Problem set 1

1. Consider a 1D lattice with lattice constant a, and a particle whose dynamics is described by the hopping Hamiltonian $\hat{H}_0 = -t \sum_n (|n\rangle\langle n+1| + |n+1\rangle\langle n|)$.

(a)Find $G_0(n_2, n_1, E)$. Suggested steps: (i) assume a finite number of lattice sites, N, and periodic boundary conditions (i.e, site $''N+1''$ is in fact site 1). Find the eigenenergies and eigenfunctions. (ii) write $G_0(n_2, n_1, E)$ in terms of eigenstates and eigenenergies. Take the limit $N \to \infty$, and evaluate the resulting integral. It's fine with me if you pick the result from a table of integrals, though you should be able to do it by hand.

(b) If E is close to the bottom of the band, the dispersion energy can be approximated (up to a constant) with a quadratic form similar to the energy of a free particle. Find the corresponding effective mass. In this limit ($a \to 0$), is $G_0(n_2, n_1, E)$ in agreement with the Green's function for a free 1D particle discussed in class?

2. Consider a spin-$\frac{1}{2}$ whose dynamics is described by the Hamiltonian $\hat{H} = -b\hat{S}_x$. Find (a) $G(\uparrow, \downarrow; E)$; (b) Fourier transform this to obtain $G(\uparrow t; \downarrow 0) = -i/\hbar \mathcal{G}(\uparrow t; \downarrow 0)$. Now check your result by solving directly the Schrödinger equation for a particle which has spin $\downarrow$ at $t = 0$, and finding the amplitude of probability to have spin $\uparrow$ at a later time t.

3. Consider an electron in a sample of volume V. The electron interacts with a set of N impurities located at random positions $\vec{R}_1, \ldots, \vec{R}_N$, through the potential $V_T(\vec{r}) = \sum_{n=1}^{N} V(\vec{r} - \vec{R}_n)$. We assume that there are very many impurities, but each one is very weak. Also, we assume that $\int d\vec{r} V(\vec{r}) = 0$.

Consider now disorder averages. By definition

$$\langle f(\vec{R}_1, \ldots, \vec{R}_N)\rangle_{dis} = \int \frac{d\vec{R}_1}{V} \cdots \int \frac{d\vec{R}_N}{V} f(\vec{R}_1, \ldots, \vec{R}_N)$$

(a) calculate $\langle V_T(\vec{r_1}) V_T(\vec{r_2})\rangle = W(\vec{r}_1 - \vec{r}_2)$. Express this in terms of the Fourier components of the impurity potential $V_{\vec{q}} = V^*_{-\vec{q}} = \int d\vec{r} e^{-i\vec{q}\vec{r}} V(\vec{r})$.

(b) calculate $\langle V_T(\vec{r_1}) V_T(\vec{r_2}) V_T(\vec{r_3}) V_T(\vec{r_4})\rangle$. Does is satisfy the factorization rule of a gaussian disorder? If not, argue in what case could we still assume that factorization to be a good approximation? Is this approximation likely to become better or worse, as you go to higher order correlations?

4. Consider an electron in an inhomogeneous external magnetic field, $\vec{B}(\vec{r})$, such that its Hamiltonian is $\mathcal{H} = \mathcal{H}_0 - \vec{b}(\vec{r}) \cdot \hat{\vec{\sigma}}$. Here, $\mathcal{H}_0 = \hat{\vec{p}}^2/2m$ is just the kinetic energy of a free electron, we expressed the spin electron in terms of Pauli matrices, $\hat{\vec{s}} = \hbar/2\hat{\vec{\sigma}}$, and I included all other constants (Bohr magneton, factor of $\hbar$ etc), in rescaling $\vec{b}(\vec{r}) \sim \vec{B}(\vec{r})$. Use appropriate diagrams and rules to express Dyson's equation for this problem, in real space. Write explicitly the expression of the second-order contribution, $G^{(2)}(\vec{r}, \sigma; \vec{r'}, \sigma'; E)$. Now, assume that $\vec{h}(\vec{r}) = h\vec{e}_x$ (i.e., uniform field in the x-direction). Switch to $\vec{k}$-space, derive the corresponding diagrams and needed rules, and use Dyson's equation to find $G_{\sigma\sigma'}(\vec{k}, E)$ exactly. Is the answer sensible?