

Entanglement Entropy of Random Spin Chains

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Recently, the entanglement entropy in condensed matter systems arise many attention because of its universal logarithmic scaling at quantum critical points. Moreover, the prefactor is related to central charge, which carries the important information critical points. Surprisingly, even in the disorder systems, where the ground state properties are much different from those of the clean systems, entanglement entropy still have the same scaling behavior but prefactor is only modified with extra $\ln 2$. This result might suggest there should be a more general treatment for entanglement entropy.

I. INTRODUCTION

Entanglement is most fundamental property of quantum computation and quantum information since it's the heart of quantum states. However, we still don't have the thorough knowledge about it. The main difficulty comes from, even though it's embarrassed to admit, there is no proper definition to quantify the concept of entanglement. Through these years, there are many different proposals nowadays, including entanglement entropy, N -site concurrence, localizable entanglement (LE) and etc.

In the condensed matter systems, people are interested in all properties of quantum many-particle ground state and their transitions between different phases. At zero temperature, the so called quantum phase transition (QPT) is a transition between different ground states due to the competition of interactions (such exchange energy J in Heisenberg model or on site interaction U in Hubbard model). In other words, this transition is driven by interactions instead of temperature (see first question of assignment 3, para and ferro magnetism transition). Nonetheless, CPT resemble many features of thermodynamic phase transitions, such as different phases have different order parameters and the correlation length diverges near quantum critical points (QCP). The key to understand CPT is correlation functions of the ground states, which reveal the very nature of many-particle states. So here comes a very interesting question: can we obtain any information of QPT from studying entanglement of these quantum many-particle states? It turns out that, quite remarkable, entanglement carry a lot of information about QCP and the ground state of quantum many-particle systems. .

The first connection between entanglement and QPT is made by A. Osterloh *et al* [1]. They studied one- and two-site concurrence of one dimensional (1D) spin chains. They found the derivative of one-site concurrence diverges when system is near QCP. Later on, G. Vidal *et al* [2] studied another quantity, entanglement entropy, in the same system. They showed this "entropy" has logarithmic scaling property near QCP. Moreover, its prefactor is related to the central charge c of QCP, which indicates different universality classes. This logarithmic scaling behavior was predicted by C.G. Callen and F. Wilczek [3] in studying black hole by conformal field theory (CFT) approach. And it's very generic in (1+1) dimension where the systems usually have this extra conformal symmetry.

It's always interesting to ask how these different ordered phase are affected when we put disorder or randomness into the system. Randomness is usually relevant in 1D, that is, any small disorder will dramatically change the properties of systems, for example, Anderson's localization. It's also true for 1D random coupling spin chains. Any small randomness will drive the ground into a phase called "random singlet phase". This phase is very different from anti- or ferromagnetic phases. Yet surprisingly, recently, G. Rafael and J.E. Moore [8] also found the logarithmic scaling of entanglement entropy in random singlet phase with exotic prefactor $c \ln 2$ where c is the central charge when there is no disorder. The main purpose of this report is to introduce their calculation.

We will support several background knowledge about entanglement and random spin chains[6],[7] in the next section. Then we can discuss G. Rafael and J.E. Moore's calculation in more detail.

II. ENTANGLEMENT ENTROPY

A. Definition and Two Spin Systems

As we mentioned before, there are many proposals to quantify the concept of entanglement. Here we will focus on one of them, entanglement entropy. If we have an entangled state $|\Psi_{AB}\rangle$ of systems A and B, which is a pure state, then the operator for density of states is $\rho = |\Psi_{AB}\rangle\langle\Psi_{AB}|$. We can get the reduced density matrix $\rho_{A(B)}$ for subsystem A (B) by tracing off the degrees of freedom for the rest part B (A), $\rho_{A(B)} = \text{Tr}_{B(A)}\rho$. Then entanglement

entropy or sometimes called von Neumann entropy of the reduced density matrix for either subsystem is:

$$S = -Tr\rho_A \ln \rho_A = -Tr\rho_B \ln \rho_B. \quad (1)$$

One interesting feature is that two subsystems share the same value of entanglement entropy, no matter how you divide them. This is the original motivation to propose entanglement entropy as a measure of entanglement since two subsystem should have the same amount of "entanglement" with each other. Here we will take two spins system for example. The most general wave function is

$$|\Psi_{12}\rangle = \cos\theta |\uparrow\downarrow\rangle + \sin\theta |\downarrow\uparrow\rangle. \quad (2)$$

Now we trace off spin 1 to get the reduced density matrix of spin 2,

$$\begin{aligned} \rho_2 &= Tr_1 |\Psi_{12}\rangle\langle\Psi_{12}| = \langle\uparrow|\Psi_{12}\rangle\langle\Psi_{12}|\uparrow\rangle + \langle\downarrow|\Psi_{12}\rangle\langle\Psi_{12}|\downarrow\rangle \\ &= \cos^2\theta |\uparrow\rangle\langle\uparrow| + \sin^2\theta |\downarrow\rangle\langle\downarrow|, \end{aligned} \quad (3)$$

then entanglement entropy between spin 1 and 2 is

$$\begin{aligned} S &= -Tr\rho_2 \ln \rho_2 = -Tr\rho_1 \ln \rho_1 \\ &= -\cos^2\theta \ln \cos^2\theta - \sin^2\theta \ln \sin^2\theta. \end{aligned} \quad (4)$$

It's clear to see that S is maximal when $\theta = \frac{\pi}{2}$, that is, when two spins form a singlet, which is as we expected.

B. Entanglement Entropy in 1D Spin Chains

Now we should apply this calculation to 1D spin one-half chains. Let's consider a XXZ and quantum Ising models on N sites with periodic boundary conditions, and the Hamiltonian is

$$H_{XXZ} = \sum_{i=1}^N J(S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + \Delta S_i^z S_{i+1}^z), \quad (5)$$

$$H_{QI} = -\sum_{i=1}^N JS_i^x S_{i+1}^x - hS_i^z, \quad (6)$$

The phase diagrams of these models are well understood. When $\Delta = 1$ for H_{XXZ} and $\lambda = \frac{J}{h} = 1$ for H_{QI} , they are QCP and systems undergo QPT between $\Delta > 1$, $\Delta < 1$ and $\lambda > 1$, $\lambda < 1$. Sometimes we call the system *critical* at QCP and *noncritical* away from QCP.

We can study the entanglement entropy between a block of L contiguous spins and the rest $N - L$ spins when the system is at its ground state $|\Psi_G\rangle$. We will have the same value either use the reduced density matrix $\rho_L = Tr_{N-L} |\Psi_G\rangle\langle\Psi_G|$ of this L spins block or $\rho_{N-L} = Tr_L |\Psi_G\rangle\langle\Psi_G|$ of the rest $N - L$ spins. Therefore we can study how entanglement entropy $S_L = -Tr\rho_L \ln \rho_L$ depends on the size of block spins L . The result is quiet neat, when the system is at QCP and L is small compared with N , entanglement entropy has universal logarithmic scaling behavior,

$$S_L = \frac{c}{3} \ln L, \quad (7)$$

where c is the central charge of QCP. In conformal field theory, central charge c plays a crucial role to determine the excitation spectrum and all critical exponents. Different QCP are governed by different central charges c . $c = 1$ when H_{XXZ} is at $\Delta = 1$ and $c = \frac{1}{2}$ when H_{QI} is at $\lambda = 1$, these basically tell us these two QPT belong to different universality classes, i.e. they have different critical exponents near QCP. Hence, entanglement entropy of man-particle ground states really carry the important information of QCP.

C. Conformal Field Theory Approach to Entanglement Entropy (Optional)

With the presence of central charge in Eq.7, it's natural to think that we should be able to derive it from conformal field theory (CFT). Although, indeed there is a direct (conformal) field theory calculation for entanglement for entangle entropy [3],[4], nonetheless, here we will follow Korepin's agreement [5] since it's easier to understand. Readers who are not so familiar with CFT can regard this section as only supplementary materials.

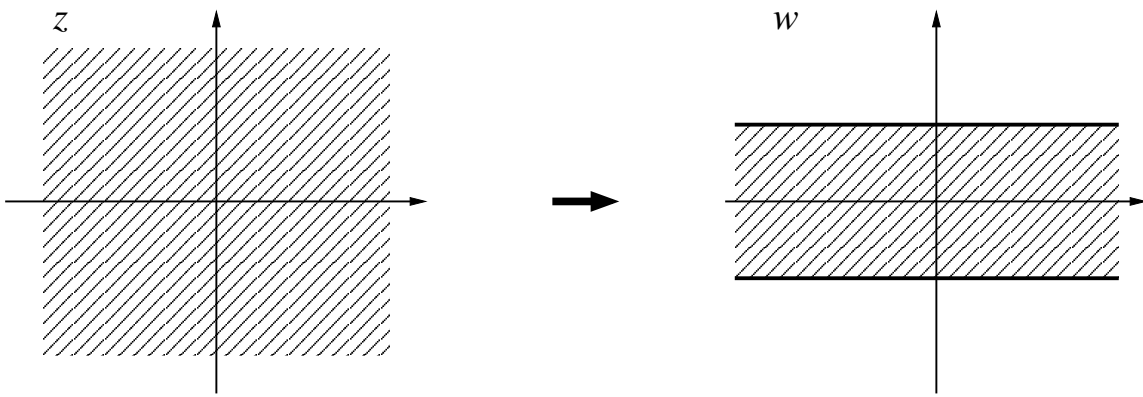


FIG. 1: Conformal mapping for z to w plane

In 1D systems with length x , the thermodynamic entropy at finite temperature T is well-known and easy to calculate,

$$S(x) = \frac{\pi T c}{3v} x, \quad T \neq 0 \quad (8)$$

where c is central charge of the system and v is the velocity scale (Fermi velocity for fermions or sound velocity for bosons). It's an extensive quantity (proportional to x) as we expected for entropy. Now we want to find out how the entanglement entropy at zero temperature depends on x ,

$$S(x) = f(x), \quad T = 0. \quad (9)$$

And somehow let's assume it's a conformal invariant quantity, that is, it's the same as finite T entropy via a conformal transformation.

In quantum 1+1 systems, real space coordinate x and $\alpha = \frac{1}{kT}$ form a 2D conformal plane. The conformal transformation connect $T = 0$ ($\alpha = \infty$) plane z to finite T plane w is (see fig.1)

$$w = \exp\left(\frac{2\pi T z}{v}\right).$$

This transformation is equivalent to replace the length scale x in Eq. 9 by the new corresponding length scale $\frac{v}{T} \sinh\left(\frac{\pi T x}{v}\right)$ in finite T systems. So we expect

$$S(x) = f\left[\frac{v}{T} \sinh\left(\frac{\pi T x}{v}\right)\right], \quad T \neq 0. \quad (10)$$

For large x , it becomes

$$S(x) = f\left[\exp\left(\frac{\pi T}{v}(x - x_0)\right)\right], \quad (11)$$

where $\frac{\pi T x_0}{v} = -\ln\left(\frac{v}{2\pi T}\right)$ and should recover Eq.8. Therefore it's easy to see that

$$S(x) = \frac{c}{3} \ln x + k, \quad T = 0, \quad (12)$$

where k is a constant.

Even the above derivation is pretty trivial but the physics meaning is deep. Thermodynamic entropy, roughly speaking, reflects the excitation spectrum of the system but entanglement entropy is resulted from ground wave function at zero temperature. It's amazing that these two properties can be connected in this subtle way.

Entanglement entropy has been discussed widely in the block hole physics field. One way to derive block entropy is through the same definition as Eq.1. So black hole entropy has the same scaling properties as entanglement entropy in all dimensions. We have to emphasize that even all these quantities have the similar name "entropy", one shouldn't mix them up together. Similar calculations can also be done in the systems without conformal symmetry through original definition Eq.1.

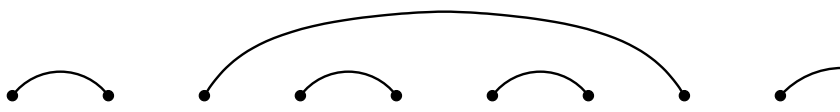


FIG. 2: A schematic for random singlet phase

III. 1D RANDOM SPIN CHAIN

We will start at $\Delta = 1$ in Eq.5, that is, the system is critical and entanglement entropy of L block spins is $S_L = \frac{1}{3} \ln L$. Now we put randomness into the system, for example, several sites are randomly replaced by different kinds of spins, so the spin coupling J in the Hamiltonian is no more a constant and instead, it will be a random distribution function for site i ,

$$H = \sum_{i=1}^N J_i (S_i^x S_{i+1}^x + S_i^y S_{i+1}^y + S_i^z S_{i+1}^z) = \sum_{i=1}^N J_i \vec{S}_i \cdot \vec{S}_{i+1}. \quad (13)$$

Before we turn randomness, we know the ground state has quasi-long-range antiferromagnetic order and gapless excitations (question 4 in assignment 3). The first question is whether this antiferromagnetic order is stable against randomness. If not, what's the new ground property when disorder is relevant. In general, the answers of these question are notoriously difficult to get. But S.K. Ma *et al* [6] they had a brilliant method to tackle this problem back to 1979. Their idea is to study how the probability distribution function of random coupling $P_\Omega(J)$ evolves as energy scale changes, where Ω is the largest coupling in the system. Many years later, D.S. Fisher [7] did a more systematic study. According their works, we know that with the presence of *any* arbitrary small randomness in the system, antiferromagnetic order no longer exists. The new ground state is called random singlet phase [see fig.2] for two spins with any distance can form a singlet bond. Now we should see the detail derivations.

The first step is to find those largest $J_i = \Omega$ couplings between spin i and $i + 1$ in the given $P_\Omega(J)$, and just diagonalize it independently of the rest of the chain. The Hamiltonian for for them is

$$H_0 = J_i \vec{S}_i \cdot \vec{S}_{i+1} \quad (14)$$

. This leads the ground state between spin i and $i + 1$ is a spin singlet state $|s\rangle$ with three excitation states forming a triplet $|t\rangle$. The corresponding energy are $E_s = -\frac{3}{4}J_i$ and $E_t = \frac{1}{4}J_i$. Then we treat the rest of the Hamiltonian as a perturbation. Spin i and $i + 1$ couple to their neighbors $i - 1$ and $i + 2$ via

$$H_1 = J_{i-1} \vec{S}_{i-1} \cdot \vec{S}_i + J_{i+1} \vec{S}_{i+1} \cdot \vec{S}_{i+2}. \quad (15)$$

Since J_i is the maximum value of the coupling constant, let's assume J_{i-1} and $J_{i+1} \ll J_i$ and treat them in perturbation, the ground state energy E_s is modified to

$$E_s + \langle s | H_1 | s \rangle + \sum \frac{|\langle s | H_1 | t \rangle|^2}{E_s - E_t} \equiv E' + \tilde{J}_{i-1, i+2} \vec{S}_{i-1} \cdot \vec{S}_{i+2}. \quad (16)$$

where

$$E' = -\frac{3}{4}J_i - \frac{3}{16J_i}(J_{i-1}^2 + J_{i+1}^2), \quad (17)$$

$$\tilde{J}_{i-1, i+2} = \frac{J_{i-1}J_{i+1}}{2J_i}. \quad (18)$$

Now we remove the spins \vec{S}_i and \vec{S}_{i+1} from the original Hamiltonian and add the effective coupling $\tilde{J}_{i-1, i+2} \vec{S}_{i-1} \cdot \vec{S}_{i+2}$ and the constant E' . The new Hamiltonian should reproduce approximately the same ground state properties as the original Hamiltonian. One can just apply the procedure again, to eliminate the spins with the next strongest couplings $\tilde{\Omega}$. After apply this procedure many times, one will alter the probability distribution $P_\Omega(J)$ as the largest cutoff coupling Ω is lowered and lowered [There is a fancy terminology for this method: real space renormalization group (RG)]. It's easier to see the evolution of $P_\Omega(J)$ in terms of new variables:

$$\beta = \ln \frac{\Omega}{J} \Gamma = \ln \frac{\Omega_0}{\Omega}, \quad (19)$$

where Ω_0 is the initial energy cutoff scale and Ω is its reduced energy scale. These variables reflect the scaling properties of the system, take Eq.18 for example and drop off a constant $\ln 2$, it's simply

$$\tilde{\beta}_{i-1,i+2} = \beta_{i-1} + \beta_{i+1}, \quad (20)$$

since the strongest bonds have $\beta = 0$. The evolution of probability distribution $P_\Omega(J)$ as Ω is lowered is basically a probability question. In terms of β and Γ , we have

$$\frac{dP_\Gamma(\beta)}{d\Gamma} = \frac{\partial P_\Gamma(\beta)}{\partial \Gamma} + P_\Gamma(0) \int d\beta_1 d\beta_2 \delta(\beta - \beta_1 - \beta_2) P_\Gamma(\beta_1) P_\Gamma(\beta_2). \quad (21)$$

The first on the right hand side is due to those bonds not strongest Ω . Their contribution to the new probability distribution is simply from changing the cutoff energy scale. The second is the contribution from those decimated spins based on Eq.20 and $P_\Gamma(0)$ is the fraction for the decimated spins at initial energy scale.

Further analysis of Eq.21 reveals an amazing property. Despite of any initial probability distribution $P_\Gamma(0)$, it always flows to the following solution, as an attractor in the differential-integral equation.

$$P_\Gamma(\beta) = \frac{1}{\Gamma} e^{-\frac{\beta}{\Gamma}}$$

This is the random singlet fixed point distribution. The physics meaning of this result is pretty generic in 1D systems. That is, randomness, or disorder is usually relevant and hence drives the original system into a new phase. (In this model, it's random singlet phase while in the random external B field systems is corresponding to Anderson's localization.) This random singlet phase (see fig.2) has many interesting properties different from those of the clean systems (Eq.5 and 6) such as the behavior of correlation function and thermodynamic quantities. It's beyond the scope of this report to discuss those interesting results. However, two important characters of random singlet phase are relevant to our final goal in calculating entanglement entropy.

1. The spin singlet can be formed at arbitrary long distance as long as Ω is low enough.
2. No singlet bond can cross each other.

It's not very difficult to understand the above two results since after two decimated spins form a singlet, they just decouple to the Hamiltonian. Therefore, the singlets formed at lower Ω can be really far from each other. Note nearest-neighbor singlets are still the majority even though singlets with long distance exist. That singlet bonds can't cross each other is just the consequence of nearest neighbor coupling in the Hamiltonian. Equipped with all the background, we can jump into the entanglement entropy of random singlet phase.

IV. ENTANGLEMENT OF RANDOM SINGLET PHASE

Before we calculate entanglement entropy of random singlet phase, let me remind you that, in the clear system, the entanglement entropy of L contiguous spins has the universal form Eq.7 at the QCP and can be derived by CFT. Starting at random singlet phase, we also attempt to calculate the entanglement entropy between a block L contiguous spins and the rest of the system (see fig.3). The precise mathematic statement to describe the relation between the typical singlet bond length λ and energy scale Ω , on average, is

$$\lambda \sim \Gamma^2. \quad (22)$$

where $\Omega = \Omega_0 e^{-\Gamma}$. So at lower Ω , the scaling variable Γ is larger so that the typical bond distance λ is also larger, as we expected.

From the first section, we know that for a singlet pair, the entanglement entropy between one spin and its partner spin is $\ln 2$. Accordingly, The entanglement entropy of a L spins segment in random singlet phase is just $\ln 2$ times the number of singlet N that connect sites inside to sites outside the segment. Apparently, N will be proportional to the total number of singlets \overline{N} and \overline{N} can be easily estimated. When we change the energy scale $\Omega \rightarrow \Omega - d\Omega$, or $\Gamma \rightarrow \Gamma + d\Gamma$, all bonds with $0 < \beta < d\Gamma$ get decimated. The average number of singlet bonds will grow by

$$d\overline{N} = P(\beta = 0) d\Gamma = \frac{d\Gamma}{\Gamma} \quad (23)$$

which leads to $\overline{N} = \ln \Gamma$. But the singlet pairs which have characteristic lengths longer than the size of segment L , won't contribute to the entanglement because they are totally outside the segment. Hence, we only include those

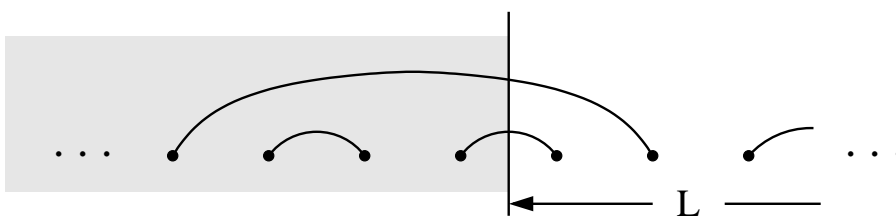


FIG. 3: The entanglement entropy of a segment is the number of singlets that connect the segment with the rest of the chain (shaded area). In this example there are two such singlets.

bonds formed up to the energy scale $\Gamma = \sqrt{L}$. So the number of singlets emanating from the segment of size L , i.e. its entanglement with the rest of the system, is

$$S_L \sim \ln 2 \cdot 2\bar{N} = \ln 2 \cdot 2 \ln \sqrt{L} + k = \ln 2 \cdot \ln L + k, \quad (24)$$

where k is a nonuniversal constant and depends on the initial probability distribution of the disorder. Even in this rough estimate, we find scaling behavior for entanglement entropy is the same as that in clean systems. In the more detail calculation, however beyond the scope of this report, the entanglement entropy of size L with the rest of the chain is

$$S_L = \frac{c \ln 2}{3} \ln L + k, \quad (25)$$

where c is the central charge of the corresponding clean systems. In other words, $c = 1$ for random Heisenberg spin chain (Eq.13) and $c = \frac{1}{2}$ for random quantum Ising chain (if J and h are random number in Eq.6).

To encapsulate, convention wisdom tells us that the physical properties behave very differently between clean and the corresponding disorder system. Nevertheless, Eq.7 and 25 show that clean and disorder systems share almost the same formula for the entanglement entropy of L spins with the rest of the chain. Moreover, in disorder systems, one can regard the prefactor as an exotic effective central charge $\tilde{c} = c \ln 2$ even though we know there is actually NO conformal symmetry in the disorder systems. Why the central charge of clean system appears in the final formula of disorder system becomes a very intriguing question. This results seems to suggest that there may be a more fundamental way to understand the logarithmic scaling behavior of entanglement entropy in 1D system. The true physics meaning of Eq.25 is still unclear. Exploring entanglement seems to point out a promising direct to understand these systems deeper.

Here we give specially thanks to Rodrigo for those figures.

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