Effects of Dissipation on Quantum Tunneling

by

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I. INTRODUCTION

The phenomenon of tunneling through a potential barrier is a well-known quantum mechanical phenomenon. In investigating the correspondence between phenomena that obey the laws of quantum mechanics and classical phenomena, it is of interest to know whether a quantity that is macroscopic in nature can undergo a tunneling process, and what will be the effects of dissipation on its tunneling. Dissipation arises from the system when it interacts with the environment surrounding it and loses energy to it through this interaction. Intuitively we would expect that dissipation would tend to suppress the tendency of a system to tunnel through a potential barrier.

More precisely, we wish to formulate and investigate a model of a 'macroscopic tunneling' process as follows. The system is described by a variable q and a Hamiltonian such that its potential has a metastable minimum, ie. a well at the origin followed by a potential barrier separating it from a continuum of states. The quasi-classical equation of motion in the region near the minimum is given by

$$M\ddot{q} + \eta\dot{q} + \frac{\partial V}{\partial q} = F_{ext}(T)$$

Here η, the phenomenological friction coefficient, represents the quasi-classical limit of the effects of system-environment interaction. The question we wish to pose is: how will the tunneling behavior of the system compare to the case where η would be zero (ie. it is completely isolated from the environment)?

In order to answer questions of this sort we need an appropriate model of the system-environment interaction. To this end we consider the system and environment conjointly to form a closed system (or "universe") in which any one degree of freedom of the environment is only weakly perturbed by the system. Then in this case it is suitable to represent the environment by a bath of harmonic oscillators with a linear coupling to the system.

There are a number of requirements that we would like to impose on the system so that to make it suitable for this kind of study. Three of them are:

1. The system must have a metastable potential well which is separated from a more stable continuum of states by a free-energy barrier.
2. When the system tunnels through the barrier, it should be macroscopically distinguishable from the state it was in while it was confined to the potential well.
3. The quantum tunneling rate due to small oscillations in the metastable well should sufficiently exceed the thermal escape rate, so that we have the criterion $\hbar\omega_0 > k_B T$

One type of system that is suitable for this kind of study is that of a DC SQUID, which typically serves as a sensitive magnetometer. The macroscopic variable of interest will be the flux Φ trapped in the ring, whose potential energy is described by the expression

$$U(\Phi) = \frac{(\Phi - \Phi_x)^2}{2L} - I_c \Phi_0 \cos(\frac{2\pi\Phi}{\Phi_0})$$

Here L is the self-inductance of the ring, I_c is the critical current of the Josephson junction, $\Phi_0 = \frac{h}{2e}$ is the flux quantum and Φ_x the externally imposed flux through the ring. We can ensure a metastable minimum if we make use of the condition $\frac{2\pi L I_C}{\Phi_0} > 1$. By suitable manipulation of Φ_x it is possible to trap the SQUID in this metastable state. In addition to the potential energy term there is also in the Hamiltonian a kinetic energy of the form $\frac{1}{2} C\dot{\Phi}^2$ where C is the capacitance of the Josephson junction. The shape of the potential in the region where tunneling occurs is made so that it is of the form of a quadratic-plus-cubic ($\alpha\Phi^2 - \beta\Phi^3$).

II. THE MODEL

In the following we use the general variable q instead of the flux Φ. The Lagrangian of the system in the absence of interaction is

$$L(q, \dot{q}) = \frac{1}{2} M\dot{q}^2 - V(q)$$

Where $V(q)$ has a well at $q = 0$ and barrier with height V_0 that has a width q_0 (for example the quadratic-plus-cubic potential discussed above).

The frequency of small oscillations around the metastable minimum, ω_0, is

$$\omega_0 \equiv | M^{-1} (d^2V / dq^2)_{q=0} |^{1/2}$$

We then add to this a term involving the harmonic oscillator bath

$$L_{osc} = \sum_\alpha (\frac{1}{2} m_\alpha \dot{x}_\alpha{}^2 - \frac{1}{2} m_\alpha \omega_\alpha{}^2 x_\alpha{}^2)$$

x_α and ω_α are the coordinates and frequencies of the harmonic oscillators respectively and the interaction term between system and environment:

$$L_{\text{int}} = -q\sum_{\alpha} C_{\alpha} x_{\alpha} + \Phi(q)$$

Where $\Phi(q) = -\frac{1}{2}M\ |\Delta\omega^2\ |\ q^2$

(This latter term comes about due to the cancellation of the frequency renormalization effect, which will not be discussed in detail here)

III. THE ANALYSIS

We are given the following Lagrangian for the system-environment interaction from which we must determine the reduced density matrix of the system:

$$L = \frac{1}{2}M\dot{q}^2 - V(q) + \sum_{\alpha}(\frac{1}{2}m_{\alpha}\dot{x}^2{}_{\alpha} - \frac{1}{2}m_{\alpha}\omega^2{}_{\alpha}x^2{}_{\alpha}) - q\sum_{\alpha} C_{\alpha}x_{\alpha} - \frac{1}{2}M\left|\Delta\omega^2\right|q^2$$

Then the reduced density matrix is (following the notation of [1])

$$K(q_i, q_f : T) \equiv \int\prod_{\alpha} dx_{\alpha i}\rho(q_i, \{x_{\alpha i}\}, q_f, \{x_{\alpha i}\} : \beta)$$

where

$$\rho(q_i, \{x_{\alpha i}\}, q_f, \{x_{\alpha f}\} : \beta) \equiv \sum_n \psi_n{}^*(q_i, \{x_{\alpha i}\})\psi_n(q_f, \{x_{\alpha f}\})\exp(-\beta E_n)$$

$$= \int_{q(0)=q_i}^{q(T)=q_f} Dq(\tau)\prod_{\alpha}\int_{x_{\alpha}(0)=x_{\alpha i}}^{x_{\alpha}(T)=x_{\alpha i}} Dx_{\alpha}(\tau)\exp(-\int_0^T Ld\tau/\hbar)$$

Then

$$K(q_i, q_f : T) = \prod_{\alpha}\int_{q(0)=q_i}^{q(T)=q_f} Dq(\tau)\exp(-\frac{S_0}{\hbar})Q_{\alpha}(T)$$

Where

$$Q_{\alpha}(T) = \int dx_{\alpha i}\int_{x(0)=x_i}^{x(T)=x_i} Dx_{\alpha}(\tau)\exp-\frac{1}{\hbar}\{\int_0^T d\tau[\frac{1}{2}m_{\alpha}(\dot{x}_{\alpha}{}^2 + \omega_{\alpha}{}^2 x_{\alpha}{}^2) + x_{\alpha}C_{\alpha}q(\tau)]\}$$

And

$$S_0 \equiv \int_0^T d\tau \{\frac{1}{2} M\dot{q}^2 + V(q)\}$$

The path integral $Q_\alpha(T)$ can be evaluated in a straightforward manner and can be simplified further if we assume that $q(\tau + T) = q(\tau)$

$$Q_\alpha(T) = \frac{1}{2} \cos ech(\frac{\omega_\alpha T}{2}) \exp\{\frac{C_\alpha{}^2}{4m_\alpha \hbar \omega_\alpha} \int_{-\infty}^{\infty} d\tau' \int_0^T d\tau \exp(-\omega_\alpha |\tau - \tau'|) q(\tau) q(\tau')\}$$

We some further manipulation we then determine that we can write the reduced density matrix as

$$K(q_i, q_f : T) = K_0(T) \int_{q(0)=q_i}^{q(T)=q_f} Dq(\tau) \exp(-S_{eff}[q(\tau)]/\hbar)$$

$$S_{eff}[q(\tau)] \equiv \int_0^T d\tau \{\frac{1}{2} M\dot{q}^2 + V(q)\} + \frac{1}{2} \int_{-\infty}^{\infty} d\tau' \int_0^T d\tau \quad (\tau - \tau')\{q(\tau) - q(\tau')\}^2$$

Where $\alpha(\tau - \tau') \equiv \sum_\alpha \frac{C_\alpha{}^2}{4m_\alpha \omega_\alpha} \exp(-\omega |\tau - \tau'|) \equiv \frac{1}{2\pi} \int_0^\infty d\omega J(\omega) \exp(-\omega |\tau - \tau'|)$

(It is assumed that the spectral density $J(\omega) \equiv \eta$, ie. of the form of ohmic dissipation)

Note that this is a positive quantity, and the second term of the effective interaction in its entirety represents the effects of dissipation on the system.

The tunneling rate Γ is seen to be proportional to
$\Gamma \propto \exp(-S_{eff}/\hbar)$

And hence it is easily seen that the positive second term implies that the effect of the dissipation is to suppress tunneling, as would be expected.

<u>References</u>

[1] Caldeira, A.O. and Leggett, A. J. “Quantum Tunneling in a Dissipative System”. *Annals of Physics*. **149**, 374-456 (1983)

[2] Takagi, Shin. *Macroscopic Quantum Tunneling.* Cambridge University Press. 1997.

[3] Leggett, A. J. “Testing the limits of quantum mechanics: motivation, state of play, prospects”. J. Phys: Condens. Matter. **14,** 2002, 415-451