973 |
| 50 | 988.04 | 5 | 999.965 |
| 60 | 983.21 | 6 | 999.941 |
| 70 | 977.79 | 7 | 999.902 |
| 80 | 971.80 | 10 | 999.700 |
| 90 | 965.31 |  |  |
| 100 | 958.35 |  |  |

## 3 Probabilities and averages.

Because of the important role of statistics, we will need to review some needed concepts before we can proceed with our main subject.

There are two main approaches to the problem of estimating how often possible outcomes of random events will occur. We may predict the frequency of allowed outcomes in repeated experiments from a priori (first principles) knowledge of the probabilities of contributing factors, making use of properties of permutations, combinations and binomial coefficients. Alternatively, we may be able to observe the possible outcomes as they occur, and we may wish to estimate a posteriori (after the fact) their probability from measured frequencies of occurrence.

In the latter case, one way to proceed is to plot the data in a bar chart (or histogram). As an example let us consider the distribution of speeds $v$ of a system of $N$ particles. We divide the range of $v$ into intervals or bins of width $\Delta$. The speed of a given particle lies in the $i^{\prime}$ th bin if its speed is between $v_{i}$ and $v_{i}+\Delta$. We then count the number $n_{i}$ in each bin and plot the result. If we only include a moderate number of particles, the resulting bar chart will typically have a ragged shape as in figure 5 a . On the other hand if we measure the speeds of a much larger number of particles it is likely to have a more regular shape as indicated in figure 5 b .

According to the law of large numbers the relative frequency after many
measurements will almost always be close to a limiting value. This limiting value is the probability $p_{i}$, of any given particle being in the $i$-th bin

$$
\begin{equation*}
p_{i}=\lim _{N \rightarrow \infty} \frac{n_{i}}{N} \tag{4}
\end{equation*}
$$

An important property of probabilities is the fact that if two possible events $i$ and $j$ are exclusive the probability that one or the other happens is

$$
\begin{equation*}
p(i . o r . j)=p(i)+p(j) \tag{5}
\end{equation*}
$$

Similarly if two possible events $i$ and $j$ are independent the probability that they both happen is

$$
\begin{equation*}
p(i . a n d . j)=p(i) p(j) \tag{6}
\end{equation*}
$$

If the two events are not independent they are said to be correlated. The probability distribution for exclusive events satisfies the normalization condition

$$
\begin{equation*}
1=\sum_{i} p_{i} \tag{7}
\end{equation*}
$$

The histogram represents a discrete distribution. The particle speed can in general take on a continuous range of values. We obtain the continuous probability distribution by taking the limit

$$
\begin{equation*}
p\left(v_{i}\right)=\lim _{\Delta \rightarrow 0} \frac{p_{i}}{\Delta} \tag{8}
\end{equation*}
$$

The continuous probability distribution associated with the particle speed may then look like figure 8(a) in the next chapter. The probability that the speed of a particle is between $v_{i}$ and $v_{i}+d v$ is $\mathrm{p}\left(v_{i}\right) d v$, and the continuous probability distribution satisfies the normalization condition

$$
\begin{equation*}
\int p(v) d v=1 \tag{9}
\end{equation*}
$$

Equation (9) is a statement of the fact that since all the particles must have some speed, the probabilities will add up to one. The mean speed is given by

$$
\begin{equation*}
\langle v\rangle=\int v p(v) d v \tag{10}
\end{equation*}
$$

Let $f(v)$ be some property associated with the speed of a particle (e.g. the particle kinetic energy is given by $f(v)=\frac{1}{2} m v^{2}$, where $m$ is the particle mass). The average value of $f(v)$ is then

$$
\begin{equation*}
\langle f(v)\rangle=\int f(v) p(v) d v \tag{11}
\end{equation*}
$$



Figure 5: Speed distribution for (a) a large (b) a small number of particles.


Figure 6: The normal distribution.

In general, we will use angular brackets $\rangle$ to describe averages. When applied to a probability distribution, the mean is the average outcome. Since we are dealing with probabilities, actual outcomes will typically deviate or fluctuate from the average. Let $x$ be a stochastic variable, i.e. a quantity that can take on different values with some probability. A useful estimate of a typical deviation of the value of $x$ from the mean is the standard deviation $\sigma$ or root mean square fluctuation

$$
\begin{equation*}
\sigma=\sqrt{\left\langle(x-\langle x\rangle)^{2}\right\rangle}=\sqrt{\left\langle x^{2}\right\rangle-\langle x\rangle^{2}} \tag{12}
\end{equation*}
$$

A closely related quantity is the variance: var $=\sigma^{2}$. A probability distribution which occurs in many context is the Gaussian or normal distribution, which can be expressed in terms of the mean and variance as

$$
\begin{equation*}
g(x)=\frac{1}{\sqrt{2 \pi(v a r)}} \exp \left[\frac{-(x-\langle x\rangle)^{2}}{2(v a r)}\right] \tag{13}
\end{equation*}
$$

The normal distribution is plotted in figure 6. The importance of this distribution is related to the central limit theorem of statistics:

Suppose $x_{i}$ is a random variable and $X$ is the sum of $N$ independent such variables all with the same mean $\langle x\rangle$ and variance
var $=\left\langle x^{2}\right\rangle-\langle x\rangle^{2}$ In the limit that $N$ gets very large the probability distribution for $X$ will be Gaussian with mean $N\langle x\rangle$ and variance $V=N$ var.

Note that we are not assuming that the probability distribution for $x$ is Gaussian - only that it has a finite mean and a variance.

### 3.1 Outcome of coin tosses

A coin is tossed $N$ times. We associate the value $x=+1$ with "heads" and $x=0$ with "tails". The average value $\langle x\rangle$ of $x$ is then $\frac{1}{2}$ if the coin is unbiased. The average value of $x^{2}$ is also $\left\langle x^{2}\right\rangle=\frac{1}{2}$. The variance associated with a single toss is thus var $=\left\langle x^{2}\right\rangle-\langle x\rangle^{2}=\frac{1}{4}$. After $N$ tosses the number of times heads has come up will be $X$. By the central limit theorem the mean value of $X$ will be $\langle X\rangle=N / 2$, the variance will be $V=N / 4$, and the probability distribution in the limit of large $N$ will be

$$
\begin{equation*}
p(X)=\sqrt{\frac{2}{\pi N}} \exp \left[\frac{-(2 X-N)^{2}}{2 N}\right] \tag{14}
\end{equation*}
$$

This result can be derived independently using the binomial distribution. There are two possible outcomes of each toss - heads or tails. After $N$ tosses there are thus $2^{N}$ possible outcomes. Of these

$$
\begin{equation*}
\frac{N!}{X!(N-X)!} \tag{15}
\end{equation*}
$$

correspond to the outcome $X$. The probability distribution for $X$ is thus in general

$$
\begin{equation*}
p(X)=\frac{N!}{2^{N} X!(N-X)!} \tag{16}
\end{equation*}
$$

If $N$ is very large we can use the Stirling approximation

$$
\begin{equation*}
\ln (z!) \simeq \frac{1}{2} \ln (2 \pi)+\left(z+\frac{1}{2}\right) \ln z-z \tag{17}
\end{equation*}
$$

as $z \rightarrow \infty$. It is left as an exercise for the reader to use (16) and (17) to derive (14).

Not all distributions have a finite mean or a variance. A famous counterexample, discussed already in the early days of probability theory, is Bernoulli's St. Petersburg paradox. Imagine that you are allowed to play the following
game against "the bank". The banker tosses a fair coin. If the outcome is "tails" you are given a ducat (or whatever they used for money in St. Petersburg towards the end of the 18 th century). If the outcome is "heads" the coin is tossed again until the outcome is "tails". If tails come up in the second trial you get 2 ducats. If tails only comes up after $n$ trials you get $2^{n-1}$ ducats.

The probability that tails comes up for the first time after $n$ trials is the probability $2^{-n+1}$ that the first $n-1$ trials yielded "heads" times the probability $\frac{1}{2}$ that the next trial gives heads, or $2^{-n}$. The mean profit is then

$$
\begin{equation*}
\langle \$\rangle=\frac{1}{2}+\frac{1}{4} 2+\ldots \frac{1}{2^{n}} 2^{n-1}+\ldots=\frac{1}{2}+\frac{1}{2}+\frac{1}{2}+\ldots=\infty \tag{18}
\end{equation*}
$$

Thus, although the probability distribution is well defined with a finite value for each possible outcome, the mean is infinite! How much would you be willing to pay to be allowed to take part in this game?

In what follows we will not encounter further "pathological" probability distributions of this type and the central limit theorem will be "safe".

### 3.2 Some properties of Gaussians

When dealing with Gaussian distributions the following integrals are useful

$$
\begin{gather*}
\int_{-\infty}^{\infty} \exp \left(-a x^{2}\right) d x=2 \int_{0}^{\infty} \exp \left(-a x^{2}\right) d x=\sqrt{\frac{\pi}{a}}  \tag{19}\\
\int_{-\infty}^{\infty} x^{2} \exp \left(-a x^{2}\right) d x=2 \int_{0}^{\infty} x^{2} \exp \left(-a x^{2}\right) d x=\frac{1}{2 a} \sqrt{\frac{\pi}{a}}  \tag{20}\\
\int_{-\infty}^{\infty} x \exp \left(-a x^{2}\right) d x=0  \tag{21}\\
\int_{0}^{\infty} x \exp \left(-a x^{2}\right) d x=\frac{1}{2 a} \tag{22}
\end{gather*}
$$

The error function is defined as

$$
\begin{equation*}
\operatorname{erf}(x)=\frac{1}{\sqrt{\pi}} \int_{-x}^{x} \exp \left(-s^{2}\right) d s \tag{23}
\end{equation*}
$$



Figure 7: Error function
and has the properties

$$
\begin{equation*}
\operatorname{erf}(0)=0 ; \quad \operatorname{erf}(1)=1 ; \quad \operatorname{erfc}(x) \equiv 1-\operatorname{erf}(x) \tag{24}
\end{equation*}
$$

where $\operatorname{erfc}(x)$ is called the complementary error function. Tables of the error function are readily available. Plots of the error function is given in figure 7.

### 3.3 Problems

## Problem 3-1.

a). Plot a histogram for the probabilities of the outcome of 5 tosses of a fair coin (5 heads, four heads one tail, etc.). Are there any significant differences between results obtained using the binomial distribution and the Gaussian approximation?
b). Estimate the probability of 1050 heads in 2000 tosses of a fair coin.
c). Estimate the probability of more than 1050 heads in 2000 tosses of a fair coin.

## Problem 3-2.

Derive (14) from (16) using the Stirling formula (17) assuming $N$ to be large.

## Problem 3-3.

(a). A person puts with equal probability either a red or a white marble into an urn. Subsequently a red marble is added to the urn. The urn is shaken and one of the two marbles is selected randomly and picked up from the urn. The selected marble is red. What is the probability that the remaining marble is red?

## Problem 3-4.

In a television game show the player is presented with three doors and told that there is a brand new luxury car behind one of them. Behind the other two doors there is nothing of value. The player is asked to guess behind which of the three doors the car is hidden and will win the car if the guess is correct. The player will tell the guess to the host. The host will then open one of the other doors. There is nothing of value behind that door. Finally the host asks if the player will change her or his guess. Is this a good idea for the player to do?

## Part II

## Kinetic theory

Thermal energy is associated with the random motion of molecules. Let us first consider a gas of low density. The molecules will move in straight trajectories until they collide with other molecules or the walls of the gas container. After a few collisions it becomes practically impossible to relate the velocity and position of the molecules to the corresponding quantities at an earlier time. The difficulty is not just the enormous amount of data required to describe a large number of particles. A more fundamental problem is the fact that after a few collisions the positions and the velocities of the particles become extremely sensitive to the initial conditions. A very similar situation occurs when throwing an unbiased die or tossing a coin. In principle, it should be possible to predict the outcome of the toss using Newton's laws and the initial velocity and position. In practice, the calculation will not be able to predict the behavior of real coins, because initial
conditions that give rise to radically different outcomes are so close together that it is difficult to specify the initial conditions and parameters of the problem with sufficient accuracy. This type of motion has been described as chaotic. Each particle is just as likely to move in any direction as in any other, and the speed of the particles is frequently changing. Therefore, if we wish to describe the properties of a gas in terms of the interaction between molecules, a statistical description is essential.

We distinguish between the random motion of a molecule and bulk (ordered) movement. An example of the latter is the flight of a solid object such as a pebble thrown in the air. A situation which is more difficult to analyze is the flow of a liquid. A "bulk" velocity can then be defined for each small region of the fluid. In all cases the separation between bulk and random motion can be made by an averaging process. The bulk motion will survive an average over molecular velocity, while the random motion averages out as noise. This averaging is reasonably straightforward when the flow is laminar (streamlined). However, when a flow is turbulent what appears as bulk motion on one length scale may be random on a larger scale.

We now assume that we can distinguish bulk and random motion, and we associate the concept of temperature with the random motion. A molecule will in general possess both kinetic and potential energy. In a gas, at low density, the molecules spend most of their time far apart. The interaction energy falls off rapidly with distance and the potential energy can often be neglected. In a solid or a liquid the potential energy plays a key role in holding the substance together, and the potential energy is often much larger in magnitude than the kinetic energy. Only the kinetic energy contributes to the temperature, however. Also, it must be stressed that not all forms of motion can be considered to be thermal, any bulk motion must be eliminated by transforming to a frame of reference in which the average velocity is zero.

Although only the random motion contributes to the temperature, bulk motion can be converted or dissipated into heat, and thus cause a rise in the temperature. From the point of view of thermodynamics the state with bulk motion is then a non-equilibrium state, and the degradation of ordered motion into heat through frictional forces constitutes approach to equilibrium. Kinetic theory is concerned with a number of dissipative processes such as diffusion, thermal conduction and viscous flow.

## 4 Maxwell velocity distribution.

The foundation of statistical physics was laid about 100 years ago. Important names ${ }^{1}$ in this context are Gibbs, Boltzmann and Maxwell. Boltzmann and Maxwell were among other things interested in the probability distribution for velocities in an ideal gas. Let $\mathbf{v}$ be the vector velocity of a particle with speed $v=|\mathbf{v}|$. If the gas is at rest, and in equilibrium with respect to its container, they found that the probability distribution for the velocity of each particle is given by

$$
\begin{equation*}
p(\mathbf{v})=\left(\frac{m}{2 \pi k_{B} T}\right)^{\frac{3}{2}} \exp \left(-\frac{m v^{2}}{2 k_{B} T}\right) \tag{25}
\end{equation*}
$$

We interpret this formula by saying that the probability that the velocity $\mathbf{v}=\left\{v_{x}, v_{y}, v_{z}\right\}$ has components between $v_{x}$ and $v_{x}+d v_{x}, v_{y}$ and $v_{y}+d v_{y}$, $v_{z}$ and $v_{z}+d v_{z}$ is

$$
\begin{equation*}
p(\mathbf{v}) d v_{x} d v_{y} d v_{z} \equiv p(\mathbf{v}) d^{3} v \tag{26}
\end{equation*}
$$

The probability distribution (25) is normalized

$$
\begin{equation*}
\int d^{3} v p(\mathbf{v})=1 \tag{27}
\end{equation*}
$$

as can be seen by substituting $a=\frac{m}{2 k_{B} T}$ into (25) and using (20) to get

$$
\begin{equation*}
\int d^{3} v \exp \left(-\frac{m v^{2}}{2 k_{B} T}\right)=4 \pi \int_{0}^{\infty} v^{2} d v e^{-a v^{2}}=\left(\frac{2 \pi k_{B} T}{m}\right)^{\frac{3}{2}} \tag{28}
\end{equation*}
$$

If there are $N$ particles in the gas the average number of particles with velocity in an interval $d^{3} v$ surrounding $\mathbf{v}$ is $f(\mathbf{v})=N p(\mathbf{v})$.

Before deriving the Maxwell velocity distribution let us introduce some related quantities. First let us consider the distribution of speeds (magnitude of the velocity). All velocities in a shell of "volume"

$$
d^{3} v=4 \pi v^{2} d v
$$

have speeds between $v$ and $v+d v$. Since the same number of particles are counted in each case we must have for the speed distribution $g(v)$

$$
g(v) d v=f(\mathbf{v}) d^{3} v
$$

[^0]

Figure 8: a) Maxwell speed distribution, b) energy distribution.
or

$$
\begin{equation*}
g(v)=4 \pi v^{2} f(\mathbf{v})=N v^{2} \sqrt{\frac{2}{\pi}}\left(\frac{m}{k_{B} T}\right)^{\frac{3}{2}} \exp \left(-\frac{m v^{2}}{2 k_{B} T}\right) \tag{29}
\end{equation*}
$$

Similarly, the distribution $n(u)$ of energies can be obtained by substituting $u=\frac{1}{2} m v^{2}$ into

$$
n(u) d u=g(v) d v
$$

giving the energy distribution

$$
\begin{equation*}
n(u)=g(v) \frac{d v}{d u}=\frac{2 \pi N}{\left(\pi k_{B} T\right)^{\frac{3}{2}}} \sqrt{u} \exp \left(-\frac{u}{k_{B} T}\right) \tag{30}
\end{equation*}
$$

The Maxwell speed and energy distribution are plotted in figure 8. The most probable speed $v_{p}$ is the one for which $\mathrm{g}(\mathrm{v})$ has a maximum. If we differentiate (29) we find that $d g / d v=0$ for

$$
\begin{equation*}
v_{p}=\sqrt{\frac{2 k_{B} T}{m}} \tag{31}
\end{equation*}
$$

The average speed can be obtained by using (22) to evaluate the integral

$$
\begin{equation*}
\langle v\rangle=\frac{1}{N} \int_{0}^{\infty} v g(v) d v=\sqrt{\frac{8 k_{B} T}{\pi m}} \tag{32}
\end{equation*}
$$

Similarly, the mean square speed is

$$
\begin{equation*}
\left\langle v^{2}\right\rangle=\frac{1}{N} \int_{0}^{\infty} v^{2} g(v) d v=\frac{3 k_{B} T}{m} \tag{33}
\end{equation*}
$$

¿From (33) we see that the mean kinetic energy is given by

$$
\begin{equation*}
\frac{1}{2} m\left\langle v^{2}\right\rangle=\frac{3}{2} k_{B} T \tag{34}
\end{equation*}
$$

This result is familiar from the thermodynamics of the ideal gas. The r.m.s. (root mean square) speed is commonly used in estimates of "typical" speeds and is given by

$$
\begin{equation*}
v_{r m s}=\sqrt{\frac{3 k_{B} T}{m}} \tag{35}
\end{equation*}
$$

We observe that the Maxwell speed distribution is skew and that there are slight differences between the most probable, the average and the r.m.s. speeds. As an example we next evaluate the average value of $1 / v$ :

$$
\begin{equation*}
\left\langle\frac{1}{v}\right\rangle=\frac{1}{N} \int_{0}^{\infty} d v \frac{1}{v} g(v)=\sqrt{\frac{2}{\pi}}\left(\frac{m}{k_{B} T}\right)^{\frac{3}{2}} \int_{0}^{\infty} d v v \exp \left(-\frac{m v^{2}}{2 k_{B} T}\right) \tag{36}
\end{equation*}
$$

which we evaluate using (22) to

$$
\begin{equation*}
\left\langle\frac{1}{v}\right\rangle=\sqrt{\frac{8 m}{\pi k_{B} T}} \int_{0}^{\infty} d x x e^{-x^{2}}=\sqrt{\frac{2 m}{\pi k_{B} T}} \tag{37}
\end{equation*}
$$

note that $\langle 1 / v\rangle$ is not the same as $1 /\langle v\rangle$ ! The fact that the average value $\langle f(v)\rangle$ of some function $f$ of the velocity is not equal to the average velocity $f(\langle v\rangle)$ will be a source of trouble when we derive formulas for transport properties such as diffusivity, viscosity and thermal conductivity. In "hand waving" derivations of these quantities the above distinction is often forgotten and formulas found in the literature sometimes differ by factors of, say, 2 or $\sqrt{\pi}$.

### 4.1 Most probable distribution, Boltzmann factor

We now derive the result (25). To do this we assume that all velocities are a priori equally probable, except that there is an overall restriction:
for an ideal gas at temperature $T$ the average translational kinetic energy should be given by (34). We shall see that the Maxwell velocity probability distribution represents the most probable distribution of velocities. Here "most probable" means that it is the distribution that can be achieved in the most possible different ways, subject to our assumptions. In order to avoid the complication of continuous distributions we first consider the case where there are $r$ possible outcomes and $N$ trials. In a typical experiment or realization, the $i$ 'th outcome will occur $n_{i}$ times. The number of ways this result can be achieved is

$$
\begin{equation*}
W\left(n_{1} \ldots n_{i} \ldots n_{r}\right)=\frac{N!}{n_{1}!n_{2}!\ldots n_{i}!\ldots n_{r}!} \tag{38}
\end{equation*}
$$

We wish to find $n_{i}, i=1,2 . . r$, that maximize (38) subject to the constraint that

$$
\begin{equation*}
\sum_{i=1}^{r} n_{i}=N \tag{39}
\end{equation*}
$$

We will also associate an energy $u_{i}$ with the $i^{\prime}$ th outcome and assume that

$$
\begin{equation*}
\sum_{i=1}^{r} n_{i} u_{i}=U_{t o t} \tag{40}
\end{equation*}
$$

where $U_{\text {tot }}$ is related to the temperature. To facilitate the algebra we introduce

$$
\begin{equation*}
S=k_{B} \ln W \tag{41}
\end{equation*}
$$

$S$ has a most important physical interpretation as the entropy ${ }^{2}$. Whenever $W$ has a maximum as a function of the arguments so has $S$. The constraints (39), (40) can be incorporated using the method of Lagrange's multipliers and our task is to maximize

$$
\begin{equation*}
S-\sum_{i=1}^{r}\left(\lambda+\alpha u_{i}\right) n_{i} \tag{42}
\end{equation*}
$$

where $\lambda$ and $\alpha$ must be chosen so as to satisfy the constraints. Differentiation with regards to $n_{i}$ then gives using (38), (41)

$$
\frac{\partial S}{\partial n_{i}}=-k_{B} \frac{\partial \ln \left(n_{i}!\right)}{\partial n_{i}}=\lambda+\alpha u_{i}
$$

[^1]Stirling's formula (17) (assuming $n_{i}$ to be large) gives

$$
\ln n_{i}+\frac{1}{2 n_{i}}+\ldots=-\left(\lambda+\alpha u_{i}\right) / k_{B}
$$

and neglecting the second term on the left hand side leads to

$$
\begin{equation*}
n_{i}=\exp \left(-\frac{\lambda}{k_{B}}\right) \exp \left(-\frac{\alpha u_{i}}{k_{B}}\right) \tag{43}
\end{equation*}
$$

Returning to the velocity distribution we let $n_{i}$ be the number of particles whose velocities are within $d^{3} v$ of $\mathbf{v}$ and have $u_{i}=\frac{1}{2} m v^{2}$. If we choose $\lambda$ so that

$$
N\left(\frac{m}{2 \pi k_{B} T}\right)^{\frac{3}{2}}=e^{-\lambda / k_{B}}
$$

and let $\alpha=1 / T$, we recover the Maxwell velocity distribution (25),

$$
n_{i}=f(\mathbf{v})=N p(\mathbf{v})
$$

Since we have already shown that the Maxwell energy distribution is correctly normalized and gives the correct average kinetic energy, this concludes the proof.

The identification of the Lagrange multiplier $\alpha$ with the inverse temperature $1 / T$ is a very important result and allows us to generalize to situations where the energy of a certain state $i$ is not just due to the kinetic energy. In general, if a set of states are a priori equally probable, but there is an overall restriction on the average energy, the probability that a particular state will be selected is

$$
\begin{equation*}
p_{i}=\text { const } \times e^{-\beta u_{i}} \tag{44}
\end{equation*}
$$

where we use the standard notation $\beta=\frac{1}{k_{B} T}$. The exponential factor in (44) is called the Boltzmann factor and the constant is given by the normalization condition that the probabilities add up to 1

$$
\begin{equation*}
\text { const }=\frac{1}{Z}=\frac{1}{\sum_{i} \exp \left(-\beta u_{i}\right)} \tag{45}
\end{equation*}
$$

The sum $Z$ over the Boltzmann factors is called the partition function and plays an important role in statistical physics.

The argument leading to $(44,45)$ is based on the identification of the logarithm of the number of ways that a certain thermodynamic state can
be realized with the entropy (41). We will also need the second law of thermodynamics. In one formulation, this law states that the change in internal energy, dU, is given by

$$
\begin{equation*}
d U=T d S \tag{46}
\end{equation*}
$$

in an infinitesimal process taking a system from one equilibrium state to another, if the process involves no work or transfer of matter. Equation (46) implies that

$$
\begin{equation*}
T=\left.\frac{\partial U}{\partial S}\right|_{N, n o w o r k} \tag{47}
\end{equation*}
$$

Let us assume that we are primarily interested in a system $\{1\}$ that

- is in contact with a heat bath $\{2\}$ at temperature $T$.
- has thermodynamic states $i$ with energy $u_{i}$ that can be realized in $w_{i}$ a priori equally probable ways.
- does not exchange work or matter with the reservoir.

The bath is much larger than system $\{1\}$, so that changes in to the energy of $\{1\}$ will have only a slight effect on the thermodynamic variables of the heat bath.

The number of ways that are compatible with energy $U_{2}=U_{t o t}-u_{i}$ is from (4.17)

$$
\begin{equation*}
W_{2}=e^{S_{2}\left(U_{2}\right) / k_{B}}=e^{S_{2}\left(U_{t o t}-u_{i}\right) / k_{B}} \tag{48}
\end{equation*}
$$

where $S_{2}\left(U_{2}\right)$ is the entropy of the bath when its energy is $U_{2}$.
We want to find the probability distribution, $P\left(u_{i}\right)$, for system $\{1\}$. Since system $\{2\}$ is much larger than $\{1\}, u_{i} \ll U_{\text {tot }}=u_{i}+U_{2}$, where $U_{\text {tot }}$ is the total energy of the two subsystems. We have

$$
P\left(U_{i}\right)=\frac{w_{i} W_{2}\left(U_{t o t}-u_{i}\right)}{\sum_{i} w_{i}\left(u_{i}\right) W_{2}\left(U_{t o t}-u_{i}\right)}
$$

We expand $S_{2}$ in a Taylor series

$$
\begin{equation*}
S_{2}\left(U_{t o t}-u_{i}\right)=S_{2}\left(U_{t o t}\right)-u_{i} \frac{\partial S_{2}}{\partial U}+\cdots \tag{49}
\end{equation*}
$$

If the heat bath is big enough we can cut off the Taylor series after the first two terms and find using (47)

$$
W_{2}\left(U_{\text {tot }}-u_{i}\right)-=W_{2}\left(U_{\text {tot }}\right) \exp \left[\frac{-u_{i}}{k_{B} T}\right]=\text { const } \times e^{-\beta u_{i}}
$$

We conclude

$$
\begin{equation*}
P\left(u_{i}\right)=\text { const } \times w_{i}\left(u_{i}\right) e^{-\beta u_{i}} \tag{50}
\end{equation*}
$$

where, normalizing the distribution, we find that the const is given by (45). The factor $e^{-\beta u_{i}}$ is the Boltzmann factor. When a system is in contact with a heat bath at a certain temperature, all possible microstates of the system are no longer equally likely. Instead, the Boltzmann factor acts as a weight factor biasing the distribution towards states with lower energy.

### 4.1.1 Isothermal atmosphere

To illustrate the use of Boltzmann factors, consider a gas in a gravitational field. We choose the potential energy to be zero at ground level. The potential energy at height $z$ will be $m g z$ with $m$ the mass of a molecule. If the temperature of the atmosphere is constant, the number of molecules per unit volume at height $h$ must be proportional to $\exp (-\beta m g z)$. With the density at ground level given as $n_{o}$ the density at height $z$ will be $n=$ $n_{o} \exp (-\beta m g z)$. The isothermal assumption is not a realistic one, because the atmosphere is generally not in thermal equilibrium, being exposed to radiation from the sun and in turn radiating excess heat into space. We will discuss a better approximation, the adiabatic atmosphere, later in this course, in the context of convection.

### 4.1.2 Diatomic gas in an electric field

Our next example is a dilute gas made up of diatomic molecules that have a permanent electric dipole moment $\vec{\mu}$. The number of dipoles is $N$, the electric field is $E$, the volume is $V$ and the temperature is $T$. The orientational energy of the dipole is (see figure 9)

$$
\begin{equation*}
U=-\vec{\mu} \cdot \mathbf{E}=-\mu E \cos \theta \tag{51}
\end{equation*}
$$

We can define an element of solid angle $d \Omega=\sin \theta d \theta d \phi$ by specifying that the polar angle is between $\theta$ and $\theta+d \theta$, while the azimuthal angle is between $\phi$ and $\phi+d \phi$. All solid angles are a priori equally probable and the probability that the dipole is oriented in a given direction is

$$
p(\theta, \phi) d \Omega=\text { const } \times \exp (-\beta U) d \Omega=A \exp (\beta \mu E \cos \theta) \sin \theta d \theta d \phi
$$

The constant $A$ is determined by the normalization condition

$$
A \int_{o}^{2 \pi} d \phi \int_{o}^{\pi} \sin \theta d \theta \exp (\beta \mu E \cos \theta)=1
$$



Figure 9: Dipole in an electric field
giving

$$
A=\frac{\beta E \mu}{4 \pi \sinh (\beta E \mu)}
$$

The orientational probability distribution is then

$$
\begin{equation*}
p(\theta, \phi) d \Omega=\frac{\beta \mu E \exp (\beta \mu E \cos \theta)}{4 \pi \sinh (\beta \mu E)} \sin \theta d \theta d \phi \tag{52}
\end{equation*}
$$

### 4.2 Problems

## Problem 4-1.

If the ratio of oxygen to nitrogen is $1 / 4$ at sea level, what do you expect the ratio to be at an altitude of 3000 m (assume an isothermal atmosphere and $T=0^{0} \mathrm{C}$ )? The mass of an oxygen molecule is 32 a.m.u., and the mass of a nitrogen molecule is 28 a.m.u., 1 a.m.u $=1.67 \times 10^{-27} \mathrm{~kg}$, and $k_{B}=1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}$.

## Problem 4-2

A certain gas consists of non-interacting molecules which are completely free to move in the $x$ - and $y$-directions, but the molecules are constrained in their motions in the $z$-direction by a harmonic oscillator potential. The kinetic plus potential energy of a single molecule of mass $m$ and velocity $v=\left(v_{x}, v_{y}, v_{z}\right)$
and position $r=(x, y, z)$ is given by

$$
\begin{equation*}
\text { Energy }=\text { K.E. }+ \text { P.E. }=\frac{m}{2}\left[v_{x}^{2}+v_{y}^{2}+v_{z}^{2}\right]+\frac{\kappa}{2} z^{2} \tag{53}
\end{equation*}
$$

(a). If the gas is in equilibrium at temperature $T$, what can one say about the following average values:

$$
\left\langle v_{x}\right\rangle,\left\langle v_{y}\right\rangle,\left\langle v_{z}\right\rangle,\left\langle v_{x}{ }^{2}\right\rangle,\left\langle v_{y}{ }^{2}\right\rangle,\left\langle v_{z}{ }^{2}\right\rangle,\left\langle v^{2}\right\rangle,\langle z\rangle,\left\langle z^{2}\right\rangle
$$

(b). What is the heat capacity of one mole of this gas?

## Problem 4.3

A zipper contains a large number $N$ of links. The energy associated with opening a link is $\epsilon>0, \mathrm{~N} \epsilon \gg k_{B} T$. The zipper can only unzip from one end while the other end remains closed. Find the mean and the variance of the number of open links.

## Problem 4.4

Calculate the root mean square fluctuation of the kinetic energy of a monatomic ideal gas about its mean value $\frac{3}{2} N K_{B} T$.

## 5 Molecular collisions.

We now discuss the effect of collisions between the molecules. Consider a gas of $n=\frac{N}{V}$ particles per unit volume. We will at first make the idealization that the molecules are hard spheres of radius $r$. The closest approach of two molecules is then a diameter $d=2 r$. The sphere of radius $d$ surrounding a given molecule is called its sphere of influence. We assume that the molecules form an ideal gas with a Maxwellian velocity distribution (25). Suppose $\mathbf{v}_{1}$ and $\mathbf{v}_{2}$ are the velocities of two molecules. Their relative velocity is then $\mathbf{v}_{r}=\mathbf{v}_{1}-\mathbf{v}_{2}$ and their center of mass velocity is $\mathbf{v}_{c m}=\frac{1}{2}\left(\mathbf{v}_{1}+\mathbf{v}_{2}\right)$. Two molecules will collide in a time interval $d t$ if the center of the second molecule happens to be located within a cylinder of height $v_{r} d t$ and radius $d$ as shown in figure 10 . The probability $\phi d t$ that a given molecule will collide with any molecule in the gas during a time interval $d t$ is then

$$
\begin{equation*}
\langle\phi\rangle d t=n \pi d^{2}\left\langle v_{r}\right\rangle d t \tag{54}
\end{equation*}
$$

where $\phi$ is the collision frequency and $\left\langle v_{r}\right\rangle$ the average relative speed. It remains to find the probability distribution for relative velocities and to
calculate $\left\langle v_{r}\right\rangle$. We have for the kinetic energy of a pair of particles

$$
\begin{equation*}
U_{12}=\frac{1}{2} m v_{1}^{2}+\frac{1}{2} m v_{2}^{2}=\frac{1}{2} M v_{c m}^{2}+\frac{1}{2} \mu v_{r}^{2} \tag{55}
\end{equation*}
$$

where $M=2 m$ is the pair mass and $\mu=\frac{1}{2} m$ is the reduced mass. All relative velocities are a priori equally probable, and we find in analogy with (25) for the distribution of relative velocities

$$
\begin{equation*}
p\left(\mathbf{v}_{r}\right)=\left(\frac{\mu}{2 \pi k_{B} T}\right)^{\frac{3}{2}} \exp \left(-\frac{\mu v_{r}^{2}}{2 k_{B} T}\right) \tag{56}
\end{equation*}
$$

This distribution differs from the Maxwellian velocity distribution only in that the reduced mass replaces the particle mass in the formula. Thus from (32) we have for the mean relative speed

$$
\begin{equation*}
\left\langle v_{r}\right\rangle=\sqrt{\frac{8 k_{B} T}{\pi \mu}}=\sqrt{2}\langle v\rangle \tag{57}
\end{equation*}
$$

The factor $\pi d^{2}$ in (54) is the equatorial cross-section of the sphere of influence. In practice the hard sphere assumption is too restrictive. We can, however, replace $\pi d^{2}$ by an effective collision cross section $\sigma$. We thus find for the collision frequency

$$
\begin{equation*}
\langle\phi\rangle=\sqrt{2} n \sigma\langle v\rangle=4 n \sigma \sqrt{\frac{k_{B} T}{\pi m}} \tag{58}
\end{equation*}
$$

A concept which is closely related to the collision frequency is the mean free path $l=\langle\lambda\rangle$, which is the average distance traveled between collisions

$$
\begin{equation*}
l=\langle\lambda\rangle=\left\langle\frac{v}{\phi}\right\rangle=\frac{1}{n \sigma \sqrt{2}} \tag{59}
\end{equation*}
$$

Note that the speed factors out in (59); the mean free path is the same for all the particles in a Maxwellian gas. In particular, the mean free path in an ideal gas of a given density is independent of the temperature. This sometimes makes $l$ the most convenient parameter to describe the collision properties of a gas.

### 5.1 Knudsen gas

It is useful to establish the order of magnitude of the mean free path under different conditions. Let us first find the mean free path of molecules of


Figure 10: Collision tube.
diameter $4 \AA$ in an ideal gas at atmospheric pressure and $T=0^{\circ} \mathrm{C}$. The number of particles per unit volume is $n=\frac{P}{k_{B} T}$, giving with $P=1.013 \times 10^{5}$ $\mathrm{N} \mathrm{m}^{-2}$

$$
\begin{equation*}
l=\frac{k_{B} T}{P \pi d^{2} \sqrt{2}}=\frac{1.38 \times 10^{-23} \times 273.16}{101.3 \times 10^{3} \times 16 \pi \times 10^{-20} \sqrt{2}}=5.210^{-8} \mathrm{~m}=520 \AA \tag{60}
\end{equation*}
$$

If we now redo the calculation for a pressure $P=10^{-9}$ atm we find that the mean free path will be by $10^{9}$ times larger or 52 m . A Knudsen gas is a gas which is at such low pressure that the mean free path is longer than the dimensions of ordinary scientific equipment used to contain it.

### 5.2 Random walk

For the discussion that follows we need to work out the probability $p(r)$ that a particle can travel a distance $r$ without undergoing any collisions. The probability that a collision occurs between $r$ and $r+d r$ is proportional to $d r$. Let the proportionality constant be $\alpha$. The probability $p(r+d r)$ that no collisions occurs in an interval of length $r+d r$ is thus $p(r+d r)=p(r)(1-\alpha d r)$ or

$$
\begin{equation*}
d p=-\alpha p d r \tag{61}
\end{equation*}
$$

Since $p(0)=1$ we find that $p(r)$ satisfies the Poisson distribution $p(r)=$ $\exp (-\alpha r)$. The mean distance between collisions is

$$
l=\int_{0}^{1} r d p=-\int_{0}^{\infty} r \frac{d p}{d r} d r=\alpha \int_{0}^{\infty} r \exp (-\alpha r) d r=\frac{1}{\alpha}
$$

giving

$$
\begin{equation*}
p(r)=\exp (-r / l) \tag{62}
\end{equation*}
$$

The picture of molecular motion which we have constructed is one in which each molecule travels in a straight line between collisions. We will make the approximation that after each collision the molecule starts in a random direction. This type of motion is called a random walk. Before considering the general 3-dimensional random walk problem let us consider the following simpler problem:

A particle moves with equal probability to the right or to the left in steps of unit length along a straight line. What is the probability that after $t$ time steps the particle has moved a distance $s$ ?

If $n_{r}$ steps are taken to the right and $n_{l}$ steps are taken to the left the net distance traveled is $s=n_{r}-n_{l}$ if we let $s>0$ correspond to motion to the right. For the desired outcome we must therefore have

$$
n_{r}=\frac{1}{2}(t+s) \quad n_{l}=\frac{1}{2}(t-s)
$$

¿From (16) the probability of this outcome is

$$
\begin{equation*}
p=\frac{t!2^{-t}}{\left[\frac{1}{2}(t-s)\right]!\left[\frac{1}{2}(t+s)!\right]} \tag{63}
\end{equation*}
$$

Example: An old exam question based on this formula is the following: A drunk leaves a beer parlor and takes random steps 1 m long in one dimension. The entrance to another pub is 5 m from the first pub. What is the probability that the drunk has passed the second pub after 8 steps?

Solution: The sought-after outcome can be reached in two ways: (i) by 8 steps in the direction of the second pub, or (ii) by seven steps towards the second pub and one step back. The total probability is the sum of the probabilities for these two outcomes

$$
p=\frac{8!2^{-8}}{8!0!}+\frac{8!2^{-8}}{7!1!}=\frac{9}{256}
$$

Example: Assume that a particle with mean free path $l$ can travel either to the left or to the right with speed $v$. After each collision the particle moves with equal probability to the left or to the right. What is the probability per unit length that the particle has traveled a distance $x$ in time $t$ ? What is the root mean square (r.m.s.) distance traveled in 1 second for a particle with speed $1000 \mathrm{~m} / \mathrm{s}$ and mean free path 1 mm ?

Solution: The mean time interval per collision is $l / v$ while the distance $x$ on the average corresponds to $s=x / l$ steps. Problems in kinetic theory normally involve situations in which enough time has elapsed for the particle to have taken many steps, e.g. in the problem before us the particle will on the average have undergone one million collisions. We can then make use of the central limit theorem of chapter 3. The mean distance traveled per step is zero while the mean square distance is

$$
\left\langle\lambda^{2}\right\rangle=\in_{0}^{1} r^{2} d p=-\int_{0}^{\infty} r^{2} \frac{d p}{d r} d r=\frac{1}{l} \int_{0}^{\infty} r^{2} \exp \left(-\frac{r}{l}\right) d r=2 l^{2} N=2 l v t
$$

The mean distance traveled after $N=t v / l$ steps is still zero but the variance is thus $2 l v t$, and the r.m.s. distance is $\sqrt{2 l v t}$. With the values provided, this distance is $\sqrt{2}$ meter. The probability distribution of distances traveled is from (13)

$$
\begin{equation*}
p(x, t)=\frac{1}{\sqrt{4 \pi l v t}} \exp \left(\frac{-x^{2}}{4 l v t}\right) \tag{64}
\end{equation*}
$$

We will use this result later when we discuss diffusion.

### 5.3 Problems

## Problem 5-1.

A small amount of a gas in which each molecule has mass $m_{1}$ is released into a gas in which there are $n$ molecules per unit volume. The mass per molecule in the second gas is $m_{2}$. The temperature is $T$ and the cross section for collisions of molecules of the two kinds is $\sigma$. Derive formulas for the mean relative velocity of molecules of the two kinds, and for the mean free path of molecules of the first kind. (Hint: (56) is still valid, but the reduced mass will no longer be $\frac{1}{2} m$ ).

## 6 Problems related to diffusion.

### 6.1 Molecules hitting a surface

In many applications one needs a formula for the mean number of particles that collide with a given surface per unit time. We assume that the spatial distribution of particles is uniform. Consider an element of volume immediately above the surface. One half of the particles in this volume have a velocity component in the direction of the surface. Let $\left\langle v_{z}\right\rangle$ be the average value of this velocity component for the particles traveling in the "right" direction. The number of particles hitting the surface per unit area is then $\frac{1}{2}\langle | v_{z}| \rangle n$. We put $v_{z}=v \cos \theta$. Averaging over solid angles for which $0<\theta \leq \pi / 2$ we find

$$
\left\langle v_{z}\right\rangle=\langle v\rangle \frac{\int_{0}^{\frac{\pi}{2}} \cos \theta \sin \theta d \theta}{\int_{0}^{\frac{\pi}{2}} \sin \theta d \theta}=\frac{1}{2}\langle v\rangle
$$

Thus, the number of particles hitting a surface per unit area and time is

$$
\begin{equation*}
j_{N}=\frac{n\langle v\rangle}{4} \tag{65}
\end{equation*}
$$

which is the required formula. We next give some examples of problems making use of this result.

In order to illustrate the usefulness of (65) in surface adsorption problem consider the following example. A clean surface is suddenly exposed to oxygen at $25^{\circ} \mathrm{C}$. Assume that every molecule striking the surface is adsorbed, and that the effective diameter of the oxygen molecule is $4 \AA$. The problem is to estimate the time interval before the surface is covered with a layer of oxygen if the pressure $P$ is (a) $10^{-6}$ Torr, (b) $10^{-10}$ Torr.

Solution: The mean number of molecules striking the surface per unit area is given by (65) where

$$
\begin{aligned}
n & =\frac{P}{k_{B} T} \\
\langle v\rangle & =\sqrt{\frac{8 k_{B} T}{\pi m}}
\end{aligned}
$$

If the atoms in the oxygen layer are arranged in a triangular lattice as in figure 11, we see that the number $\nu$ of molecules per


Figure 11: The triangular lattice.
unit surface area is $\frac{1}{2}$ molecule/(area of triangle), or

$$
\begin{equation*}
\nu=\frac{2}{\sqrt{3} d^{2}} \tag{66}
\end{equation*}
$$

Collecting terms we find

$$
\begin{equation*}
t=\frac{\nu}{j_{N}}=\frac{2}{\sqrt{3} d^{2}} \times \frac{4 k_{B} T}{P} \times \sqrt{\frac{\pi m}{8 k_{B} T}} \tag{67}
\end{equation*}
$$

Substituting $k_{B}=1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}, T=298 \mathrm{~K}, m=32 \times$ $1.67 \times 10^{-27} \mathrm{~kg}, d=4 \times 10^{-10} \mathrm{~m}, 1$ Torr= $133 \mathrm{Nm}^{-2}$ we find for the time it takes to cover the surface in the two cases (a) 2 s , (b) 5.5 hours.

Note: The oxygen monolayer need not form in the triangular structure of the figure. The assumption that all molecules will stick if they hit the surface may not be realistic. Also, during the process some molecules will evaporate etc. In this type of problem there may therefore be many "correct" answers "in the same ball park".

As a further example consider a gas at temperature $T$ inside a container. What is the average kinetic energy of molecules colliding with the container walls?

Solution: We assume a Boltzmann velocity distribution. The average kinetic energy of the molecules is from (34) given by
$\frac{3}{2} k_{B} T$. Since all the molecules from time to time hit the walls, one might expect that the average energy of the molecules striking the walls is the same. In a more careful analysis however, one would note that the faster molecules hit the walls more often and this produces a bias in the average.

Let us consider a wall at $z=0$, with the gas located below the wall $(z<0)$. In the time interval $d t$, molecules with velocity $v_{z}>0$ will hit the wall if they start within a distance $v_{z} d t$ of the wall. The probability that a molecule has velocity component $v_{z}$ is proportional to the Boltzmann factor

$$
\exp \left(-\frac{m}{2 k_{B} T} v_{z}^{2}\right)
$$

We assume that the gas is uniformly distributed spatially. The average value of $v_{z}{ }^{2}$ of the molecules hitting the wall will then be

$$
\left(v_{z}^{2}\right)_{a v}=\frac{\int_{0}^{\infty} v_{z}^{2} v_{z} d v_{z} \exp \left(-\frac{m v_{z}{ }^{2}}{2 k_{B} T}\right)}{\int_{0}^{\infty} v_{z} d v_{z} \exp \left(-\frac{m v_{z}{ }^{2}}{2 k_{B} T}\right)}
$$

If we introduce the new variable $y=\frac{m}{2 k_{B} T} v_{z}{ }^{2}, d y=\frac{m}{k_{B} T} v_{z} d v_{z}$, we get

$$
\begin{equation*}
\left(v_{z}^{2}\right)_{a v}=\frac{2 k_{B} T}{m} \frac{\int_{0}^{\infty} y e^{-y} d y}{\int_{0}^{\infty} e^{-y} d y}=\frac{2 k_{B} T}{m} \tag{68}
\end{equation*}
$$

The average value of $v_{x}{ }^{2}$ and $v_{y}{ }^{2}$ will be the same for molecules hitting the wall $\mathrm{z}=0$, as for the gas as a whole

$$
\begin{equation*}
\left(v_{x}^{2}\right)_{a v}=\left(v_{y}^{2}\right)_{a v}=\frac{k_{B} T}{m} \tag{69}
\end{equation*}
$$

giving

$$
\begin{equation*}
(K . E .)_{a v}=\frac{m}{2}\left[\left(v_{x}^{2}\right)_{a v}+\left(v_{y}^{2}\right)_{a v}+\left(v_{z}^{2}\right)_{a v}\right]=2 k_{B} T \tag{70}
\end{equation*}
$$

The average kinetic energy of the molecules hitting the walls is thus different from the average kinetic energy of the molecules in the gas. The fact that it is often hard to avoid relying on
biased samples is the source of many fallacies which occur when statistical arguments are used in ordinary life ("lies, damned lies and statistics").

As a further example consider a thin disk of radius $r$ that is embedded in a monatomic gas of pressure $P$ and temperature $T_{g}$. The disk is maintained at a temperature $T_{d}$. We assume that the gas molecules hit the disk with an average kinetic energy $2 k_{B} T_{g}$ and leave with the average kinetic energy $2 k_{B} T_{d}$. Also, assume that radiative losses are negligible and that there is no convection. We wish to estimate the rate at which heat is transferred between the gas and the disk.

Solution: The rate at which heat is transferred can be written $\dot{Q}=(\#$ of particles per unit area hitting the disk $) \times($ area $) \times($ the average energy transfer in each collision). We find

$$
\begin{equation*}
\dot{Q}=\frac{n\langle v\rangle}{4} \times 2 \pi r^{2} \times 2 k_{B}\left(T_{g}-T_{d}\right) \tag{71}
\end{equation*}
$$

with

$$
\langle v\rangle=\sqrt{\frac{8 k_{B} T_{g}}{\pi m}}
$$

and

$$
n=\frac{P}{k_{B} T_{g}}
$$

### 6.2 Effusion

Consider a container of volume $V$ which is evacuated. Suppose the container develops a small pinhole of area $A$. If the diameter of the pinhole is smaller than the mean free path of the air molecules we say that air effuses into the container. One can then assume that the rate at which molecules escape through the hole is equal to the rate at which the molecules would have hit the missing segment of the wall. If the condition is not satisfied we cannot assume that the molecules of the gas move independently, and we must solve a hydrodynamic flow problem. Assume air has molecular weight $m$, that the outside pressure is $P$ and the temperature inside and outside is $T$. We wish to
(a) derive an expression for the pressure inside the container as a function of time, assuming effusion;
(b) estimate how long it takes for the pressure to rise to $\frac{1}{2} P$.

Solution: Let the density of air outside be $n_{0}$, while the inside density is $n_{i}=\frac{N}{V}$ where $N$ is the number of molecules inside the container volume $V$. Air can effuse both in and out of the container through the hole of area $A$. We have from (65)

$$
\begin{equation*}
\dot{n}_{i}=\frac{\dot{N}}{V}=\frac{\left(n_{0}-n_{i}\right)\langle v\rangle A}{4 V} \tag{72}
\end{equation*}
$$

The solution to this differential equation is

$$
\begin{equation*}
n_{i}=n_{0}\left[1-\exp \left(-\frac{\langle v\rangle t A}{4 V}\right)\right] \tag{73}
\end{equation*}
$$

and the answer to $\mathbf{a}$ : is $p_{i}=n_{i} k_{B} T$. To solve $\mathbf{b}$ : we put $n_{i}=\frac{1}{2} n_{o}$. The time taken is thus

$$
\begin{equation*}
t=\frac{4 V}{\langle v\rangle A} \ln 2=\frac{4 V \ln 2}{A \sqrt{\frac{8 k_{B} T}{\pi m}}} \tag{74}
\end{equation*}
$$

At ordinary pressures and temperatures the mean free path is $\sim 10^{-1} \mu \mathrm{~m}$ from (60). The hole must therefore be very small indeed for (74) to be valid.

### 6.3 Diffusion

We next turn to a discussion of diffusion. The diffusion constant $D$ is important in problems in which there is a concentration gradient. We will show that a consequence of the random motion of the molecules in a gas is that a concentration gradient will give rise to a particle current

$$
\begin{equation*}
\mathbf{j}_{N}=-D \nabla n(\mathbf{r}) \tag{75}
\end{equation*}
$$

Equation (75) is known as Fick's first law. Similarly, for a time-dependent process at constant temperature we have

$$
\begin{equation*}
\frac{\partial n}{\partial t}=D \nabla^{2} n \tag{76}
\end{equation*}
$$

which is known as Fick's second law. We wish to use the kinetic theory of gases to express the diffusion constant $D$ in terms of the mean velocity $\langle v\rangle$ and mean free path $\lambda$ between collisions as

$$
\begin{equation*}
D=\frac{\langle v\rangle l}{3} \tag{77}
\end{equation*}
$$

The SI unit for $D$ is $\mathrm{m}^{2} \mathrm{~s}^{-1}$. To derive (77) let us consider the case where the density of particles $n(z)$ varies with the height $z$ in a sample while the density is constant in the $x-y$ plane. Let us monitor the flow of particles through an imaginary surface $d S$ in the $x-y$ plane at height $z=0$ (figure 12).

We estimate the flow by considering where the particles could have come from after their last collision. To get from there through the imaginary surface it must travel in the right direction. Furthermore, if a particle goes through $d S$ from below it makes a positive contribution to the current; if it comes from above the contribution is negative. The number of particles undergoing collisions in a volume element $d V$ with polar coordinates $r, \theta, \phi$ per unit time is, with $l$ the mean free path

$$
\frac{\langle v\rangle n(z)}{l} d V=\frac{\langle v\rangle n(z)}{l} r^{2} d r \sin \theta d \theta d \phi
$$

The probability that such a particle can make it to $d S$ without further collisions is $e^{-r / l}$. The probability that it is traveling in the right direction is

$$
\frac{|\cos \theta| d S}{4 \pi r^{2}}
$$

If $\theta$ is between 0 and $\pi / 2$ the particles are coming from above, and if $\theta$ is between $\pi / 2$ and $\pi$ the particle is coming from below. Collecting terms we find for the net flow

$$
\begin{equation*}
j_{N} d S=\int_{0}^{\infty} r^{2} d r\left\{\int_{\pi / 2}^{\pi} d \theta-\int_{0}^{\pi / 2} d \theta\right\} \sin \theta \int_{0}^{2 \pi} d \phi n(z) \frac{\langle v\rangle|\cos \theta|}{l 4 \pi r^{2}} d S e^{-r / l} \tag{78}
\end{equation*}
$$

We assume that the density $n(z)$ varies slowly over distances comparable to the mean free path, and expand $n(z)$ in a Taylor series

$$
\begin{equation*}
n(z)=n(0)+z \frac{\partial n}{\partial z}+\ldots=n(0)+r \cos \theta \frac{\partial n}{\partial z}+\ldots \tag{79}
\end{equation*}
$$

If we substitute (79) into (78) we note that the contribution proportional to $n(0)$ vanishes (if $n(z)$ is constant the net current should be zero). Using

$$
\begin{equation*}
\int_{0}^{\infty} r d r \exp \left(-\frac{r}{l}\right)=l^{2} \tag{80}
\end{equation*}
$$



Figure 12: Diffusion through the $x y$-plane.
and

$$
\begin{equation*}
\int_{0}^{\pi} \cos ^{2} \theta \sin \theta d \theta=\frac{2}{3} \tag{81}
\end{equation*}
$$

the integrations in (78) can be carried out for the term proportional to the derivative of $n$. We find for the current density

$$
\begin{equation*}
j_{z}=-\frac{\langle v\rangle l}{3} \frac{\partial n}{\partial z} \tag{82}
\end{equation*}
$$

or in vector notation

$$
\begin{equation*}
\mathbf{j}=-\frac{\langle v\rangle l}{3} \nabla n \tag{83}
\end{equation*}
$$

Comparing with (75) we see that the diffusion constant $D$ is given by (77).
In order to derive the time-dependent diffusion equation (76) we consider a volume element $d V=d x d y d z$. As before, we let there be a flow in the $z$ direction. If the flow into the volume element is larger than the flow out there will be a net increase in the number of particles. We have

$$
\begin{equation*}
d x d y d z \frac{\partial n}{\partial t}=d x d y(j(z)-j(z+d z))=d x d y d z \frac{\partial}{\partial z}\left(D \frac{\partial n}{\partial z}\right)=d x d y d z D \frac{\partial^{2} n}{\partial z^{2}} \tag{84}
\end{equation*}
$$

or

$$
\begin{equation*}
\frac{\partial n}{\partial t}=D \frac{\partial^{2} n}{\partial z^{2}} \tag{85}
\end{equation*}
$$

if we assume that the diffusion constant is independent of the density (which is approximately true if the density is slowly varying, but is not true in general). The three-dimensional generalization of (85) is (76).

## Einstein relation

A quantity which is closely related to the diffusion constant is the mobility $\mu$. Consider a particle which is subject to an external force $F$. This force will accelerate the particle, but from time to time this acceleration is interrupted by collisions with other particles. After each collision the particle will start out in a random direction, but will again accelerate in the direction of the force until it is interrupted by another collision and so on. The net result of this very irregular motion is a drift in the direction of the force and the average velocity of this drift is called the drift velocity $v_{d}$. The drift velocity will in general be proportional to the force $F$ and the proportionality factor is called the mobility

$$
\begin{equation*}
v_{d}=\mu F \tag{86}
\end{equation*}
$$

Let us go back to our example of the isothermal atmosphere where the equilibrium particle number density could be written

$$
n=n_{o} \exp \left(\frac{-m g z}{k_{B} T}\right)
$$

This expression for the density can be thought of as arising from two competing processes. There will be a downward current due to the drift in the gravitational field

$$
\begin{equation*}
j_{d}=-n m g \mu \tag{87}
\end{equation*}
$$

This current is compensated by a diffusion current due to the density gradient

$$
\begin{equation*}
j_{D}=-D \frac{\partial n}{\partial z}=\frac{D n_{o} m g}{k_{B} T} \exp \left(\frac{-m g z}{k_{B} T}\right)=\frac{D n m g}{k_{B} T} \tag{88}
\end{equation*}
$$

In equilibrium the two currents must cancel and we get the Einstein relation

$$
\begin{equation*}
D=\mu k_{B} T \tag{89}
\end{equation*}
$$

Equation (89) is valid under quite general circumstances.

## Solutions to the diffusion equation

We will in this course not have much to say about how to solve the one dimensional diffusion equation (85) or its three-dimensional analog (76). The details of that problem belongs to a course in partial differential equations. In chapter 5 we worked out the probability distribution for the position of a particle with speed $v$ which undergoes many collisions in one dimension (64). We first show that (64) is a solution to the diffusion equation (85) if we make the identification $D=l v$. We have

$$
\begin{align*}
\frac{\partial}{\partial x^{2}}\left(\frac{\exp \left[-\frac{x^{2}}{4 D t}\right]}{\sqrt{4 \pi D t}}\right) & =\left(-\frac{1}{2 D t}+\frac{x^{2}}{4 D^{2} t^{2}}\right)\left(\frac{\exp \left[-\frac{x^{2}}{4 D t}\right]}{\sqrt{4 \pi D t}}\right) \\
\frac{\partial}{\partial t}\left(\frac{\exp \left[-\frac{x^{2}}{4 D t}\right]}{\sqrt{4 \pi D t}}\right) & =\left(-\frac{1}{2 t}+\frac{x^{2}}{4 D t^{2}}\right)\left(\frac{\exp \left[-\frac{x^{2}}{4 D t}\right]}{\sqrt{4 \pi D t}}\right) \tag{90}
\end{align*}
$$

from which (85) follows directly. The three-dimensional probability distribution for a particle starting out at the origin and diffusing outwards is

$$
\begin{equation*}
p(\mathbf{r}, t)=\frac{1}{(4 \pi D t)^{3 / 2}} \exp \left[-\frac{r^{2}}{4 D t}\right] \tag{91}
\end{equation*}
$$

The reader may easily verify that (91) is a solution to the 3 -dimensional diffusion equation by expressing $\mathbf{r}$ in Cartesian coordinates and working out the expressions analogous to (90). From (91) we see that the typical distance traveled by a particle will be of the order $\sqrt{D t}$, which also could be expected on dimensional grounds since the diffusion constant has dimension (length) ${ }^{2}$ / time. Since the diffusion equation (76) is linear we can add solutions from several sources. Suppose we add $N_{i}$ particles at the points $\mathbf{r}_{i}$ at time $t=0$ the subsequent particle density will be

$$
\begin{equation*}
n(\mathbf{r}, t)=\sum_{i} \frac{N_{i}}{(4 \pi D t)^{3 / 2}} \exp \left(\frac{-\left(\mathbf{r}-\mathbf{r}_{i}\right)^{2}}{4 D t}\right) \tag{92}
\end{equation*}
$$

If there is a continuous distribution of initial sources the sums in (92) must be converted into integrals. Another commonly occuring problem is to find the steady state distribution after a long time has elapsed. In this case the diffusion equation (76) reduces to the Laplace equation

$$
\begin{equation*}
\nabla^{2} n(\mathbf{r})=0 \tag{93}
\end{equation*}
$$

Solutions to the Laplace equation are frequently required in electrostatics and many diffusion problems can be solved by making an analogy with a problem in electrostatics.

So far the only physical effect of diffusion is to smooth out concentration differences. In fact, diffusion, when occuring in conjunction with chemical reactions, can give rise to a rich variety of effects. Both the chemical reactions and diffusion processes supplying the chemicals to an active region occur at a certain rate. Competition between the two rates can give rise to effects such as the beautiful colored stripes of agate and many biological pattern formation processes. Similarly, if a melted mineral is allowed to cool and recrystallize, the size distribution of the crystallite grains will be governed by a competition between diffusion rates and the rate of cooling.

### 6.4 Problems

## Problem 6-1.

Gas molecules effuse through a small leak from an oven which is held at constant temperature $T$. What is the average temperature of the escaping gas if the gas is monatomic with heat capacity $\frac{3}{2} k_{B}$ per molecule?

## Problem 6-2.

Two flasks each having volume $V$ are connected by a thin tube of length $L$ and cross sectional area $A$. Initially one flask is full of pure water and the other contains "labeled water" (water in which the hydrogen is replaced by tritium) at concentration $n_{0}$. The diffusion constant of the labeled water is $D$. Tiny mixers maintain uniform concentration in each flask, but transport from one flask to the other through the connecting tube is accomplished only by diffusion.

Derive a formula for the concentration of labeled water in the two flasks as a function of time. State your assumptions.

## Problem 6-3:

Tungsten atoms (atomic weight 183.7) are measured to leave a tungsten surface at the rate of $1.14 \times 10^{-12} \mathrm{~kg} \mathrm{~s}^{-1} \mathrm{~m}^{-2}$ from a sample heated to 2000 K . Find the vapor pressure of tungsten at this temperature.

## Problem 6-4:

$10^{20}$ pollutant molecules with diffusion constant $D=1.0 \times$ $10^{-10} \mathrm{~m}^{2} \mathrm{~s}^{-1}$ are released at time $t=0$ in the center of a large pool of water. Assume that the molecules diffuse away without any stirring taking place.
(a). Plot the concentration as a function of time a distance 1 cm away from the source.
(b). When does the maximum concentration occur 1 cm away from the source.
(c). How long will it take for the concentration everywhere to reach a "safe" level of $10^{20} \mathrm{~m}^{-3}$ ?
(d). How long would it take for one half of the molecules to reach a distance 1 cm from the source. (You will need to do a numerical integration of (91) to get your answer.)

## Problem 6-5.

(a). Molecules with diffusion constant $1.0 \times 10^{-10} \mathrm{~m}^{2} / \mathrm{s}$ are released at a constant rate of $10^{10}$ molecules $/ \mathrm{s}$ in the middle of a large pool and diffuse away. What is the steady state concentration 1 cm away from the source? (Hint: The electrostatic potential associated with a charge at the origin and no charges elsewhere is of the form const/r).
(b). How long after the release has started will it take for the concentration to reach $1 / 2$ of its steady state value 1 cm away from the source?

## Problem 6-6.

A gas of $\mathrm{N}_{2}$ molecules at 300 K has a number density $N / V=$ $2.5 \times 10^{25}$ molecules per $\mathrm{m}^{3}$. A small number of $\mathrm{O}_{2}$ molecules is contained in the $\mathrm{N}_{2}$ gas. The cross section for collisions between the $\mathrm{O}_{2}$ and $\mathrm{N}_{2}$ molecules is $\sigma \simeq 4 \times 10^{-19} \mathrm{~m}^{2}$.
(a). Calculate the mean free path and the mean time between collisions for $\mathrm{O}_{2}$ under these conditions.
(b). Estimate the diffusion constant of the $\mathrm{O}_{2}$ molecules in the $\mathrm{N}_{2}$ gas.
(c). If the $O_{2}$ molecules are initially localized at one end of a tube of length 1 m containing the $\mathrm{N}_{2}-\mathrm{O}_{2}$ mixture, approximately how long would it take them to diffuse to the other end of the
tube? What type of measurement could be made to check the results of this calculation?
(d). A certain force is applied to the $\mathrm{O}_{2}$ molecules and is observed to produce a "drift velocity" of magnitude $3 \mathrm{~m} \mathrm{~s}^{-1}$. What is the magnitude of the force which produces this drift velocity?

## Problem 6.7.

A narrow tube contains ether. Initially the tube is full but the level falls slowly by evaporation. Assume that the evaporation of ether is limited by diffusion. Near the liquid surface there is equilibrium concentration of ether vapor, while at the top of the tube the concentration is essentially zero. The vapor pressure of ether at room temperature is $5.33 \times 10^{4} \mathrm{~Pa}$, the molecular weight is 74 , the density of liquid ether is $0.7 \mathrm{~g} \mathrm{~cm}^{-3}, 1$ atomic mass unit is $1.66057 \times 10^{-27} \mathrm{~kg}$.
(a). Find an equation for the depth $h$ of the ether surface below the open end as a function of time, given the diffusion constant $D$ for ether in air.
(b). Calculate the Diffusion constant $D$ if the level falls by 1 cm in 30 minutes.
(c). How long will it take for the ether level to fall to 2 cm below the open end?
(d). Estimate the diameter of an ether molecule.

## 7 Thermal conductivity and viscosity.

In this section we wish to continue our discussion of transport properties of fluids. What we do is based on the picture that a gas or fluid is made up of individual particles, each of which do their own thing, except that from time to time they collide with other particles. We want to model materials properties such as thermal conductivity and viscosity using this picture. Suppose we set up a temperature gradient in a material. If the gradient is not too large the heat flow will be proportional to the temperature gradient and the proportionality constant is the thermal conductivity. The concept of viscosity dates back to Newton, who considered laminar fluid flow. While a fluid at rest offers no resistance to shear, there will be a shear stress when
the fluid is flowing. In this type of flow the fluid is moving in parallel layers in such a way that we have a velocity profile (see figure 13).

We assume that the shear stress $P_{z y}$ is proportional to the velocity gradient. Stress is the force $F$ per unit area $A$, so if the velocity field is changing in the $z$-direction

$$
\begin{equation*}
P_{z y}=\frac{F}{A}=-\eta \frac{\partial u_{y}}{\partial z} \tag{94}
\end{equation*}
$$

The proportionality constant $\eta$ is called the viscosity. A fluid which obeys 94 is called Newtonian. The three most important fluids air, water and petroleum are all Newtonian to a very good approximation. There are, however some fairly common fluids such as muds, paints, slurries, pastes and jellies that behave differently. We will come back to the non-Newtonian fluids later in this chapter. In the SI system the unit of viscosity is $\mathrm{N} \mathrm{s} \mathrm{m}^{-2}$. A commonly used unit of viscosity is the centipoise which is $10^{-3} \mathrm{~N} \mathrm{~s} \mathrm{~m}^{-2}$. Sometimes people prefer to use the kinematic viscosity, which is viscosity divided by the mass density $\nu=\frac{\eta}{\rho}$. The SI unit of kinematic viscosity is $\mathrm{m}^{2}$ $\mathrm{s}^{-1}$. The centistoke is defined as $10^{-6} \mathrm{~m}^{2} \mathrm{~s}^{-1}$, the reason for the common use of the nonstandard units is probably that $\nu$ for water is $\approx 1$ centistoke, while $\mu$ is $\approx 1$ centipoise.

In the case of heat conductivity there is a heat current $j_{Q}$ which is proportional to the temperature gradient

$$
\begin{equation*}
j_{Q}=-\kappa \frac{\partial T}{\partial z} \tag{95}
\end{equation*}
$$

In general, we can construct an approximate theory by starting with the assumption that there is a molecular property $a(z)$ which diffuses in the $z$ direction. We assume that the property $a$ is slowly varying over distances comparable to the mean free path so that we have

$$
\begin{equation*}
a(z)=a(0)+z \frac{\partial a}{\partial z}+\ldots \tag{96}
\end{equation*}
$$

In the case of viscosity the molecular property of interest is the $y$ component (in the direction of the flow) of the average momentum per unit volume $a(z)=n m\left\langle u_{y}(z)\right\rangle$. The "current" is the shear stress $P_{z y}$ which is the net flux in the $z$ - direction of the $y$-component of momentum. We can estimate this shear stress by the same argument that we used in the previous section. In that case the molecular property which varied with $z$ was the


Figure 13: Velocity profile.
number density $n(z)$. Thus all we have to do is to replace $n(z)$ by $a(z)$ in (82) and we get using (94)

$$
P_{z y}=-\frac{\langle v\rangle l}{3} \frac{\partial a}{\partial z}=-\frac{n m\langle v\rangle l}{3} \frac{\partial\left\langle u_{y}\right\rangle}{\partial z}=\eta \frac{\partial\left\langle u_{y}\right\rangle}{\partial z}
$$

giving

$$
\begin{equation*}
\eta=\frac{n m\langle v\rangle l}{3}=\frac{m\langle v\rangle}{3 \sqrt{2} \sigma} \tag{97}
\end{equation*}
$$

where we have used (59) to relate the mean free path to the scattering cross section. Equation in (97) has the interesting consequence that the viscosity of a fluid should be independent of the density and proportional to the square root of the temperature (see(32)). Experimental results (see e.g. Flowers and Mendoza [23] for more details) on gases confirm these result at low to moderate pressures. At high pressures ( $\geq 100 \mathrm{~atm}$.) our theory is too simple minded.

We next turn to the thermal conductivity. The molecular property which is diffusing is the internal energy of the particles. A change $\Delta U$ in the internal energy per unit volume is associated with a change $\Delta T$ in the temperature. We have

$$
\Delta U=n c_{v} \Delta T
$$

where $c_{v}$ is the specific heat per particle $\left(c_{v}=\frac{3}{2} k_{B}\right.$ for a monatomic ideal gas). The heat current is thus

$$
\begin{equation*}
j_{Q}=-\frac{\langle v\rangle l}{3} n c_{v} \frac{\partial T}{\partial z}=-\kappa \frac{\partial T}{\partial z} \tag{98}
\end{equation*}
$$



Figure 14: (a) Different types of Non-newtonian fluids. (b) Non-running paint.

Using (59) we find that

$$
\begin{equation*}
\kappa=\frac{\langle v\rangle l}{3} n c_{v}=\frac{\langle v\rangle c_{v}}{\sigma 3 \sqrt{2}} \tag{99}
\end{equation*}
$$

Eq. (99) predicts that the thermal conductivity is independent of the pressure, and proportional to the square root of the temperature. Just as for the viscosity, this agrees with experiment on gases at moderate to low pressure, but there are corrections at high pressure ( $\geq 100 \mathrm{~atm}$ ). Finally from (77) and (59)

$$
\begin{equation*}
D=\frac{\langle v\rangle}{3 n \sigma \sqrt{2}} \tag{100}
\end{equation*}
$$

i.e. the diffusion constant should be inversely proportional to the density, which seems indeed to be true experimentally (see e.g. [23]).

### 7.1 Non-Newtonian fluids

As mentioned earlier there are a number of fairly common fluids that do not satisfy (94). Examples are the so-called Bingham fluids, which do no flow unless a yield stress $\tau_{0}$ is exceeded (see figure 14)

$$
\begin{array}{lc}
\text { If }\left|P_{y z}\right|>\tau_{0} & P_{y z}=\tau_{0}+\eta \frac{\partial u_{y}}{\partial z}  \tag{101}\\
\text { If }\left|P_{y z}\right|<\tau_{0} & \frac{\partial u_{z}}{\partial y}=0
\end{array}
$$

The above property of a Bingham fluid makes it useful as a paint. The threshold $\tau_{0}$ keeps the paint from running, but it can still be spread if enough force is applied. As an example consider painting a vertical wall (figure $14(\mathrm{~b})$ ). If the thickness of the layer is $\delta$ the paint will not run if

$$
P_{y z}=\rho g y<\tau_{0}
$$

for all $y<\delta$. When $\delta=\frac{\tau_{0}}{\rho g}$ it will start running where the shear stress is the largest, i.e. near the wall.

In other non-Newtonian fluids one can model the stress-strain relationship by

$$
\begin{equation*}
\left|P_{y z}\right|=k\left|\frac{\partial u_{y}}{\partial z}\right|^{n} \tag{102}
\end{equation*}
$$

If $n<1$ the fluid is called pseudoplastic, if $n>1$ it is called dilatant (see figure 14(a)).

### 7.2 Summary of results from kinetic theory

- Boltzmann factor. The probability that a system is in a state $i$ with energy $E_{i}$ at fixed $V$ and $N$ is

$$
p_{i}=\frac{1}{Z} e^{-\beta E_{i}} ; \quad Z=\sum_{i} e^{-\beta E_{i}}
$$

- Collisions. The collision frequency and mean free path are given by

$$
\langle\phi\rangle=\sqrt{2} n \sigma\langle v\rangle ; \quad l=\frac{1}{n \sigma \sqrt{2}}
$$

- Diffusion processes are governed by Fick's laws:

$$
\mathbf{j}_{N}=-D \nabla n(\mathbf{r}) ; \quad \frac{\partial n}{\partial t}=D \nabla^{2} n
$$

In kinetic theory $D$ is given by

$$
D=\frac{\langle v\rangle l}{3}
$$

- Einstein relation. The diffusion coefficient $D$ is related to the mobility through

$$
D=\mu k_{B} T
$$

- Maxwell velocity distribution for particles in an ideal gas:

$$
p(\mathbf{v})=\left(\frac{m}{2 \pi k_{B} T}\right)^{\frac{3}{2}} \exp \left(-\frac{m v^{2}}{2 k_{B} T}\right)
$$

- Speed. The average and mean square speeds are for an ideal gas

$$
\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}} ; \quad\left\langle v^{2}\right\rangle=\frac{1}{N} \int_{0}^{\infty} v^{2} g(v) d v=\frac{3 k_{B} T}{m}
$$

- Surfaces. The number of particles hitting a surface per unit are and time is

$$
j_{N}=\frac{n\langle v\rangle}{4}
$$

- Thermal conductivity, denoted $\kappa$, is the proportionality constant connecting a temperature gradient and the resulting heat current per unit area

$$
j_{Q}=-\kappa \nabla T
$$

In kinetic theory

$$
\kappa=\frac{\langle v\rangle l}{3} n c_{V}
$$

where $c_{V}$ is the specific heat per particle.

- Viscosity, denoted $\eta$, is the proportionality constant connecting a velocity gradient and the resulting shear stress in a Newtonian fluid. The kinetic viscosity is $\nu=\eta / \rho$.

$$
P_{z y}=-\eta \frac{\partial u_{y}}{\partial z}
$$

In kinetic theory

$$
\eta=\frac{n m\langle v\rangle l}{3}
$$

### 7.3 Problems

## Problem 7.1.

The thermal conductivity of air at $0^{\circ} C$ is listed as 0.023 W $\mathrm{m}^{-1} \mathrm{~K}^{-1}$, and is approximately independent of pressure.
(a) Estimate the collision cross section $\sigma$ of air molecules.
(b) There is a gap of 1 mm . between two $1 \mathrm{~m}^{2}$ window panes. The inside pane has a temperature $10^{\circ} \mathrm{C}$ and the outside pane $-10^{0} \mathrm{C}$. Calculate rate of heat loss through the window due to thermal conduction.

## Problem 7.2.

A vial of height 3 cm is filled with an aqueous suspension of water particles. The particles are spherical with radius $100 \AA$, and have a density of $1200 \mathrm{~kg} \mathrm{~m}^{-3}$ (for comparison the density of water is $1000 \mathrm{~kg} \mathrm{~m}^{-3}$ ). The sample temperature is $T=300$ K. The viscosity of water is $\eta=10^{-3} \mathrm{~Pa}$ s. The mobility of a spherical object in laminar flow is given by Stokes' law

$$
\begin{equation*}
\mu=\frac{\text { velocity }}{\text { force }}=\frac{1}{6 \pi \eta r} \tag{103}
\end{equation*}
$$

(a) Verify quantitatively that the equilibrium distribution of particles is nearly uniform over the height of the vial.
(b) The vial is placed in a centrifuge and subject to an acceleration equal to $10^{4}$ times that of gravity, acting along the height of the vial (assume that the acceleration is nearly uniform over the height of the vial). Once the new equilibrium is reached, roughly how many times more concentrated are the particles near the outer end of the vial than they were before centrifuging?
(c). Roughly how long must the sample be spun in the centrifuge before equilibrium will be reached?
(d). If the vial stands undisturbed after removal from the centrifuge, roughly how long does it take for the particles to diffuse back to their original equilibrium distribution?

## Part III

## HYDRODYNAMICS

At this stage we abandon our attempt at a microscopic discussion of the properties of fluids. Instead, we will assume that fluid quantities such as the viscosity and density are known, and concentrate on the resulting macroscopic properties.

## 8 Laminar flow through a pipe.

We will begin our discussion of hydrodynamics by discussing laminar flow and show how viscosity governs the flow properties of slowly flowing fluids.

### 8.1 Poiseuille flow

Consider, as an example, a liquid which is flowing through a narrow horizontal, cylindrical pipe of radius $r$ and length $L$. The flow is due to a pressure difference $\Delta P$ between the ends of the pipe. We will assume that the fluid is flowing in a streamline fashion with velocity parallel to the axis of the pipe. The viscosity of the fluid is $\eta$ and we assume that the velocity is zero at the walls of the pipe. In order to find the flow rate, $\dot{V}$, (units volume/time) through the pipe (see figure 15(a)).
consider an imaginary cylinder of height $L$ and radius $x<r$ which is concentric with the pipe. In the steady state the force $\Delta P \pi x^{2}$ generated by the pressure drop will balance the drag $2 \pi r \eta L d u / d x$ due to viscous forces. The velocity profile $u(x)$ satisfies the boundary condition $u(r)=0$. We find

$$
\Delta P \pi x^{2}=-\eta \frac{d u}{d x} 2 \pi x L
$$

or

$$
-x d x=\frac{2 \eta L}{\Delta P} d u
$$

This equation can be integrated to give

$$
\begin{equation*}
u(x)=\frac{\Delta P}{4 \eta L}\left(r^{2}-x^{2}\right) \tag{104}
\end{equation*}
$$

The velocity profile predicted by (104) is shown in figure $15(\mathrm{~b})$.

(a)


Figure 15: (a) Flow through a pipe. (b) Velocity profile.

To get the volume flow rate $\dot{V}$ consider a cylindrical shell of thickness $d x$. The amount of fluid flowing per unit time through the shell is

$$
d \dot{V}=2 \pi x u(x) d x=\frac{\Delta P \pi}{2 \eta L}\left(r^{2}-x^{2}\right) x d x
$$

Integrating we find for the total flow rate

$$
\begin{equation*}
\dot{V}=\int_{0}^{r} \frac{\Delta P \pi}{2 \eta L}\left(r^{2}-x^{2}\right) x d x=\frac{\Delta P \pi r^{4}}{8 \eta L} \tag{105}
\end{equation*}
$$

which is Poiseuille's formula.

### 8.2 Pipe flow for a non-Newtonian fluid

The previous result can easily be generalized to the case of non-Newtonian fluids. If the flow is laminar the flow velocity will not have any component in a direction perpendicular to the pipe axis. The pressure will then be
uniform in a cross sectional plane. Force balance then gives with $\tau(x)=P_{x z}$ the shear stress

$$
\begin{equation*}
\pi x^{2} \frac{\Delta P}{L}=2 \pi x \tau(x) \tag{106}
\end{equation*}
$$

For a power law fluid (102) we find from (106)

$$
\tau=\frac{x \Delta P}{2 L}=k\left(\frac{-d u}{d x}\right)^{n}
$$

or

$$
\begin{equation*}
-\frac{d u}{d x}=\left(\frac{x \Delta P}{2 k L}\right)^{1 / n} \tag{107}
\end{equation*}
$$

With the boundary condition $u(x)=0$ for $x=r$ we can integrate (107) to get

$$
\begin{equation*}
u(x)=\frac{n}{n+1}\left(\frac{\Delta P}{2 k L}\right)^{1 / n}\left(r^{1+n^{-1}}-x^{1+n^{-1}}\right) \tag{108}
\end{equation*}
$$

We substitute this result in the equation for the flow rate

$$
\dot{V}=\int_{0}^{r} 2 \pi x u(x)
$$

to get after some algebra

$$
\begin{equation*}
\dot{V}=\frac{\pi r^{3}}{\left(3+\frac{1}{n}\right) \eta L}\left(\frac{r \Delta P}{2 L k}\right)^{1 / n} \tag{109}
\end{equation*}
$$

which reduces to (105) in the special case $n=1$. A dilatant fluid ( $n>1$ ) acts as if there is little or no viscosity for small pressure differences, but the resistance rapidly increases if the pressure is increased ("easy does it!"). On the other hand, a pseudoplastic ( $n<1$ ) flows easier the harder one pushes. Most people have had this experience of pouring ketchup: "first there is nothing, then there is nothing and then there is everything". Blood is a pseudoplastic; this has the important consequence that it can flow with remarkable ease in very narrow capillary tubes.

Pseudoplastics are slippery. If you gently test a small puddle with your foot, it will show resistance, but if you follow up by putting down your full weight, it will yield and you may fall.

We can find the laminar flow rate of a Bingham fluid in a pipe of circular cross section. There will be a critical pressure difference to make the fluid flow at all given by

$$
\begin{equation*}
\pi r^{2} \Delta P_{c r i t}=2 \pi r L \tau_{0} \tag{110}
\end{equation*}
$$

Even if $\Delta P>\Delta P_{\text {crit }}$ the shear stress will be too small to overcome the yield threshold near the center of the pipe. The material in the center will then move like a plug, as when toothpaste is squeezed out of its tube. We define the dimensionless pressure

$$
\begin{equation*}
p=\frac{\Delta P}{\Delta P_{c r i t}} \tag{111}
\end{equation*}
$$

and we obtain $x_{0}=r / p$ for the radius of the plug. From (101) we find the differential equation for the velocity profile

$$
\begin{equation*}
-\eta \frac{d u}{d x}+\tau_{0}=\frac{x \Delta P}{2 L} \tag{112}
\end{equation*}
$$

valid for $r>x>x_{0}$. We can integrate (112) to obtain

$$
\begin{equation*}
u=\frac{1}{\eta}\left[\frac{\Delta P}{4 L}\left(r^{2}-x^{2}\right)-\tau_{0}(r-x)\right] \tag{113}
\end{equation*}
$$

for $r>x>x_{0}$. For $x<x_{0}$ we have

$$
\begin{equation*}
u=u_{0}=\frac{1}{\eta}\left[\frac{\Delta P}{4 L}\left(r^{2}-x_{0}^{2}\right)-\tau_{0}\left(r-x_{0}\right)\right]=\frac{\Delta P r^{2}}{4 \eta L}\left(1-\frac{1}{p}\right)^{2} \tag{114}
\end{equation*}
$$

We can integrate to find the flow rate

$$
\dot{V}=\int_{x_{0}}^{r} 2 \pi x u(x) d x+\pi x_{0}^{2} u_{0}
$$

and find after some algebra

$$
\begin{equation*}
\dot{V}=\frac{\Delta P \pi r^{4}}{8 \eta L}\left[1-\frac{4}{3 p}+\frac{1}{3 p^{4}}\right] \tag{115}
\end{equation*}
$$

A Newtonian fluid corresponds to the limit $p \rightarrow \infty$ In this limit (115) reduces to (105). A plot of the ratio

$$
\begin{equation*}
R={\frac{\dot{V}}{\dot{V}_{\text {Poiseuille }}}} \tag{116}
\end{equation*}
$$

vs. the reduced pressure is shown in figure 16.


Figure 16: Ratio $R$ defined by (116) vs. reduced pressure $p$ for a Bingham plastic.

### 8.3 Problems

## Problem 8.1.

Two flat solid plates of area $0.5 \mathrm{~m}^{2}$ are constrained to be parallel to each other in the horizontal $(x-y)$ plane. The plates are separated by a gap of $d=0.001 \mathrm{~m}$ of air. Suppose that the top plate is moved parallel to the stationary bottom one at a velocity $v=0.1 \mathrm{~m} \mathrm{~s}^{-1}$ along the $x$-axis. The viscosity of the air is $2.0 \times 10^{-5} \mathrm{~Pa} \mathrm{~s}^{-1}$.
(a). What is the shear force on the bottom plate?
(b). How is the force changed if the air pressure is
(i) increased from one to two atmospheres at constant temperature?
(ii) reduced to a very small value?

## Problem 8-2:

A 2 kg solid cylinder is placed inside a vertical pipe with an inside diameter of 52 mm (see Figure 17). The length of the cylinder is 200 mm and the diameter is 50 mm . The cylinder is held concentric inside the pipe by a layer of oil with viscosity of


Figure 17: Cylinder of Problem 8-2.
$9.2 \times 10^{-2} \mathrm{~Pa}$. The pressure below the cylinder is maintained at 103 kPa and above the cylinder at 100 kPa . Determine the steady state speed with which the cylinder falls. Neglect the effect of the pressure difference on the flow of oil.

## Problem 8-3.

Two square parallel plates with sides are kept apart with a clearance $c$. A pressure difference $\Delta P$ is maintained over the distance $L$ between two opposite edges. This gives rise to flow in the plane of the plates in the direction of the pressure gradient. The clearance is filled with a fluid with viscosity $\eta$. Assume laminar flow. Derive a formula analogous to (105) for the total flow rate $\dot{V}$. Assume that $c \ll L$ so that end effects can be neglected.

## 9 Dimensional analysis. Reynolds number.

Poiseuille flow represents one of a small number of situations where we can find a closed form for the solution of a flow problem. As we shall see, the laminar type of flow is not the only type that can take place, and it is not even the most common type of flow. Since analytical methods are often not
practical, we will often have to resort to empirical formulas. Dimensional analysis represents a way of reasoning which helps in constructing empirical formulas and which allows us to isolate the most important parameters. It is useful to familiarize ourselves with the method before we get involved with the details of hydrodynamics. In to order to illustrate the method let us apply it to the laminar flow problem of the previous section.

We note first that in laminar flow the fluid moves with constant speed, there is no acceleration. When constructing an empirical formula for the flow we therefore do not expect the mass density $\rho$ to play a significant role. On what do we expect the flow rate to depend? Three obvious suspects are:

- The pressure drop per unit length, $(\Delta P / L)$. One expects that the harder we push, the more fluid will get through.
- The radius $r$ of the pipe. Intuitively, the bigger the radius the more stuff will flow through.
- The viscosity $\eta$ of the fluid. The viscosity is responsible for the drag force which makes it necessary to maintain the pressure difference.

We now jump to the conclusion that these three factors are the only ones that matter, and we attempt to construct a formula for the flow rate on the form

$$
\begin{equation*}
\dot{V}=\mathrm{const}\left(\frac{\Delta P}{L}\right)^{\alpha} r^{\beta} \eta^{\gamma} \tag{117}
\end{equation*}
$$

To be a valid formula (117) has to be dimensionally consistent. This implies restrictions on the allowed values of the exponents $\alpha, \beta$ and $\gamma$. We use the notation that square brackets [] stands for "dimension of" and we reduce our units to the three basic ones of the SI system: kilogram (kg), meter (m) and second ( s ). The unit for force, the Newton, is

$$
[\mathrm{N}]=\mathrm{kg} \mathrm{~m} \mathrm{~s}^{-2}
$$

The pressure unit is the Pascal

$$
[\mathrm{Pa}]=\mathrm{Nm}^{-2}=\mathrm{kg} \mathrm{~m}^{-1} \mathrm{~s}^{-2}
$$

We have for the flow rate

$$
[\dot{\mathrm{V}}]=\mathrm{m}^{3} \mathrm{~s}^{-1}
$$

and $\left[\frac{\Delta P}{L}\right]=\mathrm{Pa} \mathrm{m}^{-1}=\mathrm{kg} \mathrm{m}^{-2} \mathrm{~s}^{-2},[r]=\mathrm{m},[\eta]=\mathrm{Pa} \mathrm{s}=\mathrm{kg} \mathrm{m}^{-1} \mathrm{~s}^{-1}$. Dimensional consistency of (117) in the case of mass, length and time requires

$$
\begin{array}{lrl}
\text { mass } \rightarrow & 0 & =\alpha+\gamma \\
\text { length } \rightarrow & 3 & =-2 \alpha+\beta-\gamma  \tag{118}\\
\text { time } \rightarrow & -1 & =-2 \alpha-\gamma
\end{array}
$$

The solution to (118) is $\alpha=1, \gamma=-1, \beta=4$. Substitution into (117) gives

$$
\begin{equation*}
\dot{V}=\mathrm{const} \frac{\Delta P r^{4}}{L \eta} \tag{119}
\end{equation*}
$$

This result is consistent with Poiseuille's formula (105) if we put const $=$ $\pi / 8$.

As a further example, let us consider the problem of finding the $d r a g$ force $F$ on an object with linear dimension $r$ at low velocities where there is laminar flow. Let us assume that the fluid is a power law fluid for which the viscous shear stress is given by (102)

$$
\left|P_{y z}\right|=k\left|\frac{\partial u_{y}}{\partial z}\right|^{n}
$$

Let us assume that the drag force can be written

$$
F=\mathrm{const} r^{\alpha} u^{\beta} k^{\gamma}
$$

We have $[F]=\mathrm{kg} \mathrm{m} \mathrm{s}^{-2},[r]=\mathrm{m},[u]=\mathrm{m} \mathrm{s}^{-1},[k]=\mathrm{kg} \mathrm{s}^{n-2} \mathrm{~m}^{-1}$. Dimensional consistency then requires

$$
\begin{array}{lll}
\text { mass } & \rightarrow 1 & =\gamma \\
\text { length } & \rightarrow 1 & =\alpha+\beta-\gamma \\
\text { time } & \rightarrow-2=-\beta+(n-2) \gamma
\end{array}
$$

with solution $\gamma=1, \beta=n, \alpha=2-n$, giving

$$
F=\text { const } k r^{2-n} u^{n}
$$

If the fluid is Newtonian, $n=1, k=\eta$ and we find

$$
\begin{equation*}
F=\text { const } \eta r u \tag{120}
\end{equation*}
$$

There is no way that the constant can be determined by dimensional arguments. However, for the special case of a spherical particle (120) is consistent with Stokes' law (103) if const $=6 \pi$.

Not all flow is laminar. Experience tells us that if the flow speed is very high the flow will be turbulent. In the case of flow through a pipe, one mechanism by which laminar flow can become unstable is if the pipe is rough. The fluid will then be accelerated around any irregularity. Whether or not the fluid can return to steady laminar flow will then depend on a competition between viscous forces, which tend to stabilize the flow, and inertia which causes eddies. In the case of our pipe flow problem, we assume that there is a critical velocity $u_{c}$ beyond which the character of the flow changes from being laminar to being turbulent. Boldened by our success in "deriving" (119) by dimensional analysis we postulate that the critical velocity depends on the inertia, viscosity $\eta$, mass density $\rho$ and the diameter $d$ of the pipe ${ }^{3}$.

$$
u_{c}=\text { const } \eta^{x} \rho^{y} d^{z}
$$

from $\left[u_{c}\right]=\mathrm{m} \mathrm{s}^{-1},[\eta]=\mathrm{kg} \mathrm{m}^{-1} \mathrm{~s}^{-1},[\rho]=\mathrm{kg} \mathrm{m}^{-3},[d]=\mathrm{m}$ we find

$$
\begin{align*}
& \text { length } \rightarrow 1=-x-3 y+z \\
& \text { mass } \rightarrow 0=x+y  \tag{121}\\
& \text { time } \rightarrow-1=-x
\end{align*}
$$

giving $x=1, y=-1, z=-1$ or

$$
\begin{equation*}
u_{c}=\text { const } \frac{\eta}{\rho d} \tag{122}
\end{equation*}
$$

We define the Reynolds number as

$$
\begin{equation*}
R e=\frac{u \rho d}{\eta} \tag{123}
\end{equation*}
$$

and note that it a dimensionless quantity. The Reynolds number occurs in many different situation other than flow in pipes. In general, $u$ has the interpretation of flow velocity, $\rho$ of mass density, $d$ of the typical dimension of object which the fluid flows in or around, and $\eta$ is the viscosity. The nature of the flow (laminar vs. turbulent) depends on the Reynolds number. In the case of pipes one typically finds that if the Reynolds number is less than 2000 the Poiseuille formula is a good approximation. To what does a Reynolds number of 2000 correspond?

Suppose we have water flowing in a pipe of diameter $10^{-2} \mathrm{~m}$. The viscosity of water is approximately $10^{-3} \mathrm{~Pa} \mathrm{~s}$, the density is $\rho=10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$. The

[^2]critical velocity is then about $0.2 \mathrm{~m} \mathrm{~s}^{-1}$. If instead we have a storm sewer of diameter 1 m , the critical velocity is $2 \mathrm{~mm} \mathrm{~s}^{-1}$, which is not very fast indeed. Air at room temperature has a viscosity of slightly less than $2 \times 10^{-5} \mathrm{~Pa} \mathrm{~s}$, at atmospheric pressure the density is $1.29 \mathrm{~kg} \mathrm{~m}^{-3}$, so the critical velocity for air is higher than for water, as it also will be for viscous fluids such as oil. Note the role of the length dimension in (123). In medicine one deals with tiny objects such as viruses and cells which swim in the blood, the movement of such objects will typically involve laminar flow. On the other hand, the flows of air and water in meteorology and oceanography are almost never laminar.

The transition from laminar to turbulent flow in a pipe is our first encounter with a hydrodynamic instability. This particular transition is analogous to a discontinuous phase transition in equilibrium thermodynamics, such as the change from a liquid to a gas. In the gas-liquid transition the density changes discontinuously, and there is typically hysteresis: e.g. vapor can be maintained for long periods in a supercooled state below the transition without condensing. Similarly, it is possible with care to maintain laminar flow at Reynolds numbers which are significantly higher than the critical number.

### 9.1 Problems

## Problem 9-1:

A cylindrical pipe is designed to carry a flow rate $\dot{V}$ (volume/time) of a fluid with viscosity $\eta$ and density $\rho$.
(a). What is the minimum diameter of the pipe if the flow is to be laminar and the critical Reynolds number is $R e$.
(b). What is then the pressure difference between two points a distance $L$ apart along the pipe.
(c). Give numerical answers if $R e=2000, \eta=10^{-3} \mathrm{~Pa} s$, $\dot{V}=10^{-4} \mathrm{~m}^{3} \mathrm{~s}^{-1}, L=1 \mathrm{~m}, \rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}$.

## Problem 9-2:

Show by dimensional analysis that for laminar flow in a pipe of a power-law fluid in which the viscous shear stress is given by

$$
\tau=k\left(\frac{-d u}{d x}\right)^{n}
$$

the flow rate will be given by

$$
\dot{V}=\text { const. } r^{3}\left(\frac{r \Delta P}{L k}\right)^{1 / n}
$$

## 10 Fluids at rest. Some problems of stability.

As we have mentioned earlier, what distinguishes a fluid from a solid is that a fluid in equilibrium cannot support any shear stress. If the system is out of equilibrium and there is a net flow, there will be viscous forces associated with changes in the velocity field, but if the fluid is at rest with regards to its surroundings there will be no viscous forces. This means that when a fluid is at rest relative to the surroundings there is only one kind of force which can be transmitted, namely the hydrostatic pressure which is always normal to any surface of contact. In many applications involving liquids it is a good approximation to assume that the density of a fluid is constant or that we are dealing with an incompressible fluid. If the mass density of such a fluid is $\rho$, and if the pressure at a free surface is $P_{s}$, then the pressure at depth z below the free surface is

$$
\begin{equation*}
P=P_{s}+\rho g z \tag{124}
\end{equation*}
$$

If the fluid is compressible and the density depends on the depth we get instead

$$
\begin{equation*}
P=P_{s}+\int_{0}^{z} \rho(z) g d z \tag{125}
\end{equation*}
$$

i.e. the pressure at depth $z$ increases by an amount which is equal to the weight of the column of fluid above it. A closely related concept is that of buoyancy. Since the pressure inside a fluid in equilibrium depends on the depth there will be a net upwards thrust on any object submerged in it. It is easy to see that (124) or (125) leads to Archimedes principle, namely that the buoyancy is a force which is directed opposite to and is equal to the weight of the fluid which is displaced by a submerged object. The buoyancy force can be thought of as applied at the center of buoyancy $B$, which is the center of gravity of the displaced fluid. If an object is submerged in a fluid, and is at equilibrium, the force of gravity must cancel the force of buoyancy. We must also consider the stability of the equilibrium. Consider first the
simple case of a balloon with a gondola submerged in air (figure 10(a) and (b)).

At a certain height the weight of the balloon + gondola will balance the force of buoyancy. Since the atmospheric density decreases with height, it is easy to see that if the balloon rises above its equilibrium height it will tend to sink, while if it sinks below the equilibrium height there will be a net upwards force. The equilibrium height is thus stable. We can next consider the stability against small torques. We see that if the center of gravity is below the center of buoyancy there will be a stabilizing torque if the balloon is rocked.

The situation is slightly more complicated if when dealing with a body which is floating on the surface of a liquid (e.g. boat or a log) . In this case an angular displacement of the body will shift the center of buoyancy from its previous position $B$ (figure 10 (c) and (d)) to its new position $B^{\prime}$. We define the metacenter $M$ as the projection of the line of action through $B^{\prime}$ on the line $B G$, where $G$ is the center of gravity. If the metacenter is above the center of gravity the floating object is stable.

### 10.1 How does a $\log$ float?

We illustrate the points made above by an example. A uniform wooden cylinder $(\log )$ has a relative density of $\rho$ For what ratios of diameter to height will it float (a) upright? (b) with the axis parallel to the water surface?

To solve this problem we must first determine the equilibrium position. We next consider an orientation which is tilted by a small angle $\theta$ from the equilibrium orientation and locate the shift in the center of buoyancy. This calculation is then used to determine the metacenter. If the metacenter is above the center of gravity, there is a restoring torque, and the equilibrium orientation is stable against small perturbations. If the metacenter is below the center of gravity, the floating object will topple.
(a). Upright cylinder.

In figure19(a) we plot the submerged part of the cylinder in a coordinate system centered on the cylinder. The submerged volume is $\pi r^{2} \rho h$, the height of the center of buoyancy is $z_{B}=\frac{1}{2} \rho h$ and we choose our coordinate system so that $x_{B}=0$. The $y$-coordinate of the center of buoyancy will depend on the tilt angle $\theta$. The volume of the thin slice in figure 19(a) of thickness dy is

$$
2 \sqrt{r^{2}-y^{2}}(y \tan \theta+\rho h) d y
$$



Figure 18: (a) Balloon floating in air. (b) Tilted balloon. (c) A floating ship. (d) Ship tilted from equilibrium.


Figure 19: (a) Submerged part of upright log. (b) Metacenter and center of buoyancy.


Figure 20: Submerged part of flat log.(a) Cross section of log. (b)Side view.

The $y$ coordinate of the center of buoyancy is thus

$$
y_{B}=\frac{2}{\pi r^{2} \rho h} \int_{-r}^{r} y \sqrt{r^{2}-y^{2}}(y \tan \theta+\rho h) d y
$$

Notice that the term independent of $\theta$ in the integrand averages to zero and introducing a new variable $\phi$ given by $y=r \cos \phi$, we have $d y=-r \sin \phi d \phi$ and obtain

$$
\begin{equation*}
y_{B}=\frac{2 r^{2} \tan \theta}{\pi \rho h} \int_{o}^{\pi} d \phi \cos ^{2} \phi \sin ^{2} \phi=\frac{r^{2} \tan \theta}{2 \pi \rho h} \int_{o}^{\pi} d \phi \sin ^{2} 2 \phi=\frac{r^{2} \tan \theta}{4 \rho h} \tag{126}
\end{equation*}
$$

The center of gravity of the cylinder is at $z_{G}=\frac{h}{2}, y_{G}=0, x_{G}=0$. From figure 19(b) the position of the metacenter on the cylindrical axis can be found as

$$
\begin{equation*}
z_{M}=\frac{\rho h}{2}+\frac{y_{B}}{\tan \theta}=\frac{1}{4}\left(2 \rho h+\frac{r^{2}}{\rho h}\right) \tag{127}
\end{equation*}
$$

The upright position is stable if $z_{M}>\frac{h}{2}$ or with $d=2 r$ if

$$
\begin{equation*}
8(1-\rho) \rho<\frac{d^{2}}{h^{2}} \tag{128}
\end{equation*}
$$

The condition (128) is plotted in figure 21(c).
(b). The log lies flat.

We first locate the equilibrium height $x_{E}$ when the cylinder floats with its axis parallel to the water surface (see figure20(a)). We find for the area A of a circle below $x_{E}$ :

$$
A=2 \int_{-r}^{x_{E}} d x \sqrt{r^{2}-x^{2}}
$$

This integral can be evaluated to yield with $x=\sin \phi$

$$
\begin{gathered}
A=2 r^{2} \int_{-\pi / 2}^{\sin ^{-1}\left(x_{E} / r\right)} \cos ^{2} \phi \\
=r^{2} \int_{-\pi / 2}^{\sin ^{-1}\left(x_{E} / r\right)}(1+\cos 2 \phi) d \phi=\left[\phi+\frac{1}{2} \sin 2 \phi\right]_{-\pi / 2}^{\sin ^{-1}\left(x_{E} / r\right)}
\end{gathered}
$$

Finally using $\sin 2 \phi=2 \sin \phi \cos \phi$ we find using the fact that at the equilibrium height we must also have $A=\rho \pi r^{2}$

$$
\begin{equation*}
A=\rho \pi r^{2}=x_{E} \sqrt{r^{2}-x_{E}^{2}}+r^{2} \sin ^{-1} \frac{x_{E}}{r}+\frac{\pi r^{2}}{2} \tag{129}
\end{equation*}
$$

This equation must be solved numerically. A straightforward method is to guess a value of $x_{E} / r$, substitute it into (129) and solve for $\rho$. The result is shown in figure 21(a).

We have chosen a coordinate system with origin at the center of the cylinder so that the center of gravity is at $x_{g}=y_{g}=z_{g}=0$. The center of buoyancy is given by the centroid of the area A of fig 20(a)

$$
\begin{equation*}
x_{B}=\frac{2}{\rho \pi r^{2}} \int_{-r}^{x_{E}} d x x\left(r^{2}-x^{2}\right)^{\frac{1}{2}}=\frac{-2\left(r^{2}-x_{E}^{2}\right)^{\frac{3}{2}}}{3 \rho \pi r^{2}} \tag{130}
\end{equation*}
$$

A plot of $x_{B}$ vs. $\rho$ is given in figure 21(b). If the cylinder is tilted by a small angle this height will stay approximately the same. The centroid of a wedge of height $h$ is at height $\frac{h}{3}$. The z-component of the buoyancy is thus at

$$
\begin{equation*}
z_{B}=-\left(\frac{1}{2}-\frac{1}{3}\right) \frac{h \Delta h}{2} 2 \sqrt{r^{2}-x_{E}^{2}} \frac{1}{\rho \pi r^{2}} \tag{131}
\end{equation*}
$$

The position with a horizontal axis is stable if the metacenter is above the center of gravity or

$$
\begin{equation*}
x_{B}-\frac{z_{B} h}{\Delta h}>0 \tag{132}
\end{equation*}
$$


(a) Horizontal log, height of surface $\mathrm{X}_{\mathrm{E}}$ vs. density $\rho$.

(b) Horizontal log, height of center of buoyancy $x_{B}$ vs. $\rho$.

(c) Stability diagram for floating log.

Figure 21:

Using (130) we find that the horizontal position is stable if

$$
\begin{equation*}
\frac{d}{h}<\sqrt{\frac{2}{1-x_{E}^{2} / r^{2}}} \tag{133}
\end{equation*}
$$

The stability situation is summarized in figure 21c. Note that for $\rho \neq 0.5$ there is a region of $d / h$ where both the upright and horizontal positions are stable. This situation is analogous to what happens in pipe flow, when the Reynolds number is close to its critical value, and both laminar and turbulent flow may be stable against small perturbations.

### 10.2 Problems

## Problem 10-1:

A hydrometer is a convenient instrument for measuring the density of a liquid from a determination of the depth at which the hydrometer floats. It is desired to construct a hydrometer for which the calibration marks for $1.00 \mathrm{gm} \mathrm{cm}^{-3}$ and $1.20 \mathrm{gm} \mathrm{cm}^{-3}$ are 10 cm apart on a stem of $1.00 \mathrm{~cm}^{2}$ cross-section (see figure 22(a)).
(a). What is the volume of the bulb below the 1.2 mark?
(b.) What is the total mass of the hydrometer?
(c.) How far from the 1.2 mark is the 1.1 mark?

## Problem 10-2:

A open pop can has diameter 7 cm , height 12 cm . It is pushed upside down into water and as a result the trapped air inside the can is isothermally compressed (see figure $22(\mathrm{~b})$ ). At a depth 25 m below the surface the force required to push the can down is zero.
(a). What is the mass of the can?
(b). Is the position of the can at depth $25 m$ stable?


Figure 22: (a) Hydrometer. (b) Submerged pop-can. (c) Dam.

## Problem 10-3:

The cross section of a dam has a rectangular shape as shown in figure 22(c). There is water seepage under the dam producing a lift pressure which is equal to the water pressure at depth $h$. The density of water and of the material in the dam is, respectively is $1000 \mathrm{~kg} \mathrm{~m}^{-3}$ and $2000 \mathrm{~kg} \mathrm{~m}^{-3}$. What is the minimum width to height ratio $w / h$ to prevent the dam from toppling about the point $P$ in the figure.

## Problem 10-4:

A vertical mine shaft extends $3000 m$ down into the earth. The pressure at the top is 100 kpa .
(a). What is the air pressure at the bottom of the shaft? Assume that the temperature is constant and equal to $-10^{\circ} \mathrm{C}$, and that the acceleration of gravity $g_{0}=9.8 \mathrm{~ms}^{-2}=$ const..
(b). The temperature increases linearly with depth from $-10^{\circ} \mathrm{C}$ at the to $30^{\circ} \mathrm{C}$ at the bottom. What is now the pressure at the bottom of the mine shaft.
(c). If the density of the earth is uniform, the acceleration g of gravity will decrease with depth $z$ as $g=g_{0}\left(1-\frac{z}{R}\right)$, where $R=6400 \mathrm{~km}$ is the radius of the earth. Show that the effect of changes in gravity will be small compared to the effect of the temperature variation.

## 11 Ideal fluids. Euler and Bernoulli equation.

At this stage we need to define some concepts and discuss some useful idealizations. We will mostly be concerned with steady flow, which is flow which does not change in time. A slightly different concept is that of uniform flow. Uniformity means that the velocities at any time do not depend on position. Steady flow can be either uniform or nonuniform. If the density of the fluid doesn't vary spatially, or in time, the fluid is incompressible. Incompressible flow in a long straight pipe will be uniform irrespective of whether it is steady or not. We have already encountered the concept of laminar flow where the movement of a volume element of fluid follows a set of nearly parallel stream lines (figure 23). A cylindrical surface of stream lines is called a stream tube. Fluid matter can only enter and leave a stream tube through its ends.


Figure 23: (a) Motion of volume element. (b) Stream tube.

In the case of laminar flow the pattern of streamlines is stabilized by viscous forces. Ideal fluids also flow along streamlines. However, we make the additional assumption that viscous forces can be neglected. The concept of ideal fluids is somewhat contradictory since from (123) the Reynolds number will be infinite if the viscosity is zero. There are, however, a number of situations in which most of the turbulence and viscous dissipation is confined to a boundary layer near the walls of the fluid. If this boundary layer is thin the ideal fluid approximation may be useful, particularly if one is only interested in a crude estimate of the parameters involved. Finally, onedimensional flow is characterized by parameters which only depend on one spatial coordinate.

We next consider the acceleration of a volume element of an ideal fluid in one-dimensional flow and the consequences of conservation laws for mass and energy. Consider a volume element containing a certain mass of fluid which moves with the flow along a streamline. Let $u(s)$ be the velocity a distance $s$ down the streamline. The velocity of this element may depend on time because the flow is non-steady (time dependent) or because the velocity changes along the streamline (non-uniform flow). The change in $u$ if the time is incremented by $d t$ is

$$
\begin{equation*}
d u=\frac{\partial u}{\partial s} d s+\frac{\partial u}{\partial t} d t \tag{134}
\end{equation*}
$$

where $d s=u d t$. We have for the acceleration

$$
\begin{equation*}
a=\frac{d u}{d t}=u \frac{\partial u}{\partial s}+\frac{\partial u}{\partial t} \tag{135}
\end{equation*}
$$

Equation (135) can be generalized to describe the derivative of any quantity $f(s, t)$ which is associated with the volume element of matter flowing along
the stream line. We have

$$
d f=\left(u \frac{\partial f}{\partial s}+\frac{\partial f}{\partial t}\right) d t
$$

Formally, we define the convective derivative as

$$
\begin{equation*}
\frac{d}{d t}=u \frac{\partial}{\partial s}+\frac{\partial}{\partial t} \tag{136}
\end{equation*}
$$

We can extend our description to flow in three dimensions. Let

$$
\mathbf{u}=u_{x} \hat{\mathbf{x}}+u_{y} \hat{\mathbf{y}}+u_{z} \hat{\mathbf{z}}
$$

and

$$
\mathbf{s}=x \hat{\mathbf{x}}+y \hat{\mathbf{y}}+z \hat{\mathbf{z}}
$$

where $\hat{\mathbf{x}}, \hat{\mathbf{y}}, \hat{\mathbf{z}}$ are unit vectors in three Cartesian directions. We have

$$
d u_{x}=\left(u_{x} \frac{\partial u_{x}}{\partial x}+u_{y} \frac{\partial u_{x}}{\partial y}+u_{z} \frac{\partial u_{x}}{\partial z}+\frac{\partial u_{x}}{\partial t}\right) d t=\left(\mathbf{u} \nabla+\frac{\partial}{\partial t}\right) u_{x} d t
$$

giving for the acceleration

$$
\begin{equation*}
\mathbf{a}=\frac{d \mathbf{u}}{d t}=(\mathbf{u} \cdot \nabla) \mathbf{u}+\frac{\partial \mathbf{u}}{\partial t} \tag{137}
\end{equation*}
$$

Similarly, the generalization to three dimensions of the convective derivative of a scalar quantity such as e.g. the temperature is

$$
\frac{d T}{d t}=(\mathbf{u} \cdot \nabla) T+\frac{\partial T}{\partial t}
$$

Let us return to the case of one-dimensional flow and assume that a volume element $\Delta d s$ inside a stream tube of thickness $d s$, cross sectional area $\Delta$, is subject to an external force per unit mass $\mathbf{f}$ and that the pressure drops by an amount $-d P$ from one end of the tube to the other. Newton's second law gives

$$
f \rho \Delta d s-d P \Delta=\rho \Delta d s \frac{d u}{d t}=\rho \Delta d s\left(u \frac{\partial u}{\partial s}+\frac{\partial u}{\partial t}\right)
$$

or if we are dealing with steady flow $\frac{\partial u}{\partial t}=0, \frac{\partial u}{\partial s}=\frac{d u}{d s}$

$$
\begin{equation*}
f=\frac{d P}{\rho d s}+u \frac{d u}{d s} \tag{138}
\end{equation*}
$$

It is easy to generalize (9-6) to three dimensional flow. In vector notation

$$
\begin{equation*}
\mathbf{f}=\frac{1}{\rho} \nabla P+(\mathbf{u} \cdot \nabla) \mathbf{u} \tag{139}
\end{equation*}
$$

Equation (139) is called the Euler equation. (If the time-dependent term, and the effect of viscous forces are included, the result is called the Navier Stokes equation. Numerical or analytical solutions to the Navier Stokes equation are beyond the scope of this course). An important special case of the Euler equation has $\rho=$ const and $f$ the force of gravity acting in the negative $z$-direction. The component of $\mathbf{f}$ along the stream tube is $-g \frac{d z}{d s}$ and we get

$$
\frac{d}{d s}\left(\frac{P}{\rho}+\frac{u^{2}}{2}+g z\right)=0
$$

or

$$
\begin{equation*}
\frac{P}{\rho}+\frac{u^{2}}{2}+g z=\text { constant } \tag{140}
\end{equation*}
$$

This equation is commonly called the Bernoulli equation. It can be thought of as a statement of conservation of energy.

We stress that (140) is only an approximation. It may be worth repeating the assumptions that have been made:

1. The flow is steady.
2. The fluid is incompressible.
3. We have derived (140) for flow along a streamline; the value of the constant may be different along a different streamline.
4. Energy dissipation due to viscous forces and eddies are neglected.

### 11.1 Darcy's law

In order to investigate the range of validity of the different approximations that we have employed let us study the simple example of the draining of a water tank through a hose. Let the cross sectional area of the tank be $A=0.5 \mathrm{~m}^{2}$ and the initial height of the water level be $h=1 \mathrm{~m}$. The tank is drained by a hose of cross sectional area $a=1 \mathrm{~cm}^{2}$, which ends $z=1$ m below the bottom of the tank. Let us first assume that water is an ideal fluid and ask:


Figure 24:
(a) What is the initial flow rate?
(b) How long does it take to empty the tank?

Solution: The pressure $P$ at the bottom of the tank is $P_{\text {atm }}+\rho g h$, where $P_{\text {atm }}$ is the atmospheric pressure and $\rho$ is the mass density of water. The flow velocity inside the water tank is negligibly small. The pressure at the end of the hose where the water escapes is $P_{\text {atm }}$. Bernoulli's equation gives for $u$, the flow velocity in the hose

$$
g(h+z)=\frac{u^{2}}{2}
$$

or $u=\sqrt{2 g(h+z)}$. Substituting numbers we find for the initial velocity

$$
u_{o}=\sqrt{2 \times 9.8 \times 2}=6.26 \mathrm{~m} \mathrm{~s}^{-1} .
$$

Therefore
(a) The initial flow rate is

$$
a u_{o}=a \sqrt{2 g(h+z)}=6.26 \times 10^{-4} \mathrm{~m}^{3} \mathrm{~s}^{-1}
$$

(b) We have $A \dot{h}=$ flow rate, or

$$
\dot{h}=-\frac{a}{A} \sqrt{2 g(h+z)}
$$

This equation can be integrated to yield

$$
\begin{equation*}
t=-\int_{h}^{0} \frac{d h}{\frac{a}{A} \sqrt{2 g(h+z)}}=\frac{A}{a} \sqrt{\frac{2}{g}}(\sqrt{h+z}-\sqrt{z}) \tag{141}
\end{equation*}
$$

Substituting numbers we find

$$
t=.5 \times 10^{4} .45 \times(\sqrt{2}-1)=932 \mathrm{~s}=15.5 \mathrm{~min}
$$

We must now ask ourselves: how realistic is the assumption that water is an ideal fluid? We note that for an ideal fluid the flow rate is independent of tube length $L$. If there are frictional losses it is obvious that the water will flow more slowly if the hose is long. What then is "long" $L=1 \mathrm{~m}$ ? $L=10$ m ? $L=100 \mathrm{~m}$ ?

One way to proceed is to modify the Bernoulli equation to take into account "frictional" losses due to viscosity or turbulence. Without such losses the equation can be written

$$
\begin{equation*}
\frac{u^{2}}{2 g}=h+z \tag{142}
\end{equation*}
$$

We refer to the right hand side, $h+z$ of 142 as the "head". A natural way to proceed is to say that losses cause the effective head to be reduced by a "frictional head" $h_{f}$. The ratio between the frictional head/length and the gravitational head has the dimension of inverse length

$$
\begin{equation*}
\left[\frac{h_{f} / l}{u^{2} / 2 g}\right]=m^{-1} \tag{143}
\end{equation*}
$$

The only other length in the problem is the diameter $d$ (or if you prefer the radius $r$ ) of the hose. It is reasonable to assume an empirical relation

$$
\begin{equation*}
\frac{h_{f}}{l}=f \frac{u^{2}}{2 d g} \tag{144}
\end{equation*}
$$

where $f$ is a dimensionless friction factor (or fudge factor) which depends on the Reynolds number of the flow, and on properties of the hose such as its roughness. Equation (144) is commonly known as Darcy's law ${ }^{4}$. In order to see that (144) is not as arbitrary as it might seem at first sight we note that Poiseuille's formula (105) can be cast in the form (144). We have for Poiseuille flow

$$
\begin{equation*}
\dot{V}=a u_{a v e}=\frac{\Delta P \pi r^{4}}{8 \eta l} \tag{145}
\end{equation*}
$$

where $u_{\text {ave }}$ is the average flow velocity and $\Delta P=\rho g h_{f}$ is the pressure drop in the pipe. We can rewrite (145)

$$
\begin{equation*}
\frac{h_{f}}{l}=\frac{32 \eta}{\rho u_{\text {ave }} d} \frac{u_{\text {ave }}{ }^{2}}{d g}=f \frac{u_{\text {ave }}{ }^{2}}{2 d g} \tag{146}
\end{equation*}
$$

with

$$
\begin{equation*}
f=\frac{64 \eta}{\rho u_{\text {ave }} d}=\frac{64}{R e} \tag{147}
\end{equation*}
$$

[^3]where we have used the definition (123) for the Reynolds number. So, what is the Reynolds number predicted by the Bernoulli equation? We have $\eta=10^{-3} \mathrm{~Pa}$ s for water,
$$
d=2 \sqrt{\frac{a}{\pi}}=1.12 \times 10^{-2} \mathrm{~m}, u_{\text {ave }}=6.26 \mathrm{~m} \mathrm{~s}^{-1}, \rho=10^{3} \mathrm{~kg} \mathrm{~m}^{-3}
$$
giving
$$
\operatorname{Re}=\frac{6.26 \times 10^{3} 1.12 \times 10^{-2}}{10^{-3}}=70000
$$

We find for the friction head

$$
\begin{equation*}
h_{f}=\frac{32(L+z)}{70000} \frac{u_{\text {ave }}{ }^{2}}{d g} \tag{148}
\end{equation*}
$$

giving $h_{f}=28 \mathrm{~cm}$ if $L=1 \mathrm{~m}, h_{f}=1.8 \mathrm{~m}$ if $L=10 \mathrm{~m}$ and $h_{f}=16 \mathrm{~m}$ if $L=100 \mathrm{~m}$. In the two first cases the correction due to frictional losses will be relatively small, while in the case of the long hose ( $L=100 \mathrm{~m}$ ) the correction will be significant ${ }^{5}$.

We must next question if the assumption of laminar flow is reasonable. In chapter9 we stated that flow in a pipe will be turbulent if the Reynolds number is larger than $\sim 2000$. Since we estimated the Reynolds number to be 70000 the assumption of laminar flow is clearly wrong.

A number of engineers have studied flow in pipes made up of different materials and with controlled roughness of the inner walls. The result when $f$ is plotted vs. the Reynolds number is commonly referred to as the Moody diagram (see figure 25)

We write the modified Bernoulli equation on the form

$$
\begin{equation*}
\frac{u^{2}}{2 g}=h+z-\frac{f(L+z)}{d} \frac{u^{2}}{2 g} \tag{149}
\end{equation*}
$$

which can be rewritten

$$
\begin{equation*}
u^{2}=\frac{2 g(h+z)}{1+\frac{f(L+z)}{d}} \tag{150}
\end{equation*}
$$

The flow is thus

$$
\begin{equation*}
\dot{V}=a u=a \sqrt{\frac{2 g(h+z)}{1+\frac{f(L+z)}{d}}} \tag{151}
\end{equation*}
$$

[^4]

Figure 25: Moody diagram. Log-log plot of friction factor f vs. Reynolds number $\rho u_{\text {ave }} d / \eta, k$ is a roughness parameter ( $k \approx 1 \mathrm{~mm}$ for concrete, $0.2-$ 1 mm for wood and 0.05 mm for commercial steel pipe).

The numerator in (151) is the flow if there are no frictional losses. The denominator

$$
\begin{equation*}
\sqrt{1+\frac{f(L+z)}{d}} \tag{152}
\end{equation*}
$$

can be thought of as a correction factor. A typical value of $f$ may be 0.03 . If $L+z=2 \mathrm{~m}$ we have a correction factor $\sim 3$, when $L+z=11 \mathrm{~m}$ the correction factor is $\sim 6$ while for $L+z=101 \mathrm{~m}$ the factor is $\sim 17$. We conclude that the Bernoulli equation should be used with caution, ideal fluids are indeed an idealization, real fluids can be much more complicated.

### 11.2 Problems

## Problem 11-1

Two water reservoirs having a difference in water surface level of 50 m are connected by a 1 km long pipe of 0.6 m diameter. Assuming the friction factor is $f=0.01$ what is the rate of flow (in $\mathrm{m}^{3} \mathrm{~s}^{-1}$ ) and shear stress at the wall of the pipe?

## Problem 11.2.

An airplane of mass 8000 kg has wing area of $400 \mathrm{~m}^{2}$. The speed of air flow is $100 \mathrm{~m} \mathrm{~s}^{-1}$ over the top of the wing and 70 $\mathrm{m} \mathrm{s}^{-1}$ below the bottom of the wing. Assume the density of air varies with height as $\rho_{0} \exp (-z / 8000 \mathrm{~m})$ with $\rho_{0}=1.3 \mathrm{~kg} \mathrm{~m}^{-3}$.
(a) What is the lift force on the airplane at low altitude.
(b) How much load could be carried if the plane were to fly at an altitude of 5000 m ?

## Problem 11-3.

Suppose the velocity coordinates of a fluid in some region is given by

$$
\mathbf{u}=\left[0.1 y^{2}, 0.2 x y, 0\right] \mathrm{m} \mathrm{~s}^{-1}
$$

The temperature in the same region is given by

$$
T=270+0.5 x t \mathrm{~K}
$$

(a) Show that the flow is irrotational in the region, i.e $\nabla \times \mathbf{u}=$ 0.
(b) Find the convective derivative of $T$.
(c) The rate of change of the temperature at the point $\mathbf{r}=[\mathrm{x}, \mathrm{y}, \mathrm{z}]$
is

$$
\frac{\partial T}{\partial t}=0.5 x \mathrm{~K} \mathrm{~s}^{-1}
$$

Explain what is the physical difference between the partial and convective derivative.

## 12 Water waves.

We are all familiar with waves on the ocean or on the surface of a lake. In the first section of this chapter we will consider ordinary water waves where the driving force is the wind. At very low speed a wind will cause a laminar shear flow. At moderate speeds of the order meters per second this flow becomes unstable and waves are generated. This effect is commonly called the Kelvin-Helmholtz instability. As it turns out, it is difficult to answer questions such as: what is the amplitude of the resulting wave? and which wavelength is selected? However, some features of water waves can be determined if we simply assume that we are dealing with a sinusoidal wave of wave number $k$. In particular we will find that with some simplifications we can determine the dispersion relation, which is the relation between the angular frequency $\omega$ and wave vector $k$. Once the dispersion relation is known, we can find the speed with which the wave propagates. In the second section of this chapter we consider solitary waves and discuss conditions for a hydraulic jump and discuss applications to tidal waves.

### 12.1 Gravity and capillary waves

Consider a channel of height $h$. We let $y$ represent the vertical position with $y=0$ the surface of water at rest. The wave propagates in the $z$-direction, with wave number $k$. The water which at rest was at height $y$, horizontal position $z$, will as a result of the wave motion be displaced by an amount $\psi$ with components $\psi_{y}, \psi_{z}$. The amplitude of the wave will depend on the height $y$ (waves are only prominent near the surface) but for a uniform channel we don't expect the amplitude of the wave to depend on $z$. We assume waves on the form

$$
\begin{align*}
& \psi_{y}(y, z, t)=A_{y}(y) \cos (\omega t-k z)  \tag{153}\\
& \psi_{z}(y, z, t)=A_{z}(y) \sin (\omega t-k z)
\end{align*}
$$

Squaring both sides of the equations and using $\sin ^{2}(x)+\cos ^{2}(x)=1$ we find

$$
\begin{equation*}
\left(\frac{\psi_{y}}{A_{y}}\right)^{2}+\left(\frac{\psi_{z}}{A_{z}}\right)^{2}=1 \tag{154}
\end{equation*}
$$

which is the equation for an ellipse. Therefore, the assumption (153) means that the particles near the surface of the water execute an elliptical path (figure 26(b)). The wave pattern during a complete cycle is described in figure 26(a).

The velocity field can be obtained by differentiating (153) with respect to time

$$
\begin{align*}
& u_{y}=-\omega A_{y} \sin (\omega t-k z)  \tag{155}\\
& u_{z}=\omega A_{z} \cos (\omega t-k z)
\end{align*}
$$

We next wish to derive some formulas for the height-dependence of the amplitude. To do this we need to make some simplifying assumptions. The first is that we assume water to be incompressible. In the next chapter we discuss flow where density changes are important. The idealization of restricting ourselves to incompressible flow is fairly harmless in the present case where we consider flow with a free surface. Mathematically, the restriction of incompressible flow can be expressed by the condition that there is no net flow in or out of a volume element (figure $27(\mathrm{a})$ ) $d V=d x d y d z$ centered at $\mathbf{r}=\{x, y, z\}$

If the velocity field is $\mathbf{u}=\left\{u_{x}, u_{y}, u_{z}\right\}$ we can write for the inflow through the $y, z$ surface at $x-\frac{d x}{2}$

$$
u_{x}\left(x-\frac{d x}{2}, y, z\right) d y d z
$$



Figure 26: (a) Displacement pattern. (b) Path of a volume element of water.
similarly, the flow through the opposite end is

$$
-u_{x}\left(x+\frac{d x}{2}, y, z\right) d y d z
$$

Adding up the contributions through all six faces yields

$$
\begin{aligned}
0 & =\left[u_{x}\left(x-\frac{d x}{2}, y, z\right)-u_{x}\left(x+\frac{d x}{2}, y, z\right)\right] d y d z \\
& +\left[u_{y}\left(x, y-\frac{d y}{2}, z\right)-u_{y}\left(x, y+\frac{d y}{2}, z\right)\right] d x d z \\
& +\left[u_{z}\left(x, y, z-\frac{d z}{2}\right)-u_{z}\left(x, y, z+\frac{d z}{2}\right)\right] d x d y
\end{aligned}
$$

Using

$$
u_{x}\left(x-\frac{d x}{2}\right)=u_{x}(x)-\frac{\partial u_{x}}{\partial x} \frac{d x}{2}
$$

and similar formulas for the other components, we find, if we put the flow out equal to the flow in

$$
-\left[\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}+\frac{\partial u_{z}}{\partial z}\right] d x d y d z=0
$$

The condition of incompressible flow is thus

$$
\begin{equation*}
\nabla \cdot \mathbf{u}=\frac{\partial u_{x}}{\partial x}+\frac{\partial u_{y}}{\partial y}+\frac{\partial u_{z}}{\partial z}=0 \tag{156}
\end{equation*}
$$

In the special case of one dimensional flow in a shallow channel we find

$$
\begin{equation*}
0=\frac{\partial u_{y}}{\partial y}+\frac{\partial u_{z}}{\partial z}=-\omega \sin (\omega t-k z)\left(\frac{d A_{y}}{d y}-k A_{z}\right) \tag{157}
\end{equation*}
$$

giving

$$
\begin{equation*}
\frac{d A_{y}}{d y}-k A_{z}=0 \tag{158}
\end{equation*}
$$

We next neglect the effect of viscosity and assume that the flow is irrotational, i.e. that there are no eddies. If there are no eddies the circulation satisfies (see figure $27(\mathrm{~b})$ )

$$
\begin{equation*}
\oint_{c} \mathbf{u} \cdot \mathbf{d s}=0 \tag{159}
\end{equation*}
$$

along any closed path.


Figure 27: (a) Divergence theorem. (b) Stokes' theorem.

In order to produce eddies one must provide angular momentum to the system, something which requires a torque. If there is no viscosity there are no shear forces associated with the wind causing the waves. The assumption of irrotational flow is clearly an idealization; in practice it means that we are neglecting any mechanism which leads to damping of the waves. From Stokes' theorem (figure 27(b))

$$
\begin{equation*}
\oint_{c} \mathbf{u} \cdot \mathbf{d s}=\int_{A}(\nabla \times \mathbf{u}) \cdot \mathbf{d} \mathbf{A}=0 \tag{160}
\end{equation*}
$$

In (160) $\mathbf{d A}$ is an element of area represented by a vector normal to the surface. If there is no circulation anywhere, (160) must hold independently of the contour c and

$$
\begin{equation*}
\nabla \times \mathbf{u}=0 \tag{161}
\end{equation*}
$$

everywhere. In our model problem,
flow in a shallow channel, everything is independent of the $x$-coordinate and (161) yields

$$
0=\frac{\partial u_{y}}{\partial z}-\frac{\partial u_{z}}{\partial y}=\omega \cos (\omega t-k z)\left[k A_{y}-\frac{d A_{z}}{d y}\right]
$$

or

$$
\begin{equation*}
\frac{d A_{z}}{d y}-k A_{y}=0 \tag{162}
\end{equation*}
$$

We differentiate (158) once more with respect to $y$ and substitute (162) to obtain

$$
\begin{equation*}
\frac{d^{2} A_{y}}{d y^{2}}-k^{2} A_{y}=0 \tag{163}
\end{equation*}
$$

The general solution to (163) is

$$
\begin{equation*}
A_{y}=a e^{k y}+b e^{-k y} \tag{164}
\end{equation*}
$$

where $a$ and $b$ are arbitrary constants. If we assume that the bottom of the channel is firm we must have $A_{y}(-h)=0$. We also assume that the amplitude of the wave near the surface is $A_{y}(0)=A$. Substitution of the boundary conditions into (164) gives two equations with two unknowns $a$ and $b$. Solving these equations we find after some algebra

$$
\begin{align*}
& A_{y}(y)=A \frac{e^{k(h+y)}-e^{-k(h+y)}}{e^{k h}-e^{-k h}}=A \frac{\sinh [k(h+y)]}{\sinh (k h)} \\
& A_{z}(y)=A \frac{e^{k(h+y)}+e^{-k(h+y)}}{e^{k h}-e^{-k h}}=A \frac{\cosh [k(h+y)]}{\sinh (k h)} \tag{165}
\end{align*}
$$

From (165) we see that the elliptical trajectories (154) always have the major axis parallel to the horizontal and that the ratio of the minor to the major axis is

$$
\tanh [k(h+y)]<1
$$

Since $A_{y}$ and $A_{z}$ in (165) have the same sign, we note from (153) that when a wave is propagating in the positive $z$ direction $\psi_{z}$ lags $90^{\circ}$ behind $\psi_{y}$ and the motion along the elliptic orbit is clockwise.

If $k h \gg 1$ we speak of deep water waves and the equations for the wave amplitudes simplify to

$$
\begin{align*}
& \psi_{y}=A e^{k y} \cos (\omega t-k z) \\
& \psi_{z}=A e^{k y} \sin (\omega t-k z) \tag{166}
\end{align*}
$$

In this case the particles in the wave will undergo a circular motion.
Another property of the wave which we would like to determine is the speed at which it travels. The ratio

$$
u_{\phi}=\frac{\omega}{k}
$$

is called the phase velocity of the wave. As we shall see, the phase velocity will depend on the wavelength. Such waves are called dispersive. In order to
determine the phase velocity we must look at the energetics of the situation, suggesting the application of the Bernoulli equation. However, this equation is only valid for steady flow. On the other hand, the wave motion (153) is manifestly time dependent. This problem can be overcome by making a coordinate transformation to a frame of reference which is moving with velocity $\omega / k$ in the $z$-direction. In this frame of reference

$$
\begin{equation*}
z^{\prime}=z-\frac{\omega t}{k} \tag{167}
\end{equation*}
$$

and we find for the components of the velocity in the moving frame

$$
\begin{align*}
u_{y}\left(y, z^{\prime}\right) & =\omega A_{y}(y) \sin \left(k z^{\prime}\right)  \tag{168}\\
u_{z^{\prime}}\left(y, z^{\prime}\right) & =\omega A_{z}(y) \cos \left(k z^{\prime}\right)-\frac{\omega}{k}
\end{align*}
$$

In the moving frame the flow is thus steady and we can apply the Bernoulli equation. Let us consider a streamline at the water surface. The height of the surface can be found by substituting $y=0$ into the first equation (165)

$$
\begin{equation*}
\psi_{y}(0)=A \cos \left(k z^{\prime}\right) \tag{169}
\end{equation*}
$$

while we have from the second equation (165)

$$
\begin{equation*}
A_{z}(0)=A \operatorname{coth}(k h) \tag{170}
\end{equation*}
$$

We find from (168)

$$
\begin{gather*}
u^{2}=u_{y}{ }^{2}+u_{z^{\prime}}{ }^{2}= \\
=\omega^{2}\left(A^{2}\left[\sin ^{2}\left(k z^{\prime}\right)+\operatorname{coth}^{2}(k h) \cos ^{2}\left(k z^{\prime}\right)\right]-\frac{2 A}{k} \operatorname{coth}(k h) \cos \left(k z^{\prime}\right)+\frac{1}{k^{2}}\right) \tag{171}
\end{gather*}
$$

We now assume that the amplitudes $A_{z}(0), A_{y}(0)$ are small, or more precisely small compared to $1 / k$ (which is proportional to the wavelength $\lambda=$ $2 \pi / k)$. This allows us to neglect the terms proportional to $A^{2}$ in (171) and we have

$$
\begin{equation*}
u^{2} \simeq \frac{\omega^{2}}{k^{2}}-\frac{2 A}{k} \operatorname{coth}(k h) \cos \left(k z^{\prime}\right) \tag{172}
\end{equation*}
$$

We must next consider the effect of surface tension. The surface tension comes about because there is an energy cost associated with the creation of a free surface

$$
\begin{equation*}
E_{\text {surface }}=\sigma \times \text { area } \tag{173}
\end{equation*}
$$



Figure 28: (a) Surface tension decreases pressure. (b) Surface tension increases pressure.(c) Radius of curvature.

Equation (173) defines the surface tension $\sigma$. The order of magnitude of $\sigma$ can be obtained by considering the binding energy $-\epsilon$ per molecule in the fluid. Crudely speaking, the molecules in the surface layer only feel $\frac{1}{2}$ of their neighbors. This gives

$$
\begin{equation*}
\sigma \simeq \frac{1}{2} \epsilon n d \tag{174}
\end{equation*}
$$

where $n$ is the number of molecules per unit volume and $d$ is the molecular diameter. Experimentally, for water at room temperature, $\sigma=0.073 \mathrm{Nm}^{-1}$.

Consider a part of a water wave which is curved upwards as in figure 12.1(a) The surface tension attempts to contract the surface and this gives rise to a negative contribution to the pressure just below the surface. If, on the other hand, the surface is curving downwards as in figure 12.1(b) the stress contracting the surface causes a positive contribution to the pressure.

To find the size of the contribution let us consider a half cylinder of radius $R$, of length $L$ (figure 12.1(c). The volume of the half cylinder is $\frac{1}{2} \pi R^{2} L$ while the area is $\pi R L$. If the radius is changed by an amount $d R$ the surface energy changes by

$$
d E=\pi L \sigma d R
$$

The work associated with this is

$$
d E=P_{s} d V=P_{s} d\left(\frac{\pi}{2} R^{2} L\right)=P_{s} \pi R L d R
$$

By comparison we find that the surface tension contributes a term

$$
\begin{equation*}
P_{s}=\frac{\sigma}{R} \tag{175}
\end{equation*}
$$

We can write for the radius of curvature of a small amplitude wave

$$
\frac{1}{R}=\frac{\partial^{2} \psi_{y}}{\partial z^{2}}
$$

allowing $R$ to be positive if the surface curves upwards, negative if it curves downwards. If $\frac{\partial^{2} \psi_{y}}{\partial z^{2}}>0$ the surface curves
upwards, otherwise it curves downwards. The pressure which enters the Bernoulli equation is thus

$$
\begin{equation*}
P=P_{a t m}-\sigma \frac{\partial^{2} \psi_{y}}{\partial z^{2}}=P_{a t m}+\sigma k^{2} A \cos k z^{\prime} \tag{176}
\end{equation*}
$$

where $P_{\text {atm }}$ is the atmospheric pressure. Adding terms we find that the Bernoulli equation for a streamline near the surface is

$$
\begin{gather*}
\text { const }=\frac{P}{\rho}+\frac{u^{2}}{2}+g A \cos \left(k z^{\prime}\right) \\
=\frac{P_{\text {atm }}}{\rho}+\frac{\sigma k^{2}}{\rho} A \cos k z^{\prime}+\frac{\omega^{2}}{2 k^{2}}-\frac{A \omega^{2}}{k} \operatorname{coth}(k h) \cos \left(k z^{\prime}\right)+g A \cos \left(k z^{\prime}\right) \tag{177}
\end{gather*}
$$

This equation must hold for all values of $z^{\prime}$. For this to be possible the terms proportional to $\cos \left(k z^{\prime}\right)$ must add to zero or

$$
\omega^{2} \operatorname{coth}(k h)=\frac{\sigma k^{3}}{\rho}+g k
$$

from which we obtain the dispersion relation

$$
\begin{equation*}
\omega=\sqrt{\left(g k+\frac{\sigma k^{3}}{\rho}\right) \tanh (k h)} \tag{178}
\end{equation*}
$$

and we have for the phase velocity

$$
\begin{equation*}
u_{\phi}=\frac{\omega}{k}=\sqrt{\left(\frac{g}{k}+\frac{\sigma k}{\rho}\right) \tanh (k h)} \tag{179}
\end{equation*}
$$

The phase velocity is not the only velocity of interest, we must also consider the group velocity

$$
\begin{equation*}
u_{g}=\frac{d \omega}{d k}=\frac{1}{2}\left[g+\frac{3 \sigma k^{2}}{\rho}+\left(g+\frac{\sigma k^{2}}{\rho}\right) \frac{k h}{\sinh (k h) \cosh (k h)}\right] \sqrt{\frac{\tanh (k h)}{g k+\frac{\sigma k^{3}}{\rho}}} \tag{180}
\end{equation*}
$$

In most situations the waves which propagate along the water surface do not have a single frequency but will be made up of a group or superposition of waves with slightly different frequencies. Let us consider a superposition of two waves of frequencies $\omega_{1}$ and $\omega_{2}$ and wave numbers $k_{1}$ and $k_{2}$. Using the trigonometric identity

$$
\begin{equation*}
\cos a+\cos b=2 \cos \left[\frac{1}{2}(a+b)\right] \cos \left[\frac{1}{2}(a-b)\right] \tag{181}
\end{equation*}
$$

we have

$$
\begin{aligned}
\cos \left(\omega_{1} t-k_{1} z\right)+ & \cos \left(\omega_{2} t-k_{2} z\right)=2 \cos \left[\frac{1}{2}\left\{\left(\omega_{1}+\omega_{2}\right) t-\left(k_{1}+k_{2}\right) z\right\}\right] \\
& \times \cos \left[\frac{1}{2}\left\{\left(\omega_{1}-\omega_{2}\right) t-\left(k_{1}-k_{2}\right) z\right\}\right]
\end{aligned}
$$

The resulting wave pattern is one in which beats modulate the carrier frequency $\frac{1}{2}\left(\omega_{1}+\omega_{2}\right)$. The velocity of propagation of the beats is

$$
\begin{equation*}
\frac{\omega_{1}-\omega_{2}}{k_{1}-k_{2}} \simeq \frac{d \omega}{d k}=u_{g} \tag{182}
\end{equation*}
$$

if $\omega_{1}, \omega_{2}$ are close to $\omega$ and $k_{1}, k_{2}$ are close to $k$. If we have a situation in which a calm surface is disturbed, such as when a stone is thrown into a pond, a group of waves is produced. This group is made up of individual waves of a range of frequencies. The component waves propagate with the phase velocity while the disturbance itself propagates with the group velocity. As long as the frequency range and the amplitude is not too large the group velocity will be given by (180).

Let us now consider some special cases:
If $k h \gg 1, \tanh (k h) \simeq 1$, and we have deep water waves. For deep water waves

$$
\begin{equation*}
\omega=\sqrt{g k+\frac{\sigma k^{3}}{\rho}} \tag{183}
\end{equation*}
$$



Figure 29: (a) Phase and group velocity for deep water waves.

There are two limiting cases, if

$$
g k \gg \frac{\sigma k^{3}}{\rho}
$$

we have gravity waves. In the opposite limit

$$
g k \ll \frac{\sigma k^{3}}{\rho}
$$

we talk about ripples or capillary waves. Using $\sigma=0.073 \mathrm{~N} \mathrm{~m}^{-1}, g=9.81$ $\mathrm{m} \mathrm{s}^{-2}, \rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}$ we find that

$$
g k=\frac{\sigma k^{3}}{\rho}
$$

when $\lambda=\frac{2 \pi}{k}=0.017 \mathrm{~m}=1.7 \mathrm{~cm}$. For gravity waves

$$
\begin{equation*}
\omega=\sqrt{g k} \tag{184}
\end{equation*}
$$

The phase velocity of gravity waves is thus

$$
\begin{equation*}
u_{\phi}=\sqrt{\frac{g}{k}} \tag{185}
\end{equation*}
$$

while for the group velocity

$$
\begin{equation*}
u_{g}=\frac{1}{2} u_{\phi} \tag{186}
\end{equation*}
$$

For capillary waves

$$
\begin{equation*}
\omega=k^{3 / 2} \sqrt{\frac{\sigma}{\rho}} \tag{187}
\end{equation*}
$$

The phase velocity is now

$$
\begin{equation*}
u_{\phi}=\sqrt{\frac{k \sigma}{\rho}} \tag{188}
\end{equation*}
$$

and the group velocity

$$
\begin{equation*}
u_{g}=\frac{3}{2} u_{\phi} \tag{189}
\end{equation*}
$$

¿From these equations we see that gravity waves move faster the longer the wavelength; also a group of waves will travel slower than the individual waves. This situation is reversed for ripples. In the latter case short wavelength waves move the fastest and the group will travel faster than the phase. In figure 29 we plot, for deep water waves, the behavior of the phase and group velocities in the transition region between gravity and capillary waves.

### 12.2 The hydraulic jump

In the limit that $k h \ll 1$ (shallow waves) we have

$$
\tanh (k h) \simeq k h
$$

and we see that for gravity waves in a shallow channel $\omega=k \sqrt{g h}$

$$
\begin{equation*}
u_{g}=u_{\phi}=\sqrt{g h} \tag{190}
\end{equation*}
$$

We note that both the phase and group velocities will be slower in a shallow channel than in a deep channel. This has a consequence that waves tend to pile up as they reach a gently sloping beach and they eventually break (the surf is up!). Equation (190) implies that the speed of gravity waves in a shallow channel is independent of the wave vector i.e. is dispersion-less; the group and phase velocities are the same. This allows for the possibility of a solitary wave traveling along a river or canal without being dispersed. Striking example of this are tidal bores and tidal waves following earthquakes (tsunamis). In Canada, the most well known example of a tidal bore

## Tidal wave



Figure 30: Hydraulic jump in moving frame
occurs on the Peticodiac river in New Brunswick, while there have been tsunamis causing considerable damage in the Alberni inlet on Vancouver Island. For a bore to occur one needs very high tides and a river with gently sloping bottom entering a funnel shaped estuary. A bore can be thought of as a hydraulic shock front moving upstream. Behind the front the water flows upstream for some time before the current reverses. Since the channel is deeper behind the bore the wave speed is faster behind the front than before it. This means that energy is fed into the front and prevents it from dissolving. The ratio of the speed of the front to the wave velocity in front of it is called the Froude number. As we shall see, it is a general condition for the formation of shocks that the Froude number is greater than one. If the Froude number is less than about 1.7 the bore will have an undular shape. For a larger Froude number the wave will break, and the larger the Froude number the more violent the front will be. Clearly tidal bores represent a complicated phenomenon which depends much on the details of the local conditions.

We will here consider the simple case of a hydraulic jump where an undisturbed shallow channel is invaded by an elevation wave or tidal wave. In order to have a manageable problem we neglect friction between the flowing water and the channel walls, and also assume a uniform velocity profile through the channel. Consider a frame of reference in which the front of the hydraulic jump is at rest (Figure 30). We will assume the situation is stationary. This assumption is not realistic, since after some
time the height of the tidal wave will decrease, but for a sufficiently short time period we can take the amplitude to be constant. In the front frame, the undisturbed region is flowing towards it with velocity $u_{1}$, and we let $u_{2}$ be the flow velocity behind the front. Conservation of mass give for the volume flow per unit channel width

$$
\begin{equation*}
\dot{V}=h_{1} u_{1}=h_{2} u_{2} \tag{191}
\end{equation*}
$$

¿From (124) we find for the net pressure force per unit width $w$ of the channel

$$
\frac{F}{w}=\int_{0}^{h} \rho g z d z=\frac{1}{2} \rho g h^{2}
$$

The momentum balance equation can then be written as

$$
\begin{equation*}
F_{1}-F_{2}=(\text { mass flow rate }) \times\left(u_{2}-u_{1}\right) \tag{192}
\end{equation*}
$$

The mass flow rate per unit channel width is

$$
j_{M}=\rho \dot{V}
$$

We can rewrite (192)

$$
\begin{equation*}
\rho u_{1}^{2} h_{1}+\frac{1}{2} \rho g h_{1}^{2}=\rho u_{2}^{2} h_{2}+\frac{1}{2} \rho g h_{2}^{2} \tag{193}
\end{equation*}
$$

Combining (193) and (191) one finds after a little algebra

$$
\begin{equation*}
0=\left(h_{1}-h_{2}\right)\left[h_{1}+h_{2}-\frac{2 \dot{V}^{2}}{g h_{1} h_{2}}\right] \tag{194}
\end{equation*}
$$

Equation (194) has a trivial solution $h_{1}=h_{2}$, and also a nontrivial solution in which a height difference $h_{2}-h_{1} \neq 0$ is maintained. Solving 194 for $h_{2}$ we find for the nontrivial solution

$$
\begin{equation*}
h_{2}=-\frac{h_{1}}{2}+\sqrt{\frac{h_{1}^{2}}{4}+\frac{2 \dot{V}^{2}}{g h_{1}}} \tag{195}
\end{equation*}
$$

(when solving the quadratic equation for $h_{2}$ we discard one root which gives a negative value for $h_{2}$ ). The balance equations are perfectly symmetric with respect to $h_{1}$ and $h_{2}$; they permit both a positive and negative surge.

Only the positive surge $h_{1}<h_{2}$ turns out to be stable. To see this consider the energy flow per unit width of the cannel

$$
\begin{equation*}
j_{E}=\int_{0}^{h}\left(\rho g z+\frac{1}{2} \rho u^{2}\right) u d z=\frac{j_{M}\left(g h+u^{2}\right)}{2} \tag{196}
\end{equation*}
$$

We find from (193), (196) and (191)

$$
\left(j_{E}\right)_{1}-\left(j_{E}\right)_{2}=g j_{M}\left(h_{1}^{2}+h_{2}^{2}\right) \frac{h_{2}-h_{1}}{4 h_{1} h_{2}}
$$

The turbulence and viscous shear generated at the jump makes the process dissipative; energy is not conserved. Rather we must have $\left(j_{E}\right)_{1}-\left(j_{E}\right)_{2}>0$. As a consequence only the solution $h_{2}>h_{1}$ is allowed. The unphysical solution $h_{1}>h_{2}$ corresponds making a film of a tidal wave coming in and showing the movie backwards. Clearly something would appear to be "wrong".

### 12.3 Problems

## Problem 12-1.

Breaking waves. The theory of section 12.1 is only valid for small amplitudes. If the height of the wave is increased it will eventually break. To see why this is bound to happen consider the amplitude equations (166) for deep waves. In these equations $z$ and $y$ represent the coordinates of a particle when the fluid is at rest. At the surface $y=0$ the actual $z$ coordinate of the particle will be

$$
\zeta=z+A \sin (\omega t-k z)
$$

where z is the position when the particle is at rest. The y coordinate of the particle will be

$$
\eta=A \cos (\omega t-k z)
$$

If we plot the surface at some instant $t$ (i.e. $\eta$ versus $\zeta$ ) we see that the wave will fold back on itself if $\frac{\partial \zeta}{\partial z}$ is negative (Figure 31). This happens if $A k>1$. Since the phase velocity is $\omega / k$ and the velocity amplitude is $A \omega$ a criterion for the amplitude for which a wave about to break is that the horizontal speed is equal to the phase velocity. Estimate the height of a deep water wave of wavelength 10 m which is just about to break.


Figure 31:

## Problem 12-2.

Shallow-draught canal boats pulled by horses are most easy to pull if they are made to travel on their own bow wave. Treat the bow wave as a shallow water wave. If the maximum speed at which the horses can pull is $4 \mathrm{~m} \mathrm{~s}^{-1}$, what is the maximum canal depth at which this trick will work?

## Problem 12-3.

A tidal wave of height $h_{2}-h_{1}=5 \mathrm{~m}$ is surging up a 5 m deep channel. How fast is it traveling? What is the Froude number of the front? What is the Froude number behind the tidal wave?

## Problem 12-3.

A ship traveling with the wind overtakes waves with wavelength 20 m every 5 s . How fast is the ship going?

## Problem 12-4.

A bay is open towards deep water at one end and closed at the other. The average depth is 50 m . The resonance frequency for long wavelength oscillations is 12.5 hrs . What is the effective length of the bay?

## Problem 12-5.

A shallow lake is 100 km long. Sometimes one can observe an end to end oscillations of the surface called a seiche with period 5.5 hrs . What is the effective depth of the lake.

## 13 Compressible flow.

In the previous chapter we made the idealization that the fluid is incompressible, and we also neglected any possible change in the internal energy of the fluid causing changes in temperature along the flow. In this chapter we discuss sound waves, which are compression and dilation waves, and more extreme conditions such as shocks. To do this we must go beyond the Bernoulli equation which assumes incompressible flow and which doesn't take into account heat and work exchanged with the surroundings. For this purpose we must make contact with thermodynamics.

The first law of thermodynamics is a restatement of the principle of energy conservation which distinguishes between two forms of energy; heat and work.

$$
\begin{equation*}
\Delta Q=\Delta E+\Delta W \tag{197}
\end{equation*}
$$

Here $\Delta Q$ is the heat given to the system, $\Delta E$ is the change in the internal energy of the working substance and $\Delta W$ is the work done by the system. Let us consider a mass element of fluid $\Delta m$ which upstream has cross section $A_{1}$ and length $l_{1}$. The internal energy per unit mass is initially $e_{1}$. In order to get this mass element into our system we must do the work $P_{1} A_{1} l_{1}$. Since this is work done on the mass element the contribution to $\Delta W$ in (197) is negative. Eventually, after having done the work $\Delta W$ and having received the heat $\Delta Q$, the mass elements escape downstream, performing the final work $P_{2} A_{2} l_{2}$. The height upstream is $z_{1}, z_{2}$ downstream. Conservation of energy then gives
$\Delta Q-\Delta W=\Delta m\left[\left(e_{2}-e_{1}\right)+\frac{1}{2}\left(u_{2}{ }^{2}-u_{1}{ }^{2}\right)+g\left(z_{2}-z_{1}\right)\right]+P_{2} A_{2} l_{2}-P_{1} A_{1} l_{1}$
Dividing by $\Delta m$ we get the balance equation

$$
\begin{equation*}
\frac{\Delta Q}{\Delta m}-\frac{\Delta W}{\Delta m}=e_{2}-e_{1}+\frac{1}{2}\left(u_{2}^{2}-u_{1}^{2}\right)+g\left(z_{2}-z_{1}\right)+\frac{P_{2}}{\rho_{2}}-\frac{P_{1}}{\rho_{1}} \tag{198}
\end{equation*}
$$

An important special case is a free adiabatic processes for which $\Delta Q$ and $\Delta W$ are zero. In the case of steady flow we have along a streamline, from

$$
\begin{equation*}
\frac{P}{\rho}+\frac{u^{2}}{2}+g z+e=\text { constant } \tag{198}
\end{equation*}
$$

It is convenient at this stage to introduce the enthalpy

$$
H=E+P V
$$

The enthalpy per unit mass is

$$
h=e+\frac{P}{\rho}
$$

The balance equation (199) is then

$$
\begin{equation*}
h+\frac{u^{2}}{2}+g z=\text { constant } \tag{200}
\end{equation*}
$$

Note the similarity between (200) and the Bernoulli equation. The difference is that we now take into account the effect of conversion of kinetic energy into heat. We must then replace the pressure term in the Bernoulli equation by an enthalpy term. Consider an ideal gas working substance. Let $C_{P}$ be the specific heat per mole at constant pressure and $C_{V}$ the specific heat per mole at constant volume. You will have learned from an elementary thermodynamics course that

$$
C_{P}=C_{V}+R
$$

where $R$ is the gas constant. The internal energy per mole is

$$
E=C_{V} T
$$

and the enthalpy per mole is

$$
H=C_{P} T
$$

In the case of air, the gravity term in (200) can usually be neglected and we find

$$
\begin{equation*}
c_{P} T+\frac{1}{2} u^{2} \simeq \mathrm{constant} \tag{201}
\end{equation*}
$$

where $c_{p}$ is the specific heat per unit mass.

### 13.1 Stagnation temperature

Consider a projectile or an airplane traveling through air at constant speed $u$. The temperature far away is $T$. Near the tip of the nose cone the air will be moving with approximately the same speed as the projectile while in a frame of reference moving with the projectile the speed far away is $u$ (figure 32). Equation (201) then predicts that the stagnation temperature $T_{s}$ just in front of the cone will be given by

$$
c_{p} T+\frac{1}{2} u^{2}=c_{p} T_{s}
$$



Figure 32:
or

$$
T_{s}-T=\frac{u^{2}}{2 c_{p}}
$$

For air $c_{p} \simeq 1.0 \mathrm{~kJ} \mathrm{~kg}^{-1} \mathrm{~K}^{-1}$ ). We see that for an airplane traveling at 200 $\mathrm{m} \mathrm{s}^{-1}$ the temperature rise will be about 20 K . A bullet from a high speed rifle may travel at the speed of $1 \mathrm{~km} \mathrm{~s}^{-1}$. The stagnation temperature will then be quite high, about 500 above the ambient temperature. Of course, we are dealing with a very idealized situation. The assumption of adiabatic flow will not always be that accurate - there may be a considerable amount of heat transport away from the cone.

### 13.2 Sound waves

Sound waves are small amplitude elastic waves that can propagate in a fluid or solid. In the case of fluids long wavelength sound waves are longitudinal, and the waves consist of an alternating pattern of rarefactions and compressions. For ordinary sound the changes in pressure tend to be very small. The intensity of a sound wave is often measured in decibels

$$
I_{d B}=20 \log _{10}\left(\frac{P_{e}}{P_{r e f}}\right)
$$

The reference pressure is

$$
P_{r e f}=2 \times 10^{-10} b a r
$$



Figure 33:
and $P_{e}$ is the root mean square pressure amplitude $(1 / \sqrt{2}$ of the peak excess pressure). At the pain level of 120 dB the peak excess pressure will be $2 \sqrt{2} \times 10^{-4}$ bar, which is small compared to the ambient pressure $\simeq 1$ bar. The mechanism for a sound wave is that gas motion generates a change in the density, which causes a change in pressure. There will then be an unbalanced force which accelerates the gas and causes the cycle to repeat. We write for the pressure and density

$$
\begin{aligned}
P & =P_{a}+P_{e} \\
\rho & =\rho_{a}+\rho_{e}
\end{aligned}
$$

where the subscript $a$ stands for average while $e$ stands for excess.
The relationship between changes in density and pressure depends on the properties of the medium in which the sound waves propagate. We will assume that the processes are fast enough that they can be considered to be adiabatic, i.e. without any heat transport. Since the amplitudes $P_{e}$ and $\rho_{e}$ are small compared to the ambient conditions we assume that they are proportional to each other

$$
\begin{equation*}
P_{e}=\kappa \rho_{e} \tag{202}
\end{equation*}
$$

with

$$
\begin{equation*}
\kappa=\left.\frac{\partial P}{\partial \rho}\right|_{\text {adiabatic }} \tag{203}
\end{equation*}
$$

Let us consider a column of air of cross-section $A$ (Figure 33).
When the air is at rest in equilibrium this column extends from $x$ to $x+d x$. We assume that a sound wave is traveling in the $x$-direction and that
at some instant the left end of the column is displaced an amount $\xi(x, t)$ which is small compared to the wavelength of the sound wave. Conservation of mass gives

$$
A \rho_{a} d x=A\left(\rho_{a}+\rho_{e}\right)(d \xi+d x)
$$

or

$$
\rho_{e} d x=-\left(\rho_{a}+\rho_{e}\right) d \xi \simeq-\rho_{a} d \xi
$$

if $\rho_{e} \ll \rho_{a}$, giving

$$
\begin{equation*}
\rho_{e} \simeq-\rho_{a} \frac{\partial \xi}{\partial x} \tag{204}
\end{equation*}
$$

The column is subject to a net force

$$
A[P(x+\xi, t)-P(x+\xi+d x+d \xi, t)] \approx-A \frac{\partial P_{e}(x, t)}{\partial x} d x
$$

causing an acceleration

$$
\begin{equation*}
A d x \rho_{a} \frac{\partial^{2} \xi}{\partial t^{2}}=-A \kappa d x \frac{\partial \rho_{e}}{\partial x}=A \kappa d x \frac{\partial^{2} \xi}{\partial x^{2}} \rho_{a} \tag{205}
\end{equation*}
$$

using (202) and (204). From (205) we get the wave equation

$$
\begin{equation*}
\frac{\partial^{2} \xi}{\partial t^{2}}=\kappa \frac{\partial^{2} \xi}{\partial x^{2}} \tag{206}
\end{equation*}
$$

It is easy to see that (206) admits solutions in the form of traveling waves of the form $f(x \pm c t)$, where $c=\sqrt{\kappa}$, and it can be shown that the general solution to (206) can be written on the form

$$
\xi(x, t)=f_{1}(x-c t)+f_{2}(x+c t)
$$

We next wish to calculate the sound velocity assuming the waves travel in an ideal gas and that the sound vibrations are adiabatic. For an adiabatic process

$$
P V^{\gamma}=\text { constant }
$$

with $\gamma=C_{P} / C_{V}$ or

$$
P=\text { Const. } \rho^{\gamma}
$$

Differentiating we find

$$
\left.\frac{\partial P}{\partial \rho}\right|_{\text {adiabatic }}=\frac{\gamma P}{\rho}=\frac{\gamma k_{B} T}{m}=\frac{\gamma R T}{\mu}
$$

where $m$ is the mass of a molecule and $\mu$ the molecular weight. Using (203) we finally get

$$
\begin{equation*}
c=\sqrt{\frac{\gamma k_{B} T}{m}}=\sqrt{\frac{\gamma R T}{\mu}} \tag{207}
\end{equation*}
$$

It is interesting to compare the speed of sound (207) with typical molecular speeds. ¿From (35) we have for the rms speed

$$
v_{r m s}=\sqrt{\frac{3 k_{B} T}{m}}=\sqrt{\frac{3}{\gamma}} c
$$

Since $\gamma \simeq 1.4$ for air we see that the rms speed and the sound speed are quite comparable.

### 13.3 Shock waves

As we discussed earlier, sound waves are typically encountered as small amplitude waves. We have also met phenomena such as tidal waves where the amplitudes are large. As we shall see, a condition for maintaining a sharp shock front is that the wave velocity of the front is faster than the wave velocity ahead of it. In the case of tidal waves this came about because the wave velocity in a shallow channel increases with height and the water level is higher behind the bore. We will here consider a shock wave in air, caused by e.g. an explosion. The released heat causes the temperature to rise with the result that the sound velocity is larger behind the shock than in front of it.

We will assume that the conditions around the shock change slowly in time (however, as the volume traversed by the shock is increased the excess temperature and pressure will diminish). This slow time evolution takes place in the rest frame of the shock front. If you are in the laboratory frame and get hit by an oncoming shock wave things are not changing slowly at all. We will not here attempt to discuss the time evolution of a shock wave, instead we will only try to develop an equation of state which relates the conditions behind the shock to those ahead of it. We will wish to answer the question: for a given shock speed what is the excess temperature and pressure behind the shock, given the conditions in the undisturbed region in front of the shock. As we shall see this question can be answered from the conservation laws of the problem. We proceed in a manner which is quite similar to what we did in the case of the hydraulic jump, with one important difference. When discussing the hydraulic jump we could ignore the effect


Figure 34:
of the heating of the water in the disturbed region behind the front. In the present case the effect of the dissipated heat is quite important and we need to consider the enthalpy balance in addition to the conservation laws for mass and momentum.

We use subscript 1 to describe the condition ahead of the shock while subscript 2 describes conditions behind the front. The speed $u_{1}$ of the air ahead of the shock is then equal to the shock front speed in the laboratory frame.

Conservation of mass gives rise to the condition

$$
\begin{equation*}
\rho_{1} u_{1}=\rho_{2} u_{2} \tag{208}
\end{equation*}
$$

where $\rho$ is the density, and $u$ the speed. Conservation of momentum tells us (see also the discussion in connection with (192))that with $P$ the pressure

$$
\begin{equation*}
\left(P_{1}-P_{2}\right) \times \text { area }=\text { mass flow rate } \times\left(u_{2}-u_{1}\right)=\operatorname{area} \times \rho_{2} u_{2} \times\left(u_{2}-u_{1}\right) \tag{209}
\end{equation*}
$$

Dividing by the area and using (208) we find

$$
\begin{equation*}
P_{1}-P_{2}=\rho_{2} u_{2}^{2}-\rho_{1} u_{1}^{2} \tag{210}
\end{equation*}
$$

Neglecting the effect of gravity we get from the energy balance (200)

$$
\begin{equation*}
h_{1}+\frac{1}{2} u_{1}^{2}=h_{2}+\frac{1}{2} u_{2}^{2} \tag{211}
\end{equation*}
$$



Figure 35: Ratios between physical quantities on the two sides of the shock front

We define the Mach numbers

$$
\begin{equation*}
M_{1}=\frac{u_{1}}{c_{1}} ; M_{2}=\frac{u_{2}}{c_{2}} \tag{212}
\end{equation*}
$$

noting that the sound velocities $c_{1}, c_{2}$ may be different on the two sides of the front. We further assume that the working substance is an ideal gas for which the enthalpy per unit mass is $h=c_{P} T$. The sound velocity is given by

$$
\begin{equation*}
c=\sqrt{\gamma R T / \mu}=\sqrt{\gamma P / \rho} \tag{213}
\end{equation*}
$$

Equation (211) can be rewritten in terms of the Mach numbers

$$
c_{P} T_{1}+M_{1}^{2} \frac{\gamma R T_{1}}{2 \mu}=c_{P} T_{2}+M_{2}^{2} \frac{\gamma R T_{2}}{2 \mu}
$$

or

$$
\frac{T_{2}}{T_{1}}=\frac{c_{P}+\frac{M_{1}{ }^{2} \gamma R}{2 \mu}}{c_{P}+\frac{M_{2}{ }^{2} \gamma R}{2 \mu}}
$$

Using $c_{P}=\gamma c_{V}$ and $\mathrm{R} / \mu=c_{P}-c_{V}=\frac{\gamma-1}{\gamma} c_{P}$, we find

$$
\begin{equation*}
\frac{T_{2}}{T_{1}}=\frac{1+\frac{1}{2}(\gamma-1) M_{1}^{2}}{1+\frac{1}{2}(\gamma-1) M_{2}^{2}} \tag{214}
\end{equation*}
$$

¿From the ideal gas law we also have

$$
\frac{T_{2}}{T_{1}}=\frac{P_{2} \rho_{1}}{P_{1} \rho_{2}}
$$

¿From (208) and (212) we find

$$
\frac{T_{2}}{T_{1}}=\frac{P_{2} u_{2}}{P_{1} u_{1}}=\frac{P_{2} M_{2}}{P_{1} M_{1}} \sqrt{\frac{\gamma R T_{2}}{\gamma R T_{1}}}
$$

giving

$$
\begin{equation*}
\frac{T_{2}}{T_{1}}=\left(\frac{P_{2} M_{2}}{P_{1} M_{1}}\right)^{2} \tag{215}
\end{equation*}
$$

¿From (210)

$$
\begin{equation*}
P_{1}-P_{2}=\rho_{2} M_{2}^{2} c_{2}^{2}-\rho_{1} M_{1}^{2} c_{1}^{2} \tag{216}
\end{equation*}
$$

We use the expression (213) for the sound velocity and (216) becomes

$$
\begin{equation*}
\frac{P_{2}}{P_{1}}=\frac{1+\gamma M_{1}{ }^{2}}{1+\gamma M_{2}{ }^{2}} \tag{217}
\end{equation*}
$$

Combining (214), (215) and (217), we finally get

$$
\begin{equation*}
\frac{1+\frac{1}{2}(\gamma-1) M_{1}{ }^{2}}{1+\frac{1}{2}(\gamma-1) M_{2}{ }^{2}}=\frac{\left(1+\gamma M_{1}{ }^{2}\right)^{2}}{\left(1+\gamma M_{2}{ }^{2}\right)^{2}} \frac{M_{2}{ }^{2}}{M_{1}{ }^{2}} \tag{218}
\end{equation*}
$$

Equation (218) relates the Mach numbers on the two sides of the shock. It has the trivial solution $M_{1}=M_{2}$ which corresponds to uniform flow without any shock. Numerical analysis also shows that there are also other solutions. If we hold one of the Mach numbers fixed, say $M_{2}$, there will always be one solution $M_{1}<1$ if $M_{2}>1$ and one with $M_{1}>1$ if $M_{2}<1$. We argued at the beginning of this section that for a front to be self sustained it must move faster than the sound velocity in the medium in front of it, i.e. only the solution $M_{1}>1$ is physically acceptable. This situation is completely analogous to what we had in the case of the hydraulic jump, where we only accepted the solution where the tidal wave moved faster than the wave velocity in the undisturbed channel. For each solution $M_{2}$ and with a knowledge of the temperature $T_{1}$, density $\rho_{1}$ and pressure $P_{1}$ in the undisturbed region in front of the shock we can use (215), (217) and the ideal gas law to calculate $T_{2}, \rho_{2}, P_{2}$. If one of these quantities, e.g. $T_{2}$ or $P_{2}$, is known from the properties of the explosion that caused the shock all the other quantities can then, in principle, be calculated.

The argument that only solutions with $M_{1}>1$ are physically acceptable can be put on a more formal basis by considering the second law of thermodynamics. Since the process is adiabatic the entropy of the system must increase as the shock passes by; we must have for the entropy per unit mass $s_{2}>s_{1}$. It can be shown that this is only possible if $M_{1}>1$. A plot of the numerical properties of the shock parameters is given in figure 35 .

### 13.4 Problems

## Problem 13-1.

(a.) The bulk modulus of a material can be defined as

$$
B=\rho \frac{\partial P}{\partial \rho}
$$

Given that $B=2.1 \times 10^{9} \mathrm{~Pa}, \rho=10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$. Find the sound velocity in water.
(b) Consider a liquid (like beer) in which gas bubbles are uniformly distributed throughout the fluid. Let the (adiabatic) bulk modulus of the pure fluid be $B_{l}$ and the bulk modulus of the gas be $B_{g}$. The density of the pure fluid is $\rho_{l}$ and the density of the gas is $\rho_{g}$. Find the sound velocity of the brew assuming it contains a volume fraction $a$ of gas.
(c) Estimate the sound velocity of a mixture of water and 1 volume \% air.

## 14 Physical similarity and modeling.

We will conclude our discussion of fluid mechanics by discussing dimensional analysis in more detail. The underlying idea is straight forward although the simple-mindedness of the arguments sometimes makes the practitioner somewhat uneasy. Each particular flow pattern arises because of the interplay of a number of forces and pseudo-forces due to inertia, pressure differences, viscosity, external forces such as gravity, surface tension, elasticity and stresses due to thermal expansion and buoyancy forces. Two systems are physically similar if the ratio between the relevant forces is the same in both systems. It is then possible to study a given system by building a scale model of the system in question. One can measure the properties of the model, and if it satisfies conditions of physical similarity one can rescale the measured properties and apply the results to the actual system of interest. Even if one is not actually building a model system, dimensional analysis often represents a convenient first line of attack on a complicated problem.
Let us assume that

$$
\begin{aligned}
& \alpha=\text { thermal expansion coefficient } \\
& g=\text { acceleration of gravity } \\
& \kappa=\text { thermal diffusivity } \\
& L=\text { length parameter } \\
& u=\text { typical velocity } \\
& \eta=\text { viscosity } \\
& \rho=\text { mass density } \\
& \sigma=\text { thermal conductivity }
\end{aligned}
$$

Inertial pseudo-forces can be described as mass times acceleration. Often the acceleration is centripetal in origin and of the order $u^{2} / L$, while the mass is $\rho L^{3}$. The resulting forces then scale as

$$
\begin{equation*}
\text { Inertial force } \sim \rho u^{2} L^{2} \tag{219}
\end{equation*}
$$

Viscous stress (force per area) is given by the viscosity times the velocity gradient, which is typically proportional to $u / L$. The resulting force thus scales as

$$
\begin{equation*}
\text { Viscous force } \sim L u \eta \tag{220}
\end{equation*}
$$

The ratio between the inertial and viscous forces is the

$$
\begin{equation*}
\text { Reynolds number }=\frac{\rho u L}{\eta} \tag{221}
\end{equation*}
$$

### 14.1 Flow past a submerged object:

Let $L$ be a typical dimension of an object moving in a fluid with speed $u$. We assume that the flow around the object can be taken to be incompressible and assume a drag force law

$$
\begin{equation*}
F=\operatorname{Lu\eta f}(R e) \tag{222}
\end{equation*}
$$

where $f$ is a function of the geometry of the object. In the case of a spherical object at low Reynolds number, the drag force is given by Stokes' law

$$
\begin{equation*}
F=3 \pi d u \eta \tag{223}
\end{equation*}
$$

where the diameter plays the role of length parameter. Therefore, for a spherical object we have $f(R e)=3 \pi$, in the limit $R e \rightarrow 0$. Note that we could equally well have used the inertial force expression (219) in constructing the drag law

$$
\begin{equation*}
F=\rho u^{2} L^{2} h(R e) \tag{224}
\end{equation*}
$$

By comparing (222) and (224) we find that

$$
f(R e)=\operatorname{Re} h(R e)
$$

Suppose we wish to estimate the drag force on a given submerged object of arbitrary shape by building a scale model to scale, say, 1:5. For tests with
the model to represent a situation which is physically similar, the Reynolds number must be the same. If the fluid is the same we see from (221) that the flow velocity must be increased by a factor of 5 . The force on the model and the real object would, from (222) then be the same.

In the case of a sphere the drag force is given by Stokes' law for very low Reynolds number $R e \leq 2$. In the range $20<R e<2000$ the drag coefficient $h(R e)$ is approximately proportional to the inverse square root of Re. For Reynolds numbers up to about $10^{5} h(R e)$ stays about constant. If the Reynolds number is increased coefficient there is a sudden drop in the drag coefficient and the behavior then depends not only on the Reynolds number but on the roughness of the sphere's surface [2]. This behavior is analogous to the dependence of the friction factor $f$ on the roughness in the case of flow in pipes. The drag coefficient in the case of flow past a long cylinder behaves in a similar way to the drag coefficient for a sphere [1] [47].

If the velocity of flow approaches that of sound, the flow can no longer be taken to be incompressible, and physical similarity requires that both the Mach and the Reynolds number be the same. If the model is to be tested in the same medium as the object it simulates, the two must be of the same size. However, if the model is tested in a pressurized wind tunnel one can use a smaller model with the density of air $\times$ size constant. The sound velocity will be approximately independent of the density.

### 14.2 Convection

Buoyancy forces are important in convection problems. If a fluid is heated from below, thermal expansion may cause the density of the fluid to increase with height. The fluid is then unstable and convection will set in.

### 14.3 Convection in the atmosphere

Under most conditions the temperature of the atmosphere decreases with altitude. If this leads to a situation in which denser air lies atop of less dense air the atmosphere will become unstable against convection. The density is determined by the local temperature and pressure while the pressure will decrease with increasing altitude.

Let us begin by making a crude estimate of the critical temperature lapse rate

$$
\begin{equation*}
l=-\frac{\partial T}{\partial z} \tag{225}
\end{equation*}
$$

at the on set of convection. Here $z$ is the altitude. The mass density $\rho=$ $M N / V$, where M is the mass of a molecule, given by

$$
\begin{equation*}
\rho=\frac{M P}{k_{B} T} \tag{226}
\end{equation*}
$$

We have

$$
\begin{equation*}
\frac{d P}{d z}=-g \rho \tag{227}
\end{equation*}
$$

Differentiating (226) and substituting (227) gives

$$
\frac{d \rho}{d z}=\frac{M}{k_{B} T} \frac{d P}{d z}-\frac{M P}{k_{B} T^{2}} \frac{d T}{d z}=\frac{\rho}{T}\left(l-\frac{M g}{k_{B}}\right)
$$

With $M=29$ a.m.u., 1 a.m.u. $1.66 \times 10^{-27} \mathrm{~kg}, k_{B}=1.381 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}$, we find that the critical lapse rate is $0.034 \mathrm{~K} \mathrm{~m}^{-1}$ or approximately 3 degrees per 100 m . This is an overestimate of the critical lapse rate.

To get a more realistic estimate consider an element of the gas which contains a mass $d M$. At equilibrium this mass will occupy a volume $d V$. Suppose that by a chance fluctuation the element is displaced by a small amount upwards. The element will then expand approximately adiabatically to the equilibrium volume at the pressure of the new location. If our volume element is heavier than the gas that it is replacing it will sink back to where it came from and the atmosphere is stable, if it is lighter it will rise. It follows that we have marginal stability for the adiabatic lapse rate given by (see Chapter 13) $c_{p} T+g z=$ const, where $c_{p}$ is the specific heat at constant pressure per unit mass (approx. $3.5 k_{B} / M$ for air). The critical lapse rate is then 3.5 times smaller than calculated above or about 1 K per 100 m .

It is interesting to speculate why the observed lapse rate in the atmosphere is close to the adiabatic rate. If the atmosphere had been at equilibrium it would be isothermal. It is kept out of equilibrium by the incoming radiation from the sun. If the oceans and the ground are more efficient in absorbing sunlight than the atmosphere, a temperature gradient will be established. In the steady state the magnitude of the temperature gradient is determined by the ability of the atmosphere to transport away the excess heat. Heat conduction (Chapter 7) is an inefficient way of transporting heat and if this were the only mechanism the adiabatic lapse rate would be exceeded. On the other hand convection is a very efficient mechanism and once in place the rapid mixing may reduce the lapse rate below its critical value. This dichotomy gives rise to a self-organized pattern of changing weather
near marginal stability. In order to understand the convection pattern in the atmosphere one must add in the effects of evaporation and condensation of water and also take into account Coriolis forces due to the earth's rotation.

### 14.4 Rayleigh Bénard convection

Consider next a liquid heated from below. Let us assume that the liquid is nearly incompressible, i.e. the only density variations are due to thermal expansion. The buoyancy force will be $\Delta \rho g L^{3}$ where

$$
\Delta \rho=\alpha \rho(\Delta T)
$$

and $\alpha$ is the bulk thermal expansion coefficient. We thus have

$$
\text { Buoyancy forces } \simeq \alpha(\Delta T) g L^{3} \rho
$$

The viscous force is again given by (220), but for this formula to be useful we must estimate the convective velocity. Without convection the time dependence of the temperature will be governed by the diffusion equation

$$
\begin{equation*}
\frac{\partial T}{\partial t}=D \nabla^{2} T \tag{228}
\end{equation*}
$$

where $D$ is the thermal diffusivity.
With convection we must replace the partial derivative with respect to time by the convective derivative

$$
\begin{equation*}
\frac{\partial T}{\partial t}+\mathbf{u} \cdot \nabla T=D \nabla^{2} T \tag{229}
\end{equation*}
$$

For steady state, $\frac{\partial T}{\partial t}=0$ and if we assume that $\nabla T \sim(\Delta T) / L, \nabla^{2} T \sim$ $(\Delta T) / L^{2}$ we find

$$
u \sim \frac{D}{L}
$$

and we find that the viscous force is proportional to $\eta D$. The Rayleigh number is defined as

$$
\begin{equation*}
R a=\frac{F_{\text {buoyant }}}{F_{\text {viscous }}}=\frac{\alpha(\Delta T) \rho g L^{3}}{\eta D} \tag{230}
\end{equation*}
$$

Without convection the heat flow is given by

$$
\begin{equation*}
\dot{Q}=\frac{A D(\Delta T)}{L} \tag{231}
\end{equation*}
$$

The ratio between the actual heat flow and the heat flow without convection is called the Nusselt number, $N u$. In the case of Rayleigh-Bénard convection due to heating from below we have physical similarity if the Rayleigh numbers of two different situations are the same or $N u=N u(R a)$.

Convection is the driving mechanism for the great ocean currents which together with atmospheric convection are crucial in determining the earth's climate. In order to study convection in the oceans one consider density changes due to variations in salinity. Also, because of the great depth of the oceans one needs to worry about the compressibility of water. Finally the fact that the thermal expansion coefficient of water is not monotonic, leading to a maximum of the density of water at $4{ }^{\circ} \mathrm{C}$ must be taken into account. However, modeling of ocean currents represent a very important problem on which a great deal of effort is spent. The main reason for this interest is worry about the effects of global warming.

### 14.5 Wave resistance of a ship

We have seen in chapter 12 that gravity forces are often the dominant ones in the case of water waves. We have typically

$$
\begin{equation*}
\text { Gravitational forces } \sim \rho g L^{3} \tag{232}
\end{equation*}
$$

The square root of the ratio between the inertial and gravitational forces is defined as the Froude number

$$
\begin{equation*}
F r=\sqrt{\frac{u^{2}}{L g}} \tag{233}
\end{equation*}
$$

This definition is compatible with the definition of the Froude number in section 12.2 where we discussed the hydraulic jump. If we let the characteristic length $L$ be the depth of undisturbed channel, the Froude number (233) is just the ratio of the velocity of the bore to the velocity $\sqrt{L g}$ of a gravity wave in the shallow channel, which was how we defined the Froude number in section 12.2.

The wave resistance of a ship is a function of the Froude number, although there are also other factors that contribute to the drag, such as skin resistance and eddy production, which depend on the Reynolds number. If we want to estimate the drag force due to the waves produced in the wake by making a model, we select a speed for the model so that the Froude
numbers are the same and we assume the drag force $F$ due to generation of the waves in the wake

$$
\begin{equation*}
F_{\text {waves }}=\rho u^{2} L^{2} f(F r) \tag{234}
\end{equation*}
$$

where $f$ is some function that depends on the geometry. Suppose we build a scale model $1 / 25$. For the Froude numbers to be the same the model should then be tested at $1 / 5$ of the operating speed of the ship. From (234) the drag on the model due to waves is then $1 /(25)^{3}$ that of the actual ship.

These examples should give some of the flavor of arguments based on physical similarity. The arguments are not very rigorous and it is obvious they have to be used with some caution. However, when combined with experience these type of arguments are very useful, particularly when the physical situation is to complicated to yield to direct calculations.

### 14.6 Problems

## Problem 14-1.

A 1 mm deep layer of water has temperature $\approx 20 \mathrm{C}$. The thermal expansion coefficient of water is $2 \times 10^{-4} \mathrm{~K}^{-1}$ at this temperature, the thermal conductivity is $\kappa=0.0014 \mathrm{cal} \mathrm{cm}^{-1}$ $\mathrm{s} \mathrm{K}{ }^{-1}$, while the viscosity is $10^{-3} \mathrm{~Pa}$ s. The critical Rayleigh number for convection with a free surface is approximately 1100. One calorie is 4.1840 J .
(a) What is the critical temperature difference between top and bottom for onset of convection?
(b) The compressibility $\frac{-1}{V} \frac{\partial V}{\partial P}$ of water is about $4.5 \times 10^{10}$ $\mathrm{Pa}^{-1}$. Is the approximation of neglecting the density change due to changes in pressure justified?
(c) Redo the calculations under (a) and (b) for the case of a 2 m deep swimming pool.

## Part IV

## Review

## 15 Old exams and other problems.

1. 

(a). For the dipolar gas in subsection 4.1.2 show that the average dipole moment parallel to the field is

$$
\begin{equation*}
\langle\mu \cos \theta\rangle=\mu\left(\operatorname{coth}(\beta \mu E)-\frac{1}{\beta \mu E}\right) \tag{235}
\end{equation*}
$$

(b). Show that the average Polarization (dipole moment per unit volume) for small fields is

$$
\begin{equation*}
P=\frac{N \beta \mu^{2} E}{3 V} \tag{236}
\end{equation*}
$$

(Hint: for small arguments the hyperbolic cotangent has the Taylor expansion

$$
\left.\operatorname{coth}(x)=\frac{1}{x}+\frac{x}{3}+\ldots\right)
$$

2. A fluid of viscosity $\eta$ exhibits laminar flow in a cylindrical pipe of radius $r$ and length $L$. For a pressure drop $\Delta P$, it may be shown that the flow velocity $u(x)$ at a distance $x$ from the cylinder axis is given by

$$
u(x)=\frac{\Delta P}{4 \eta L}\left(r^{2}-x^{2}\right)
$$

(a). State the assumptions which yields this equation.
(b). The viscosity of $\mathrm{H}_{2} \mathrm{O}$ at $20^{\circ} \mathrm{C}$ in c.g.s. units is approximately 0.01 poise. What does this correspond to in SI units?
(c). Calculate the total flow rate (volume per second) if the flowing fluid is water at $20^{\circ} \mathrm{C}, \Delta P=0.1 \mathrm{~atm} \simeq 10^{4} \mathrm{~N} \mathrm{~m}^{-2}, L=1 \mathrm{~m}, r=0.1 \mathrm{~cm}$.
(d). Calculate the shear stress exerted on the pipe wall and the total force exerted by the flowing water on the pipe under the conditions in (c).
3. Consider a dispersion of spherical particles of radius $r$, density $\rho$, in a fluid with viscosity $\eta$ and density $\rho_{1}<\rho$. The particles feel a gravitational force $m g$ (in the $z$-direction) and a buoyancy force. If $r$ is sufficiently small
the drag on particles moving with velocity v will be given by Stokes' law $F=6 \pi r v \eta$.
(a). If the number $n$ of particles per unit volume is $n_{o}$, at height $z=0$, what is the equilibrium value of $n$ at height $h$ ?
(b). Derive a formula for the particle diffusion constant by balancing the drift due to gravity and diffusion.
(c). If $\rho=2 \times 10^{3} \mathrm{~kg} \mathrm{~m}^{-3}, \rho_{1}=10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$ and the viscosity of the fluid is $10^{-3} \mathrm{~kg} \mathrm{~m}^{-1} s^{-1}$, and $\mathrm{T}=300 \mathrm{~K}$, for what particle radius will the number of particles per unit volume at $h=1 \mathrm{~m}$ be $\frac{1}{2} n_{0}$ ? What is the diffusion constant under these conditions?
4. A sphere of radius $r$ is moving inside a fluid with velocity $v$. The drag force is measured to be $F$. What must the velocity of a sphere of radius $2 r$ be for the motion to be dynamically similar? What is then the drag force?
5. It was shown in class that the frequency $\omega$ and wave vector $k$ are related by

$$
\omega^{2}=\left(g k+\frac{\sigma k^{3}}{\rho}\right) \tanh (k h)
$$

Assume that the surface tension of water is $\sigma=0.073 \mathrm{~N} \mathrm{~m}^{-1}$.
(a). Derive a formula for the behavior of the phase and group velocity as a function of the wavelength in the general case.
(b). Find the wavelength for which the phase velocity has a minimum in the case of deep water waves. Give a numerical answer.
(c). Show that for long wavelengths the effect of surface tension is negligible. Explain qualitatively why the amplitude of an ocean swell increases as it approaches a sloping beach.
(d). What is the group velocity of water waves of wavelength 100 m (i) in the middle of the ocean? (ii) in a channel of depth 5 m ?
6. Two thin soap bubbles of radii $R_{1}=9.9 \mathrm{~cm}, R_{2}=10.1 \mathrm{~cm}$ are connected by a tube of length $l=10 \mathrm{~cm}$ and diameter 1 mm as shown in Figure 36 Assume that the surface tension of the film increases the pressure inside the bubble by an amount $\frac{4 \sigma}{R}$ where $\sigma=0.075 \mathrm{~N} \mathrm{~m}^{-1}$. The outside pressure is 1 atm and the temperature is 300 K , the viscosity of air is $2 \times 10^{-5}$ $\mathrm{kg} \mathrm{m}^{-1} \mathrm{~s}^{-1}$.
(a). What is the pressure difference between the two bubbles? Which bubble has the highest pressure?


Figure 36: Soap bubbles of Problem 5.
(b). Assuming Poiseuille flow what is the initial flow rate in $\mathrm{gm} \mathrm{s}^{-1}$ or $\mathrm{kg} \mathrm{s}^{-1}$ ?
(c). Is the assumption of Poiseuille flow reasonable (assume the critical Reynolds number based on the diameter to be 2000)?
(d). What happens to the bubbles eventually (assume they don't evaporate!)?

## Midterm examination October 1992

Closed book exam

1. Consider a suspension of spherical particles of radius $r$, density $\rho$, in a fluid with density $\rho_{1}<\rho$. The particles experience a gravitational force and a buoyancy force. The temperature is $T$.
(a). If the number of particles per unit volume is $n=n_{o}$ at height $z=0$, what is the equilibrium value of $n$ at height $z=h$.
(b). If $\rho=2 \times 10^{3} \mathrm{~kg} \mathrm{~m}^{-3}, \rho_{1}=10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$ and $\mathrm{T}=300 \mathrm{~K}$ for what particle radius will the number of particles per unit volume at $h=1 \mathrm{~m}$ be $\frac{1}{2} n_{o}$.
2. For an ideal gas the mean free path is

$$
l=\frac{1}{\sqrt{2} n \sigma}
$$

the diffusion constant is

$$
D=\frac{\langle v\rangle l}{3}
$$

the viscosity is

$$
\eta=\frac{m\langle v\rangle}{3 \sqrt{2} \sigma}
$$

and the mean speed is

$$
\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}}
$$

(a). If a gas is kept at constant temperature and the pressure is doubled, by what factors will the (i) mean free path (ii) viscosity (iii) diffusion constant change?
(b). Estimate scattering cross section $\sigma$ and the diameter of the molecules of a gas with mass density $1.29 \mathrm{~kg} \mathrm{~m}^{-3}$ at $10^{5} \mathrm{~Pa}$ and 300 K , if the viscosity is $2 \times 10^{-5} \mathrm{Pas}$.

The Boltzmann constant is $1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}$.

## Midterm Examination October 1993

## Closed book exam. Allowed aid: Calculator

1:
(a)

A gas is made up of $N_{1}$ molecules of molecules of mass $m_{1}$ and $N_{2}$ molecules of mass $m_{2}$. A few of these molecules are adsorbed at the surface of the gas container with probability $p_{1}$ and $p_{2}$ respectively each time they hit the wall. Once the molecules are adsorbed the probability that they escape is negligible.

What will the ratio of concentrations of the two adsorbed species be on the surface?
(b)

Molecules are injected into a medium from a point source and diffuse away. The steady state concentration will be proportional to $1 / r$ where $r$ is the distance to the source. The diffusion constant is $D$ and the rate at which molecules are injected is $\dot{N}$. Find the steady state concentration $n(r)$ of the injected molecules.

2:
A gas of mass density $1.29 \mathrm{~kg} \mathrm{~m}^{-3}$ at $1 . \times 10^{5} \mathrm{~Pa}$ pressure and 300 K has viscosity $2 \times 10^{-5} \mathrm{~Pa}$ s. Find the diffusion constant.

Some formulas:

$$
j_{N}=\frac{n\langle v\rangle}{4} ; \quad k_{B}=1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}
$$

For an ideal gas the mean free path and the diffusion constant are

$$
l=\frac{1}{\sqrt{2} n \sigma} ; \quad D=\frac{\langle v\rangle l}{3}
$$

and the viscosity and mean speed are

$$
\eta=\frac{m\langle v\rangle}{3 \sqrt{2} \sigma} ; \quad\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}}
$$

## Midterm Examination October 1994

Allowed aid: One double sided "cheat sheet", Calculator Some formulas and useful formulas are listed on page 2 of the exam

1:
Liquid ether at $T=300 \mathrm{~K}$ has an equilibrium vapor pressure of $5.33 \times 10^{4}$ Pa , the mass density of the liquid is $\rho=700 \mathrm{~kg} \mathrm{~m}^{-3}$. The molecular weight of ether is 74. At equilibrium an equal number of molecules hit the liquid from above as leave the surface by evaporation.
(a)

How many ether molecules leave the surface by evaporation per s and $\mathrm{m}^{2}$ of free surface?
(b) Assume that near the ether surface the ether vapor is in equilibrium with the liquid but that ether molecules 1 cm above the surface are blown away so that the ether concentration is effectively zero. Assume also that ether molecules diffuse from the liquid surface to a height 1 cm above. What is the steady state concentration of ether molecules 0.5 cm above the liquid surface?
(c) What is the net diffusion current (molecules per second and square meter) 0.5 cm above the liquid surface. The diffusion constant of ether in air is approximately $2 \times 10^{-5} \mathrm{~m}^{2} \mathrm{~s}^{-1}$

2: A sealed container contains an ideal gas at $\mathrm{T}=300 \mathrm{~K}$ and $10^{5} \mathrm{~Pa}$ pressure. The gas is made up of molecules of diameter $3 \AA$ and molecular weight 30 .
(a) Calculate the diffusion constant $D$ and viscosity $\eta$ of the gas.
(b) What happens to the diffusion constant and the viscosity if the temperature is lowered by $10 \%$, but the volume of the container stays constant?
(c) At time $t=0$ a small amount of different molecules of the same size and mass as the original molecules of the gas are released from a point source inside the container and start diffusing. At what time after the release will the diffusion current 1 mm away from the source reach a maximum?

Some values of constants
1 a.m.u. $=1.67 \times 10^{-27} \mathrm{~kg} k_{B}=1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}$
Some formulas:

$$
\begin{gathered}
l=\frac{1}{n \sigma \sqrt{2}} ; \quad D=\frac{\langle v\rangle l}{3} \\
\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}} ; \quad P V=N k_{B} T \\
\eta=\frac{n m\langle v\rangle l}{3} \\
n(\mathbf{r}, t)=\frac{N}{(4 \pi D t)^{3 / 2}} \exp \left[-\frac{r^{2}}{4 D t}\right] \\
j_{N}=\frac{n\langle v\rangle}{4} ; \quad j_{N}=-D \nabla n
\end{gathered}
$$

## Midterm Examination October 1995

Allowed aids: One double sided "cheat sheet", Calculator Some constants and useful formulas are listed at the end of the exam 1:
(a)

A sphere of radius $r$, temperature $T$ is embedded in a gas with molecules of mass $m$ and specific heat per molecule $5 / 2 k_{B}$. The pressure is $P$ and the temperature of the gas is $T_{g}$. Assume that gas molecules hit the sphere with mean kinetic energy $3 k_{B} T_{g}$ and then leave with mean energy $3 k_{B} T$. Estimate the rate at which heat is transferred between the gas and the sphere assuming that collisions between gas molecules and the sphere is the only
mechanism of heat exchange.
(b)

Assuming that initially (time $t=0$ ) the temperature difference is $\Delta T$. Derive an expression for the temperature of the sphere at later times $t$. Assume that the heat capacity of the sphere is $C$ (units $[\mathrm{J} / \mathrm{K}]$ ).
(c)

How long will it take for the temperature difference to halve, assuming the temperature of the gas is kept constant? Also, assume that $\mathrm{r}=1 \mathrm{~cm}, C=15$ $\mathrm{J} \mathrm{K}^{-1}, P=10^{5} \mathrm{~Pa}, T_{g}=300 \mathrm{~K}, T=275 \mathrm{~K}, m=29$ a.m.u.

2: A container contains an ideal gas at $T=300 \mathrm{~K}$, pressure $10^{5} \mathrm{~Pa}$. The gas is made up of molecules of diameter $3 \AA$ and molecular weight 32 a.m.u.
(a)

Calculate the diffusion constant $D$ of the gas.
(b)

Estimate how long it will take for a typical molecule to diffuse 1 m .
(c)

What is the mean distance that a molecule has diffused in 1 s ?
Some values of constants
1 a.m.u. $=1.67 \times 10^{-27} \mathrm{~kg}$
$k_{B}=1.38 \times 10^{-23} \mathrm{~J} / \mathrm{K}$.
Some formulas:

$$
\begin{gathered}
l=\frac{1}{n \sigma \sqrt{2}} ; \quad D=\frac{\langle v\rangle l}{3} \\
\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}} ; \quad P V=N k_{B} T \\
\eta=\frac{n m\langle v\rangle l}{3} \\
n(\mathbf{r}, t)=\frac{N}{(4 \pi D t)^{3 / 2}} \exp \left[-\frac{r^{2}}{4 D t}\right] \\
j_{N}=\frac{n\langle v\rangle}{4} ; \quad j_{N}=-D \nabla n \\
\int_{0}^{\infty} r^{3} e^{-a r^{2}} d r=\frac{1}{2 a^{2}}
\end{gathered}
$$

## Christmas examination December 1991

Allowed aids: "cheat sheet", calculator. Please note that some formulas and numerical values of some physical constants can be found at the end of the examination. Time 2 hours. ANSWER ALL 4 QUESTIONS.

1. A wave with wavelength 100 m is traveling on the surface of a deep ocean
(a). What is the phase velocity?
(b). What is the group velocity?
(c). Estimate the phase and group velocity of a wave with the same wavelength in a $1 m$ deep channel.
2. Fine dust is dispersed uniformly in a room with a 3 m high ceiling. It takes about a minute for the dust to settle. Assume that the particles are made of an inert material of density $\rho=2 \times 10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$ and that the particles are spherical.
(a). Estimate the radius of the particles.
(b). Check that in equilibrium almost all dust will settle.
3. A cylindrical drum of length $\mathrm{L}=10 \mathrm{~cm}$, radius $\mathrm{r}=10 \mathrm{~cm}$ can rotate inside a fixed concentric casing with clearance $c=1 \mathrm{~mm}$. The space between the drum and the casing is filled with water. Assume laminar flow.
(a) What is the power required to rotate the cylinder with angular velocity $\omega=1 \mathrm{rad} \mathrm{s}^{-1}$ ? Neglect frictional forces at the ends.
(b) What is the force of friction on the cylinder walls if the drum is pulled with velocity $v=10 \mathrm{~cm} \mathrm{~s}^{-1}$ parallel to the cylindrical axis which is assumed to be horizontal.
(c) The force pulling the cylinder is inactivated. If the mass of the cylinder is 0.1 kg how long will it take for the drum under the conditions in question (b) above to slow down to $1 \mathrm{~cm} \mathrm{~s}^{-1}$.
4. A valuable irregularly shaped object (VISO) of mass 100 kg is to be dropped from high altitude. You are required to estimate the velocity with which it is falling just before it hits the ground. The object is dense enough that the buoyancy of the air can be neglected.
(a). You have at your disposal a $1: 10$ scale model and a wind tunnel where you can measure the force on a suspended object and adjust and
measure the wind speed. What should the force on the model be to achieve physical similarity?
(b). What is then the ratio between the terminal speed of the VISO and the air speed in the wind tunnel.
(c). The wind tunnel has broken down, but you are still required to estimate the terminal speed of the VISO. You have a friend with a water tank. You are able to pull the model with adjustable speed and to measure the drag force in the tank correcting for buoyancy. What force would you use and what is the ratio between the flow speed and the VISO terminal velocity in air?

## SOME FORMULAS

$$
\begin{gathered}
F=u \eta r 6 \pi \\
R e=\frac{u \rho d}{\eta} \\
\omega=\sqrt{\left(g k+\frac{\sigma k^{3}}{\rho}\right) \tanh (k h)}
\end{gathered}
$$

$g=9.8 \mathrm{~m} \mathrm{~s}^{-2}, \eta=2.0 \times 10^{-5} \mathrm{~Pa} \mathrm{~s}$ for air and $\eta=10^{-3} \mathrm{Pas}$ for water, $k_{B}=1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}, \rho=10^{3} \mathrm{~kg} \mathrm{~m}^{-3}$ for water, $\rho=1.29 \mathrm{~kg} \mathrm{~m}^{-3}$ for air.

## Solution to 1991 Christmas Exam

Problem 1:
For deep water waves $\tanh (k h) \simeq 1$ and

$$
\omega \simeq \sqrt{g k+\frac{\sigma k^{3}}{\rho}}
$$

If the wavelength is as large as 100 m , the second term inside the square root can be neglected and

$$
\omega \simeq \sqrt{g k}
$$

(a) The phase velocity is

$$
u_{\phi}=\frac{\omega}{k}=\sqrt{\frac{g}{k}}=\sqrt{\frac{g \lambda}{2 \pi}}=\sqrt{\frac{9.8 \times 100}{2 \pi}}=12.49 \mathrm{~m} \mathrm{~s}^{-1}
$$

(b) The group velocity is

$$
u_{g}=\frac{d \omega}{d k}=\frac{1}{2} \sqrt{\frac{g}{k}}=6.24 \mathrm{~m} \mathrm{~s}^{-1}
$$

(c) For shallow water gravity waves $\tanh (k h) \simeq k h$ and $\omega=k \sqrt{g h}$ we have

$$
u_{\phi}=u_{g}=\sqrt{g h}=\sqrt{1 \times 9.8}=3.13 \mathrm{~ms}^{-1}
$$

Problem 2:
(a) If the Boltzmann factor

$$
\exp \left(-\frac{m g h}{k_{B} T}\right) \ll 1
$$

the steady state will be characterized by balance between the forces of gravity and friction $m g=6 \pi u \eta r$ We have

$$
m=\frac{4 \pi \rho r^{3}}{3}
$$

giving

$$
u=\frac{m g}{6 \pi \eta r}=\frac{2 \rho r^{2} g}{9 \eta}
$$

We estimate the clearing time as

$$
t=\frac{h}{u}=\frac{9 \eta h}{2 \rho r^{2} g}
$$

Solving for the radius we obtain

$$
r=\sqrt{\frac{9 \eta h}{2 \rho t g}}=\sqrt{\frac{9 \times 2.10^{-5} \times 3}{2 \times 2.10^{3} \times 60 \times 9.8}}=1.5 \times 10^{-5} \mathrm{~m}=15 \mu \mathrm{~m}
$$

(b) We need to check the Boltzmann factor. We have assuming $T \simeq$ 300K

$$
\frac{m g h}{k_{B} T}=\frac{4 \pi \rho r^{3} g h}{3 k_{B} T}=\frac{4 \pi \times\left(1.510^{-5}\right)^{3} \times 2.10^{3} \times 9.8}{3 \times 1.38 \times 10^{-23} \times 300} h=0.66 \times 10^{11} \frac{h}{\text { meter }}
$$

For distances of the order of meters the Boltzmann factor will be extremely small. Even for distances as small as a particle radius the Boltzmann factor will be very small.

## Problem 3:

(a) The velocity gradient is $\omega r / c$, the frictional force per unit area is thus $\omega r \eta / c$. The area of the cylinder is $2 \pi r L$, and we get for the frictional torque

$$
\frac{\omega r^{3}}{c} \eta 2 \pi L
$$

The required power is thus

$$
\frac{\omega^{2} r^{3}}{c} \eta 2 \pi L=\frac{1^{2} \times 0.1^{3} \times 10^{-3} \times 2 \pi \times 0.1}{10^{-3}}=6.28 \times 10^{-4} \mathrm{~W}
$$

(b) The frictional force per unit area is now $v \eta / c$ giving

$$
F=\frac{v}{c} 2 \pi \eta r L=\frac{0.1}{0.001} 2 \pi \times 10^{-3} \times 0.1 \times 0.1=6.28 \times 10^{-3} \mathrm{~N}
$$

(c) From Newton's second law we have for deceleration

$$
m \dot{v}=-\frac{v}{c} 2 \pi \eta r L
$$

with solution

$$
v=v_{o} \exp \left\{-\frac{2 \pi \eta r L t}{m c}\right\}
$$

Solving for $t$ we find

$$
t=\frac{m c \ln \frac{v_{o}}{v}}{2 \pi \eta r L}=\frac{0.1 \times 10^{-3} \ln 10}{2 \pi \times 10^{-3} \times 0.1 \times 0.1}=3.7 \mathrm{~s}
$$

Problem 4:
For physical similarity the Reynolds number has to be the same for the VISO and for the model. We let subscript $v$ refer to the VISO and $m$ to the model. Equating the Reynolds numbers gives

$$
\frac{u_{v} \rho_{v} r_{v}}{\eta_{v}}=\frac{u_{m} \rho_{m} r_{m}}{\eta_{m}}
$$

where $r_{v}$ and $r_{m}$ is the linear dimension of the VISO and the model respectively. It doesn't matter how this quantity is measured, as long as it is done the same way for both objects, We write the drag force in the form

$$
F=u \eta r f(R e)
$$

(we could equally well have written $F=u^{2} \rho r^{2} \tilde{f}(R e)$ ). The function $f(R e)$ (or $\tilde{f}(R e)$ ) is unknown, but the value would be the same for all flows with the same Reynolds number. Thus,

$$
\frac{F_{v}}{F_{m}}=\frac{u_{v} \rho_{v} r_{v}}{u_{m} \rho_{m} r_{m}}
$$

(a) If the experiment with the model is made in air $\rho_{v}=\rho_{m}, \eta_{v}=\eta_{m}$, and when the VISO has reached its terminal velocity $F_{v}=m_{v} g=980 \mathrm{~N}$. Since $r_{v}=10 r_{m}$ we find $u_{m}=10 u_{v}$. We have $F_{m}=F_{v}=980 \mathrm{~N}$.
(b) The terminal speed of the VISO is $1 / 10$ of the speed in the wind tunnel when the drag force is $980 N$.
(c) We have $\rho_{v}=1.29 \mathrm{~kg} \mathrm{~m}^{-3}, \rho_{m}=1000 \mathrm{~kg} \mathrm{~m}^{-3}, \eta_{v}=2 . \times 10^{-5} \mathrm{~Pa} \mathrm{~s}$, $\eta_{m}=10^{-3} \mathrm{~Pa}$ s. We have

$$
u_{m} r_{m} \frac{1000}{10^{-3}}=u_{v} r_{v} \frac{1.29}{2 \times 10^{-5}}
$$

giving $u_{m}=0.64 u_{v}$. The ratio of the forces is

$$
\frac{F_{m}}{F_{v}}=\frac{0.64 \times 0.1 \times 10^{-3}}{210^{-5}}=3.2
$$

Thus the speed of the model should be adjusted until the drag force is $3.2 \times 980=3136 \mathrm{~N}$. The terminal velocity of the VISO will be $1 / 0.64=1.56$ times the speed of the model when the drag force on it is 3136 N .

## Final exam December 1992

Allowed aids: 1 double sided "cheat sheet", calculator. Time 2 hours. Answer all 4 questions.

1: It was shown in class that, subject to some simplifying assumptions, deep water waves obey a dispersion relation in which the frequency $\omega$ is related to the wave vector k through

$$
\omega=\sqrt{g k+\frac{\sigma k^{3}}{\rho}}
$$

where g is the acceleration of gravity, $\sigma$ is the surface tension and $\rho$ is the mass density.
(a). Describe in words the approximations made in deriving the formula.
(b). Calculate the wave length for which the phase velocity is a minimum.
(c). What is the value of the group velocity when the phase velocity is a minimum? We have $\rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}$ and $\sigma=0.073 \mathrm{~N} \mathrm{~m}^{-1}$ for water, and $g=9.8 \mathrm{~m}^{2} \mathrm{~s}^{-1}$.

2: Water is flowing steadily in a horizontal cylindrical pipe of length $L$ inner radius $r$. To compensate for viscous and turbulent losses a pressure difference $\delta P=\rho g h_{f}$ is maintained between the ends with $h_{f}$ the "frictional head". Assume that the flow rate is $\dot{V}$ and that $h_{f}$ is given by Darcy's law

$$
h_{f}=f \frac{L u^{2}}{r g}
$$

where the mean speed is $u=\frac{\dot{V}}{\pi r^{2}}$ and $f$ is a dimensionless parameter.
(a). What is the force on the pipe?
(b). Derive a formula for the shear stress exerted on the inner surface of the pipe by the flowing water.
(c). Give a numerical answer for the net force in (a) assuming $\dot{V}=10^{-4}$ $\mathrm{m}^{3} \mathrm{~s}^{-1}, f=10^{-2}, r=10^{-2} \mathrm{~m}, L=1 \mathrm{~m}$.

3: A container of volume $V$ is filled with a gas at temperature $T$, pressure $P$, made up of molecules of mass $m$. According to kinetic theory the number of particles hitting the walls of the container per unit time and surface area is $j_{N}=n\langle v\rangle / 4$ where $n$ is the number of molecules per unit volume and

$$
\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}}
$$

is the mean speed. The mean free path of the molecules is

$$
\langle\lambda\rangle=\frac{1}{n \sigma \sqrt{2}}
$$

where $\sigma$ is the collision cross section. Gas escapes into outside vacuum through a small circular hole. The area of the hole is $A$.
(a). Derive a formula for the rate of change of pressure with respect to time due to the escape of molecules.
(b). For the kinetic theory to be valid the distribution of particle velocities must not be effected by the hole. This requires that the radius of the
hole should not be larger than the mean free path of the molecules. What is the range of pressures, given $r$, for which the formula derived in part (a) is valid?
(c). Give numerical estimate of the maximum area of the hole for the conditions under (b) to be valid assuming $P=100 \mathrm{k} \mathrm{Pa}, \sigma=20 \times 10^{-20}$ $\mathrm{m}^{2}, T=300 \mathrm{~K}, k_{B}=1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}$.

4: It is desired to measure the viscosity $\eta$ of a fluid by dropping a steel ball of radius $r$ in the fluid and observing the terminal velocity $u$ assuming Stokes' law $F=6 \pi \eta r u$. The density of the fluid is $\rho_{f}$ and that of the ball $\rho_{b}$.
(a). Derive a formula for the viscosity as a function of $\rho_{f}, \rho_{b}, u, r$.
(b). For turbulent drag corrections not to be important, the Reynolds number $R e=\rho_{f} u r / \eta$ must not exceed a critical value $R$. What restrictions does this put on the size of the ball?
(c). If the ball is too small the experiment won't work too well. Why not?

## Final Exam, December 1992 Solution

1
a)

The most important assumptions are:
(i). Viscous and turbulent losses are neglected.
(ii). The fluid is incompressible.
(iii). The wave amplitude is small compared with the wavelength.
(iv). The channel is much deeper than the wavelength.
b)

The phase velocity

$$
\begin{equation*}
v_{\phi}=\frac{\omega}{k}=\sqrt{\frac{g}{k}+\frac{\sigma k}{\rho}} \tag{237}
\end{equation*}
$$

has a minimum when

$$
\begin{equation*}
\frac{d v_{\phi}}{d k}=0=\frac{-\frac{g}{k^{2}}+\frac{\sigma}{\rho}}{2 \sqrt{\frac{g}{k}+\frac{\sigma k}{\rho}}} \tag{238}
\end{equation*}
$$

i.e. for

$$
\begin{equation*}
k=k_{0}=\sqrt{\frac{g \rho}{\sigma}} \tag{239}
\end{equation*}
$$

giving for the wavelength $\lambda=2 \pi \sqrt{\frac{\sigma}{g \rho}}$.
Substituting the values for $g, \sigma, \rho$ we find $\lambda=0.017 \mathrm{~m}$.
c)

The group velocity is given by

$$
\begin{equation*}
v_{g}=\frac{d \omega}{d k}=\frac{g+\frac{3 \sigma k^{2}}{\rho}}{2 \sqrt{g k+\frac{\sigma k^{3}}{\rho}}}=\sqrt{2 \sqrt{\frac{g \sigma}{\rho}}} \tag{240}
\end{equation*}
$$

Substituting numerical values for $g, \sigma, \rho$ we find $v_{g}=0.23 \mathrm{~m} \mathrm{~s}^{-1}$ 2
a)

We have for the net force

$$
\begin{equation*}
F=\delta P \pi r^{2}=\rho g \pi r^{2} h_{f}=\rho \pi r f L \frac{u^{2}}{g} \tag{241}
\end{equation*}
$$

Using

$$
\begin{equation*}
u=\frac{\dot{V}}{\pi r^{2}} \tag{242}
\end{equation*}
$$

we find the desired result

$$
\begin{equation*}
F=\frac{\rho f L \dot{V}^{2}}{\pi r^{3}} \tag{243}
\end{equation*}
$$

b)

The shear stress is

$$
\begin{equation*}
\tau=\frac{F}{2 \pi r L}=\frac{\rho f \dot{V}^{2}}{2 \pi^{2} r^{4}} \tag{244}
\end{equation*}
$$

c)

$$
\begin{equation*}
F=\frac{1000 \times 10^{-2} \times 1 \times 10^{-6}}{\pi 10^{-6}}=3.2 \mathrm{~N} \tag{245}
\end{equation*}
$$

3
a)

Starting with

$$
\begin{equation*}
P=\frac{N k_{B} T}{V} \tag{246}
\end{equation*}
$$

we find

$$
\begin{equation*}
\dot{P}=-\frac{k_{B} T N<v>A}{4 V^{2}}=-\frac{P A<v>}{4 V} \tag{247}
\end{equation*}
$$

b)

If

$$
\begin{equation*}
\lambda=\frac{1}{n \sigma \sqrt{2}}=\frac{k_{B} T}{P \sigma \sqrt{2}} \tag{248}
\end{equation*}
$$

we see that for $\lambda>r$ we must have

$$
\begin{equation*}
P<\frac{k_{B} T}{r \sigma \sqrt{2}} \tag{249}
\end{equation*}
$$

c)

We have

$$
\begin{equation*}
\lambda=-\frac{k_{B} T}{P \sigma \sqrt{2}}=1.4 \times 10^{-7} \mathrm{~m} \tag{250}
\end{equation*}
$$

If $r=\lambda$ we have

$$
\begin{equation*}
A=\pi r^{2}=6.710^{-14} \mathrm{~m}^{2} \tag{251}
\end{equation*}
$$

4
a)

The force on the steel ball is

$$
\begin{equation*}
F=\left(\rho_{b}-\rho_{f}\right) g \frac{4 \pi r^{3}}{3}=6 \pi \eta r u \tag{252}
\end{equation*}
$$

giving for the viscosity

$$
\begin{equation*}
\eta=\frac{2 g\left(\rho_{b}-\rho_{f}\right) r^{2}}{9 u} \tag{253}
\end{equation*}
$$

b)

Substitution of

$$
\begin{equation*}
u=\left(\rho_{b}-\rho_{f}\right) \frac{2 r^{2} g}{9 \eta} \tag{254}
\end{equation*}
$$

into

$$
\begin{equation*}
R e_{c r i t}=\frac{\rho_{f} u r}{\eta} \tag{255}
\end{equation*}
$$

gives

$$
\begin{equation*}
r<\left(\operatorname{Re}_{c r i t} \frac{9 \eta^{2}}{2 \rho_{f}\left(\rho_{b}-\rho_{f}\right) g}\right)^{1 / 3} \tag{256}
\end{equation*}
$$

c)

If the ball is too small the velocity will fluctuate due to Brownian motion and it will be hard to measure an average speed.

## Final exam December 1993

Allowed aid: Calculator, 2 page "cheat sheet"
Answer 4 out of the 5 questions. Time $21 / 2$ hours.
1:
A virus of spherical shape and mass $10^{-18} \mathrm{~kg}$, density $\rho=1000 \mathrm{~kg} \mathrm{~m}^{-3}$, is floating in air at the temperature 300 K , pressure 100 k Pa . The mass of an air molecule is $4.84 \times 10^{-26} \mathrm{~kg}$. Boltzmann's constant is $k_{B}=1.3810^{-23} \mathrm{~J}$ $\mathrm{K}^{-1}$.
(a). What is the r.m.s. speed of a virus? Of an air molecule?
(b). What is the mean time between air-molecule virus collisions?
(c). Calculate the r.m.s. momentum of the virus and of an air molecule. Assume that the r.m.s. momentum transfer in each collision is approximately equal to the r.m.s. momentum of an air molecule. How many collisions will typically be needed for a virus to change its direction?
(d). How far will the virus typically travel between each time it changes direction?

## 2:

The cross section of a dam has the shape of an equilateral triangle with side $l$. There is water seepage under the dam producing a lift pressure which is equal to the water pressure at depth $h=l \sqrt{3} / 2$ where $\rho_{w}$ is the density of water. What is the minimum density $\rho_{d}$ of the dam material for it to be stable against toppling about the point $P$ in figure 37 , when the water level reaches the apex of the triangle?

3:
A circular disk is rotating with angular velocity $\omega$. There is a clearance $c$ to a stationary disk below. The radius of the disk is $r$ and the viscosity of the fluid between the disks is $\eta$. Assume laminar flow.
(a). What is the flow velocity at height $y<c$ above the stationary disk when the radial distance from the axis is $x$.
(c). How much heat must be transported away per unit time to prevent the temperature from rising.

4:
The bulk thermal expansion coefficient of water at $20^{\circ} \mathrm{C}$ is $0.2 \times 10^{-3}$ $\mathrm{K}^{-1}$. The compressibility of water at that temperature is $4.5 \times 10^{-10} \mathrm{~Pa}^{-1}$.


Figure 37: Figure to problem 2 on 1993 exam.
(a). What is the critical thermal lapse rate (temperature change per unit height) for which the density of water heated from below will increase with height.
(b.) The thermal conductivity is $K=0.0014 \mathrm{cal} \mathrm{cm}^{-1} \mathrm{~s}^{-1} \mathrm{~K}^{-1}$ and 1 $\mathrm{cal}=4.18 \mathrm{~J}$. The specific heat of water is $1 \mathrm{cal} \mathrm{g}^{-1} \mathrm{~K}^{-1}$ and the density of water approximately $1 \mathrm{~g} \mathrm{~cm}^{-3}$. What is the heat flow (Watt m${ }^{-2}$ ) for the critical lapse rate.
(c.) If the critical lapse rate is exceeded convection may set in. Are there conditions where the lapse rate exceeds the rate calculated under (a) but there is no convection? If convection takes place will the heat flow increase or decrease assuming that the temperature difference is maintained?

## 5:

In a certain fluid the viscous stress is related to the velocity gradient by

$$
\tau=k\left(-\frac{\partial u_{x}}{\partial y}\right)^{n}
$$

(a). Use dimensional analysis to show that the laminar flow rate in a pipe can be written

$$
\dot{V}=c r^{3}\left(\frac{r \Delta P}{L k}\right)^{1 / n}
$$

where $L$ is the length of the pipe $r$ is the radius, $\Delta P$ is the pressure drop and c is a dimensionless constant.
(b). Use the result found under (a) to find a formula for the friction
factor $f$ in Darcy's law

$$
\frac{h_{f}}{L}=f \frac{u^{2}}{r g}
$$

where $h_{f}$ is the frictional head.

## Solution to final Exam December 93

1:
(a)

$$
\frac{1}{2} M V_{r m s}=\frac{3}{2} k_{B} T
$$

For the virus:

$$
V_{r m s}=\sqrt{\frac{3 \times 1.381 \times 10^{-23} \times 300}{10^{-18}}}=0.111 \mathrm{~m} \mathrm{~s}^{-1}
$$

For the air molecules:

$$
v_{r m s}=\sqrt{\frac{3 \times 300 \times 1.381 \times 10^{-23}}{4.84 \times 10^{-26}}}=507 \mathrm{~m} \mathrm{~s}^{-1}
$$

(b)

The radius of the virus can be obtained from

$$
\frac{4 \pi \rho r^{3}}{3}=M
$$

giving

$$
r=\left(\frac{3 \times 10^{-18}}{4 \pi \times 1000}\right)^{1 / 3}=6.2 \times 10^{-8} \mathrm{~m}
$$

The number of air molecules per unit volume is with P the pressure

$$
n=\frac{P}{k_{B} T}
$$

The mean time $\tau$ between collisions can be obtained, either by arguing that

$$
\frac{1}{\tau}=n \pi r^{2}\langle v\rangle
$$

or by saying that

$$
\frac{1}{4} n\langle v\rangle
$$

air molecules hit per unit area of the virus. Since the area is $4 \pi r^{2}$ the two approaches give the same result. If we use $v_{r m s}$ for $\langle v\rangle$ we get

$$
\tau=\frac{k_{B} T}{P \pi r^{2} v_{r m s}}=\frac{300 \times 1.381 \times 10^{-23}}{10^{5} \pi\left(0.62 \times 10^{-8}\right)^{2} \times 507}=6.76 \times 10^{-15} \mathrm{~s}
$$

If instead we use the more accurate formula

$$
\langle v\rangle=\sqrt{\frac{8 k_{B} T}{\pi m}}
$$

we get

$$
\begin{equation*}
\tau=1.08 \times 10^{-14} \mathrm{~s} \tag{257}
\end{equation*}
$$

(c)

The rms momentum of the virus is

$$
P_{r m s}=M V_{r m s}=1.11 \times 10^{-19} \mathrm{~kg} \mathrm{~m} \mathrm{~s}^{-1}
$$

while for the air molecules

$$
p_{r m s}=m v_{r m s}=2.45 \times 10^{-23} \mathrm{kgm} \mathrm{~s}^{-1}
$$

The number of collisions typically needed is

$$
N_{c} \approx \frac{P_{r m s}^{2}}{p_{r m s}^{2}}=2 \times 10^{7}
$$

(d)

$$
l \approx V_{r m s} \tau N_{c}=1.56 \times 10^{-8} \mathrm{~m}
$$

If we use the more accurate value 257 we obtain

$$
l=2.5 \times^{-8} \mathrm{~m}
$$

2:
The height of the apex is

$$
h=\frac{\sqrt{3} l}{2}
$$



The cross-sectional area of the dam is

$$
\frac{h l}{2}=\frac{\sqrt{3} l^{2}}{4}
$$

We need to calculate the torque about the point $P$ (see figure). If counterclockwise torques are considered to be positive we have for the torque per unit width due to the weight of the dam

$$
T_{d}=\frac{\sqrt{3} l^{3} \rho_{d}}{8}
$$

The pressure of the seeping water is

$$
P=\frac{\sqrt{3} \lg \rho_{w}}{2}
$$

The upwards force per unit width is thus

$$
F_{w}=\frac{\sqrt{3} l^{2} g \rho_{w}}{2}
$$

The opposing torque due to seeping water is

$$
T_{w}=-\frac{\sqrt{3} l^{3} \rho_{w}}{4}
$$

The pressure on the side of the dam acts perpendicular to the surface. The torque due to the pressure on the side of the dam is

$$
T_{s}=\rho_{w} g \int_{-l / 2}^{l / 2} d x x\left(x+\frac{l}{2}\right) \frac{\sqrt{3}}{2}=\rho_{w} g \frac{\sqrt{3} l^{3}}{24}
$$

The condition $T_{d}+T_{w}+T_{s}>0$ gives

$$
\frac{2 \rho_{w}-\rho_{d}}{\rho_{w}}<\frac{1}{3}
$$

If $\rho_{d}>\frac{5}{3} \rho_{w}$, the dam will be stable against toppling. We may also wish to check if the downwards forces are enough to overcome the upwards lift due to the seeping water. The downwards force due to the pressure on the side of the dam is (the downwards component of the pressure is $\frac{1}{2}$ the pressure)

$$
F_{s}=\frac{\rho_{w} g}{2} \int_{0}^{l} d x \frac{x \sqrt{3}}{2}=\frac{\sqrt{3} l^{2} g \rho_{w}}{8}
$$

Giving for the net upwards force

$$
F_{w}-F_{s}-F_{d}=\frac{\sqrt{3} l^{2} g \rho_{w}}{2}-\frac{\sqrt{3} l^{2} g \rho_{d}}{4}-\frac{\sqrt{3} l^{2} g \rho_{w}}{8}
$$

or $\rho_{d}>\frac{3}{2} \rho_{w}$. If this condition is not met the seeping water may lift the dam allowing the pressure along the side to move it sideways. However, $\rho_{d}>\frac{5}{3} \rho_{w}$ is the more severe restriction.

3:
(a).

The flow velocity at height $y<c$ above the stationary disk at radial distance $x$ is

$$
\frac{\omega y x}{c}
$$

(b).

The shear stress at radial distance x is

$$
\tau=\frac{\eta \omega x}{c}
$$

The dissipated power is

$$
\int_{0}^{r} d x \omega x \frac{\eta \omega x}{c} 2 \pi x=\frac{\pi \omega^{2} r^{4} \eta}{2 c}
$$

4:
(a).

The rate of change of the density with height $h$ is given by

$$
\frac{d \rho(P, T)}{d h}=\frac{\partial \rho}{\partial P} \frac{\partial P}{\partial h}+\frac{\partial \rho}{\partial T} \frac{\partial T}{\partial h}
$$

The compressibility is given by

$$
\kappa=\frac{1}{\rho} \frac{\partial \rho}{\partial P}
$$

The thermal expansion coefficient is

$$
\alpha=\frac{1}{V} \frac{\partial V}{\partial T}=-\frac{1}{\rho} \frac{\partial \rho}{\partial T}
$$

and we have

$$
\frac{\partial P}{\partial h}=-\rho g
$$

and

$$
\frac{\partial T}{\partial h}=-l
$$

where $l$ is the thermal lapse rate. Collecting terms gives

$$
\frac{d \rho}{d h}=\rho(-\kappa \rho g+\alpha l)
$$

The critical lapse rate is thus

$$
l=\frac{\kappa \rho g}{\alpha}=\frac{4.5 \times 10^{-10} \times 1000 \times 9.8}{0.2 \times 10-3}=0.022 \mathrm{~K} \mathrm{~m}^{-1}
$$

(b).

The thermal conductivity in SI units is $K=0.5852 \mathrm{~J} \mathrm{~m}^{-1} \mathrm{~s}^{-1} \mathrm{~K}^{-1}$, the heat flow is thus $K l=0.013 \mathrm{~W} \mathrm{~m}^{-2}$.
(c).

It is possible to have a fluid which is stable against convection even though the density increases with height, if the layer of fluid is thin enough. The condition for convection for an incompressible fluid is that the Rayleigh number must exceed a critical value.

If convection sets in, while the temperature difference is maintained, the heat flow will increase.

5 :
(a.)

We assume that

$$
\dot{V}=c r^{\alpha}\left(\frac{\Delta P}{L}\right)^{\beta} k^{\gamma}
$$

¿From

$$
k=\tau\left(-\frac{\partial u_{x}}{\partial L}\right)^{-n}
$$

we find

$$
\begin{gathered}
{[k]=[\tau]\left[s^{-1}\right]^{-n}=[P a][s]} \\
{[r]=[m]} \\
\frac{\Delta P}{L}=[P a][m]^{-1} \\
{[\dot{V}]=[m]^{3}[s]^{-1}}
\end{gathered}
$$

and
Pressure $\rightarrow 0=\beta+\delta$
Seconds $\rightarrow-1=n \delta$
Meters $\rightarrow 3=\alpha-\beta$
The solution is $\delta=-1 / n, \beta=1 / n, \alpha=3+1 / n$.

$$
\begin{equation*}
\dot{V}=c r^{3}\left(\frac{r \Delta P}{L k}\right)^{1 / n} \tag{258}
\end{equation*}
$$

(b).

We have

$$
\rho g h_{f} g=\Delta P ; \quad \dot{V}=\pi r^{2} u
$$

Substitution into (258) gives

$$
\frac{h_{f}}{L}=\frac{u^{2}}{r g}\left(\frac{\pi}{c}\right)^{n} \frac{k}{\rho u^{2-n} r^{n}}
$$

or

$$
f=\left(\frac{\pi}{c}\right)^{n} \frac{k}{\rho u^{2-n} r^{n}}
$$

## Final exam. December 1994

Allowed aid: Calculator, 2 page "cheat sheet"
Answer 4 out of the 5 questions. Time $21 / 2$ hours.
If you answer more than 4 questions you will only be given credit for the 4 best answers.

1:

A vial of height 3 cm is filled with an aqueous suspension. The particles in the suspension are spherical with radius $100 \AA$, and have a density of
$2000 \mathrm{~kg} \mathrm{~m}^{-3}$ (twice that of water). The sample temperature is $\mathrm{T}=300 \mathrm{~K}$. The viscosity of water is $\eta=10^{-3} \mathrm{~Pa} \mathrm{~s}$. The mobility of a spherical object in laminar flow is given by Stokes' law

$$
\mu=\frac{\text { velocity }}{\text { force }}=\frac{1}{6 \pi \eta r}
$$

(a) Verify quantitatively that the equilibrium distribution of particles is nearly uniform over the height of the vial. Boltzmann's constant is $k_{B}=$ $1.3810^{-23} \mathrm{~J} \mathrm{~K}^{-1}$.
(b) The vial is placed in a centrifuge and subject to an acceleration equal to $10^{4}$ times that of gravity, acting along the height of the vial (assume that the acceleration is nearly uniform over the height of the vial). Once the new equilibrium is reached, roughly how many times more concentrated are the particles near the outermost end of the vial (where the particles tend to concentrate) end of the vial than they were before centrifuging?
(c) Roughly how long must the sample be spun in the centrifuge before equilibrium will be reached?

## 2:

A cylinder of length $L$ and radius $R$ can rotate inside a fixed concentric casing with a clearance, $c$, which is much less than $R$. The space between the drum and the casing is filled with a lubricant with viscosity $\eta$. Assuming laminar flow
(a) What is the power required to rotate the cylinder with angular velocity $\omega$ ? Neglect end effects.
(b) Calculate the power loss due to viscous forces at the ends assuming that the clearance at top and bottom is $c$ and that the flat surfaces are parallel.

## 3:

Water is flowing steadily in a horizontal cylindrical pipe of length $L$ inner diameter $d$. To compensate for viscous and turbulent losses a pressure difference $\delta P=\rho g h_{f}$ is maintained between the ends with $h_{f}$ the "frictional head". Assume that the flow rate is $\dot{V}$ and that $h_{f}$ is given by Darcy's law

$$
h_{f}=f \frac{L u^{2}}{2 d g}
$$

where the mean speed is $u=\frac{4 V}{\pi d^{2}}$ and $f$ is a dimensionless parameter.
(a) What is the force exercised by the water on the pipe?
(b) Derive a formula for the shear stress exerted on the inner surface of the pipe by the flowing water.
(c) Give a numerical answer for the net force in (a) assuming $\dot{V}=10^{-4}$ $\mathrm{m}^{3} \mathrm{~s}^{-}, f=10^{-2}, d=10^{-2} \mathrm{~m}, L=1 \mathrm{~m}$.

4:
A hot air balloon contains a volume $V=4 \pi R^{3} / 3$ of air at temperature $T_{h}>T_{a}$, where $T_{a}$ is the ambient temperature. The pressure inside is approximately the same as the pressure $P$ outside. The payload of the balloon (exclusive of the hot air) is $M$. The average mass of an air molecule is $m$.
a Find a formula for the minimum value of $V=V_{\text {min }}$ for the balloon to rise assuming the other constants are given.
b The drag force on a rising balloon is $F=\rho u^{2} R^{2} h$ where $h$ is a constant, $\rho$ is the density of the outside air and $u$ is the speed with which the balloon rises. Find a formula for the steady state speed with which the balloon will rise if $V>V_{m i n}$.

## 5:

Imagine that a hole has been drilled to the center of a planet of radius $R$. If the acceleration of gravity at the surface of the planet is $g$ it will, at distance $r<R$ from the center, be $\frac{g r}{R}$. The pressure at the surface is $P$, and the atmosphere on the planet consists of molecules of mass $m$.
a Find the pressure $P(r)$, for $r<R$, assuming an isothermal ideal gas at temperature $T$.
b Give a numerical answer for the pressure at the center of the planet $(r=0)$ assuming $P=10^{5} \mathrm{~Pa}, m=5 \times 10^{-26} \mathrm{~kg}, g=10 \mathrm{~m} \mathrm{~s}^{-2}, k_{B}=$ $1.38 \times 10^{-23} \mathrm{~J} \mathrm{~K}^{-1}, R=6 \times 10^{6} \mathrm{~m}, T=300 \mathrm{~K}$.

## Final exam. December 1995

Allowed aids: One double sided "cheat sheet", Calculator.
Some constants and useful formulas are listed at the end of the exam.
Time 2 Hours. Answer 4 out of the 5 questions. If you answer all 5 questions you will be given credit for the 4 best answers.

1:
The density of the fluid in a body of water depends on the pressure $P$ and the temperature $T$. Assume that with increasing depth $z$ we have $T=T_{0}+l z$, where $l$ is the lapse rate, and that the compressibility $K=\frac{\partial \rho}{\rho \partial P}$, and thermal expansion coefficient $\alpha=-\frac{\partial \rho}{\rho \partial T}$ are constant, as is the acceleration of gravity $g$.
(a)

Under what conditions will the density $\rho$ decrease with depth (leading to overturning of the water)?
(b)

Estimate numerically the critical lapse rate beyond which the density will decrease with depth.
2 :
It is observed that raindrops with diameter approximately $10^{-3} \mathrm{~m}$ fall towards the ground with a terminal speed proportional to the diameter of the drop. The proportionality constant is found to be $2 \times 10^{3} \mathrm{~s}^{-1}$ if the diameter is given in meters.
(a)

What is the drag force on a raindrop of diameter 2 mm traveling at its terminal speed? What is the terminal speed?
(b)

Find the form of the drag coefficient $h(R e)$ if the drag force is of the form

$$
F=\rho_{a i r} u^{2} d^{2} h(R e)
$$

$u$ is the velocity of the drop, $d$ the diameter of the drop and the viscosity of air is $\eta=2 \times 10^{-5} \mathrm{~Pa}$ s and the Reynolds number is defined as $R e=\rho_{a i r} d u / \eta$. 3:
(a)

A shallow lake is 100 km long and 50 m deep. Sometimes one can observe an end-to-end oscillation of the surface called a seiche. What is the period of this oscillation?
(b)

Describe the physical significance of the Reynolds number, the Froude number and the Mach number. When are these numbers important in problems where one wishes to establish physical similarity between a system and a scale model of the system.
4:
It is desired to pump a fluid with density $\rho$ through a horizontal pipe with
a volume flow rate $\dot{V}$. The pipe has a circular cross section with diameter $d$ . Assuming that the frictional head is given by Darcy's law

$$
\frac{h_{f}}{l}=\frac{f u^{2}}{2 d g}
$$

where $u$ is the average flow velocity and $f$ can be taken to be constant.
(a)

What is the shear stress between the fluid and the walls of the pipe.
(b)

What is the power required to pump the fluid.
5:
(a)

A dam has a rectangular cross section with height $h$ and width $w$. The density of the material in the dam is $\rho_{D}$, while the density of the water is $\rho_{W}$. The water behind the dam is filled to level $x$. There is water seepage under the dam producing a lift pressure equal to the water pressure at depth $h$. What is the minimum height to width ratio $w / h$ to prevent the dam from toppling?
(b)

Assuming $x=1.0 h, w / h=1$, what is the minimum density ratio $\rho_{D} / \rho_{W}$ to prevent the dam from toppling?

Some formulas and values of constants
$\rho_{\text {water }}=1000 \mathrm{~kg} \mathrm{~m}^{-3}, \rho_{\text {air }}=1.29 \mathrm{~kg} \mathrm{~m}^{-3}$.
$g=9.81 \mathrm{~m} \mathrm{~s}^{-2}$.
The thermal expansion coefficient of water is $\alpha=2 \times 10^{-4} \mathrm{~K}^{-1}$.
The compressibility of water is $K=4.5 \times 10^{-10} \mathrm{~Pa}^{-1}$.
The shallow water wave velocity is $v=\sqrt{g h}$, where $h$ is the depth.

## 16 Suggested topics for term paper.

The essays are due on the last day of classes.
One type of essay which you may write is to review a topic, e.g. in the style of a serious newsmagazine such as "The Economist". Alternatively, you may use a more informal style and instead describe some calculations and experiments of your own. You may collaborate with a friend on researching a topic, but each student should write her/his own essay. Typically, when you do the background reading for one of these topics there will be some things you understand and some that you don't. When you write the report
concentrate on the parts you understand. Try not to exceed 10 typed pages including figures.

1. Patterned ground. Research how geometrical patterns are created by convection cells in water logged soil that undergoes repeated freezethaw cycles. A possible starting point is the pictorial essay by Kranz et al. [31], see also [18, 25, 28] .
2. The wake pattern from a boat. Discuss what determines the pattern of wakes after a moving boat [16, 59].
3. Why a fluid flows faster when the tube is pinched. A Jearl Walker "Amateur Scientist" note [58] describes a very counter-intuitive experiment by Gatzek. You may wish to verify the results by experiments of your own.
4. How and why bacteria swim, or life at low Reynolds number. For a number of readable articles on the motion of bacteria see $[6,7,9,10,11,24,45]$. A book on hydrodynamics at low Reynolds number is [29]. Collective effects may also be important. For the rôle of convection, and how bacteria and algae can set up convection cells in order to get more nutrients see [30]. There is much more material in the references cited above than you can cover in a single term paper, so you should specialize on some aspect of the problem, but try not too rely too much on a single reference.
5. Fluid uptake by a blotter. For a simple demonstration of scaling behavior that you can reproduce yourself see [5, 42].
6. The flight of beer bubbles. You may start with the articles of Crawford [17] or Coghlan [14]. Shafer and Zare [49] claim that "The deformation, oscillation, wandering and ultimate breakup of a rising, rapidly growing bubble make beer bubble dynamics a rich phenomenon, worthy of study".
7. Convective instabilities. A possible starting point is the review by Normand et al. [40]. Much of this article contains advanced material which requires a strong math background. But, there is a long introductory section which contain many historical references which could form the basis of an interesting paper.
8. The hydraulic jump in the kitchen sink. A jet of water that falls on a horizontal plane will spread out radially in a thin layer. This layer will be surrounded by a circular hydraulic jump [26, 41, 60]. Discuss the factors that determine the position of the jump and its height. Supplement your theories with experimental observations of your own.
9. Tidal bores. A possible starting point is the Scientific American article by Lynch [32].
10. Bathtub vortices. Will the bathroom vortex spin in the opposite sense in the Northern and the Southern hemispheres. If so what will happen at the equator? For studies on these problems see [53, 8, 55, $15,54,33,37,50]$.
11. How objects float. The problem of finding the orientation of a floating object can be solved for non-cylindrical objects too. The results are sometimes counter-intuitive, see [20, 21].
12. Wind energy conversion. For a possible starting point when discussing modern windmills see [19].
13. Motion of a falling paper. Why is the motion of a falling sheet of paper so irregular and unpredictable? For a very idealized model see [52].
14. Traffic flow. This is a topic where a kind of "hydrodynamics" influences our daily lives. The book by Braun et al. [13] contains a series of readable sections on how to model traffic flow. Arnott and Small [3] give some interesting examples of how attempted remedies for traffic congestion may have the opposite of the intended effect. Migowsky et al. [39] describe cellular automata modeling of the effects of individual driving patterns and accidents affect overall flow in a freeway model. An important book is that of Prigogine and Herman [44]. As expected, considering the economic importance of the problem, the literature on traffic flow is enormous, and you should be able to find many useful references yourself.

## 17 Values of some constants

atomic mass unit, 1 a.m.u. $=1.67 \times 10^{-27} \mathrm{~kg}$
Ångström, $1 \AA=10^{-10} \mathrm{~m}$
Avogadro's number, $N_{A}=6.023 \times 10^{23}$
Boltzmann constant, $k_{B}, 1.38 \times 10^{-23} \mathrm{JK}^{-1}$
Calorie, cal, 4.184J
Velocity of light, $c=3 \times 10^{8} \mathrm{~m} \mathrm{~s}^{-1}$

## References

[1] E. Achenbach 1971, Influence of surface roughness on the cross-flow around a circular cylinder, J. Fluid Mech. 46321.
[2] E. Achenbach 1972, Experiments on the flow past spheres at very high Reynolds numbers, J. Fluid Mech. 54565.
[3] R. Arnott and K. Small 1994, The economics of traffic congestion American Scientist 82 446-55.
[4] D.J. Barber and R. Loudon 1989, An introduction to the properties of condensed matter, Cambridge: University Press.
[5] C.P. Bean et al. 1991, Fluid uptake by a blotter, Amer. J. Phys. 59533.
[6] H.C. Berg 1975, Bacterial behavior, Nature 254389.
[7] H.C. Berg and E.M. Purcell 1977, Physics of Chemoreception, Biophysical Journal 20193.
[8] A.M. Binnie 1964, Some experiments on the bath-tub vortex J. Mech. Eng. Sci. 6 256-7.
[9] R.P. Blakemore and R.B. Frankel 1981, Magnetic Navigation in Bacteria, Scientific American December Issue 42 .
[10] R. Blakemore 1975, Magnetotactic Bacteria Science 190, 377.
[11] R.P. Blakemore, R.B. Frankel and A J Kalmijn, 1980, South seeking magnetotactic bacteria in the southern hemisphere Nature 286384.
[12] L. Boltzmann 1992, A German professor's trip to el dorado, Physics Today January issue p44.
[13] M. Braun, C. S. Coleman and D. A. Drew 1978, eds. Differential equation models Modules in applied mathematics Vol. 1 Springer Verlag.
[14] Coghlan A. 1991, The flight of beer bubbles from first principles, New Scientist, 9 Nov. issue p24.
[15] W. Cope 1983, The bathtub vortex Am. Sci. 71566.
[16] F.S. Crawford 1984, Elementary derivation of the wake pattern of a boat, American J. Physics 52782.
[17] F. S. Crawford 1990, Hot water, fresh beer and salt, Am. J. Phys. 58 1033.
[18] J.G. Dash 1989, Science 246,1591 (1989) Thermomolecular pressure in surface melting motivation for frost heave, Science 246, 1591.
[19] J.B. Dragt 1993, Wind energy conversion Europhysics News 2427.
[20] Paul Erdös, Gerard Schibler and Rou Herndon 1992, Floating equilibrium of symmetrical objects and the breaking of symmetry Part 1 Prisms, Amer. J. Phys 60335.
[21] Paul Erdös, Gerard Schibler and Rou Herndon 1992, Floating equilibrium of symmetrical objects and the breaking of symmetry Part 2: The cube the octahedron and the tetrahedron, Amer. J. Phys 60345.
[22] R.P. Feynman, R.B. Leighton and M.Sands 1964, The Feynman lectures on physics, Addison-Wesley.
[23] B.H. Flowers and E. Mendoza 1970, Properties of matter, John Wiley\&Sons.
[24] R B Frankel, R.P Blakemore, FF Torres de Araujo, DMS Esquivel and J. Danon 1981, Magnetotactic bacteria at the geomagnetic equator, Science 2121269.
[25] K.J. Gleason et al.1986, Geometrical aspects of sorted patterned ground in recurrently frozen soil, Science, 232, 216.
[26] R.P. Godwin 1993, The hydraulic jump (shocks and viscous flow in the kitchen sink), Amer. J. Phys. 61829.
[27] E.S.R Gopal 1974, Statistical mechanics and properties of matter, John Wiley\&Sons.
[28] B. Hallet 1990, Self-organization in freezing soils:from microscopic ice lenses to patterned ground, Can. J. Phys. 68842.
[29] John Happel and Howard Brenner 1973, Low Reynolds number hydrodynamics-with special applications to particulate media, Nordhoff second rev. ed.
[30] J.O. Kessler and N.A. Hill 1995, Microbial consumption patterns in Spatio-temporal Patterns, Ed. P.E. Cladis and P.Palffy-Muhoray, SFI studies of complexity, Addison Wesley.
[31] W.B. Kranz, K.J. Gleason and N. Caine 1988, Patterned Ground, Scientific American December issue, page 68.
[32] D.K. Lynch 1982, Tidal bores, Scientific American October issue page 146.
[33] D.L. Kelly, B.W. Martin and E.S. Taylor 1964, A further note on the bathroom vortex, J. Fluid Mech. 19 539-42.
[34] C. Kittel and H. Krömer 1980, Thermal physics, W.H. Freeman and Company.
[35] M. J. Klein 1990, The physics of J, Willard Gibbs in his time, Physics Today Sept. issue p40.
[36] I. G. Main 1984, Vibrations and waves in physics, Cambridge University Press.
[37] A.W. Marris 1967, Theory of the bathtub vortex, J.Appl. Mech. 34 1115.
[38] B. S. Massey 1989 Mechanics of fluids, fifth ed., van Nostrand.
[39] S. Migowsky, T. Wanschura, P. Rujian, 1994, Competition and cooperation on a toy autobahn model, Zeitschrift fur Physik B-Condensed Matter. 95, 407-414.
[40] C. Normand, Y. Pomeau and M. G. Velarde 1977, Convective instability: A physicist's approach, Reviews of Modern Physics 49581.
[41] R.G. Olsson and E.T Turkdogdan 1966, Radial spread of a liquid stream on a horizontal plate, Nature 211813.
[42] G.C. Peiris and K. Tennakone 1980, Amer. J. Physics 48415.
[43] D. Pneuli and C. Gutfinger 1992, Fluid Mechanics, Cambridge University Press.
[44] I. Prigogine and R. Herman 1971. Kinetic Theory of Vehicular Traffic. Elsevier, New York.
[45] E.M. Purcell 1977,Life at low Reynolds number, Amer. J. Phys. 453.
[46] L. Reichl 1980, A modern course in statistical physics, University of Texas Press.
[47] A. Roshko 1961, Experiments on the flow past a circular cylinder at very high Reynolds number, J. Fluid Mech 10345.
[48] R.H. Sabersky, A.J. Acosta and E.G. Hauptmann 1989, Fluid flow; a first course in fluid mechanics, third edition McMillan .
[49] N. Shafer and R. Zare 1991, Physics Today October issue.
[50] Merwin Sibulkin 1983, A note on the bathroom vortex and the earth's rotation, Am. Sci. 71 352-3.
[51] D. Tabor 1991, Gases, liquids and solids and other states of matter, third edition Cambridge University Press.
[52] Y. Tanabe and K. Kaneko 1994, Behavior of a falling paper, Phys. Rev. Lett. 73 1372-5.
[53] A. H. Shapiro 1962, Bath-tub vortices, Nature 196 1080-81.
[54] M. Sibulkin 1962, A note on the bathroom vortex, J.Fluid Mech. 14 21-4.
[55] L.M. Trefethen, R.W. Bilger, P.T. Fink, R.E. Luxton and R.I. Tanner 1965, The bath-tub vortex in th southern hemisphere, Nature 2071084 5.
[56] Loup Verlet 1967, Computer experiments on classical fluids. I Thermodynamical properties of Lennard-Jones Molecules, Phys Rev 15998.
[57] Loup Verlet 1968, Computer experiments on classical fluids. II Equilibrium correlation functions Phys. Rev. 165201.
[58] J. Walker 1984, Why a fluid flows faster when the tube is pinched, Scientific American "The Amateur Scientist" October issue.
[59] J. Walker 1988, Scientific American "The Amateur Scientist" February issue 124 .
[60] E.J. Watson 1964, The radial spread of a liquid jet over a horizontal plane, J. Fluid Mech. 20481.

## Index

A posteriori probability, 13
Adiabatic lapse rate, 111
Adiabatic process, 98, 101, 102
Archimedes principle, 63
Arnott R., 145
Atmospheric convection, 110
Average speed, 23, 51
Average value, 14
Balzarini, D., 5
Bar chart, 13
Barber, D.J., 10
Bathtub vortex, 145
Beer bubbles, 144
Bernoulli equation, 75, 88, 90
Bingham fluid, 50, 55
Binomial distribution, 17, 19
Blood, 55
Bloom, M., 5
Boltzmann factor, 24-30, 50, 125
Boltzmann, L., 22
Boundary layer, 73
Braun M., 145
Breaking waves, 96
Brownian motion, 131
Bulk modulus, 107
Buoyancy, 63, 64, 68
Buoyancy force, 110
Capillary waves, 92
Center of buoyancy, $63,64,68$
Centipoise, 47
Centistoke, 47
Central limit theorem, 16
Cobb, R., 5
Coexistence
water with steam, 12

Coexisting phases, 7
Coin tosses, 17
Collision cross section, 31
Collision frequency, 31, 50
Compressibility, 6, 115, 132
Compressible flow, 98-108
Conservation of energy, 75
Conservation of mass, 104
Conservation of momentum, 104
Continuous distribution, 14
Convection, 110-113
Convection cells, 144
Convective derivative, $74,82,112$
Convective instabilities, 144
Coriolis forces, 112
Correlated events, 14
Coulomb potential, 9
Course
assignments, 4
marking scheme, 5
midterm, 4
prerequisites, 4
term paper, 5
text, 5
UBC calendar description, 5
Critical point, 7
Dam stability of, 72, 132
Darcy's law, 75-81
Davis H., 5
Deep water waves, $87,91,124$
Diffusion, 4, 39-46
Dilatant fluid, 55
Dimensional analysis, 58-63
Dipolar gas, 116
Dipole moment, 28

Discrete distributions, 14
Dispersion relation, 82, 90
Dispersion-less waves, 93
Dissipation, 96
Dissipative processes, 21
Drag coefficient, 110
Drag force, $53,59,60,109,113$, $117,124,126,129$

Eddy production, 113
Effusion, 38
Einstein relation, 42, 51
Energy distribution, 23
Enthalpy, 98
Enthalpy balance, 104
Entropy
increase of, 107
Equation of state, 7
Euler equation, 75
Exclusive events, 14
Feynman, R.P., 5
Fick's laws, 50
First law of thermodynamics, 98
Flowers, B.H. , 5
Fluids, 7
Friction factor, 78
Frictional head, 78
Froude number, 94, 113
Fudge factor, 78
Gas, 6
Gatzek W.J., 144
Gaussian distribution, 16, 18, 19
Gibbs, J.W., 22
Glasses, 7
Gopal, E.S.R., 5
Gravity waves, 92
Group velocity, 91, 125, 130
Gutfinger, C., 5

Heat, 98
Herman I., 145
Histogram, 13, 19
Hydraulic jump, 93-96, 145
Hydrodynamic instability, 62
Hydrometer, 70
Hydrostatic pressure, 63
Hysteresis, 62
Ideal fluids, 73
Incompressible flow, 72,83
Incompressible fluid, 63
Independent events, 14
Inertial pseudoforces, 109
Irrotational flow, 85
Isothermal atmosphere, 28, 29
Kelvin-Helmholtz instability, 82
Kinematic viscosity, 47, 51
Kittel, C, 5
Klein, M.J., 22
Knudsen gas, 31
Krömer, H., 5
Kranz W.B., 144
Laminar flow, 46, 53-58, 72
Latent heat, 7
Law of large numbers, 13
Lennard-Jones potential, 9
Liquids, 6
Longitudinal waves, 100
Loudon, R., 10
Lynch D.K., 145
Mach number, 106, 110
Main, I.G., 5
Marginal stability, 111
Mass conservation, 104
Massey, B.S., 5

Maxwell velocity distribution, 2224, 51
Mean free path, 31, 50
Mean square speed, 24, 51
Mendoza, E., 5
Metacenter, 64, 68
Migowsky S., 145
Mobility, 51
Molecular collisions, 30-34
Molecular dynamics, 10
Molecules, 5
Momentum conservation, 104
Monolayer, 36
Moody diagram, 79
Murray, D., 5
Navier Stokes equation, 75
Neuman, M., 5
Newtonian fluid, 47, 51
Non-Newtonian fluids, 50, 54-56
Normal distribution, 16
Normalization condition, 14
Normand C., 144
Nusselt number, 113
Ocean current, 113
One-dimensional flow, 73
Pattern formation, 4
Pauli exclusion principle, 9
Phase transition, 7
Phase velocity, 87, 90, 124, 129
Physical similarity, 108-115
Pneuli, D., 5
Poiseuille flow, 53-54, 78
Poiseuille's formula, 60, 78
Polarization, 116
Power law fluid, 55, 60
Prigogine I., 145
Probability
a priori, 13
Probability theory, 13-20
Pseudoplastic, 55
Random walk, 32-34
Rayleigh Bénard convection, 112
Rayleigh number, 112
Rayleigh-Bénard convection, 113
Reduced mass, 31
Reichl, L., 5
Relative velocity probability distribution, 30
Reynolds number, 61, 62, 70, 73, $78,79,109,110,113,118$, $126,129,144$
Rigidity, 6
Ripples, 92
Root mean square speed, 24
Sabersky, R.H., 5
Seepage, 72, 132
Seiche, 97
Self-bound system, 6
Shafer N., 144
Shallow water gravity waves, 125
Shock waves, 103-107
Skew distributions, 24
Skin resistance, 113
Small amplitude approximation, 88
Small K., 145
Solids, 7
Sound speed, 103
Sound waves, 100-103
Speed distribution, 22
Speed of sound, 103
Sphere of influence, 30
St. Petersburg paradox, 17
Stagnation temperature, 99
Standard deviation, 16

Steady flow, 72, 88
Stirling approximation, 17, 20
Stochastic variable, 16
Stoke' theorem, 86
Stokes' law, 52, 60, 109, 117, 129
Stream lines, 72
Stream tube, 72
Stress, 47
Surf, 93
Surface tension, 88
for water, 89
Tabor, D., 5
Thermal conductivity, 48, 51
Thermal diffusivity, 112
Tidal bores, 93, 145
Transport properties, 46
Triangular lattice, 35
Tsunami, 93
Turbulent flow, 21, 61, 62, 70, 79, 128

Uniform flow, 72
Van der Waals attraction, 9
Vapor pressure, 7
Variance, 16
Velocity distribution, 22
Velocity profile, 47
Verlet L., 10
Viscosity, 4, 6, 7, 10, 24, 46-52
Viscous force, 109
Wakes, 144
Walker J., 144
Water
density of, 11, 113
Water evaporation and condensation, 112
Water waves, 82-97

Wave equation, 102
Wave vector, 82
Wind energy, 145
Wind tunnel
pressurized, 110
Work, 98
Zare R., 144
Zipper, 30


[^0]:    ${ }^{1}$ Readers who are interested in the history of thermodynamics will enjoy the article by M. J. Klein [35] on the US physicist Josiah Willard Gibbs 1839-1903. I also recommend Boltzmann's entertaining description of his "sabbatical" in California [12].

[^1]:    ${ }^{2}$ In statistical mechanics (41) can be thought of as the definition of entropy. The realization that the entropy defined this way agrees with the thermodynamic entropy was a major discovery by Boltzmann. In recognition of this, the formula is inscribed on Boltzmann's tombstone in Vienna.

[^2]:    ${ }^{3}$ We could equally well have used the radius here, but it is more conventional to use the diameter to describe the dimension of a pipe

[^3]:    ${ }^{4}$ Our definition of the friction factor f agrees with [48], but differs with that of [38] by a factor of 4

[^4]:    ${ }^{5}$ We stress that our discussion is only qualitative - we have neglected losses occurring at the intake to the hose, the kink after $z$ and the nozzle where the water escapes.

