The Henon-Heiles Hamiltonian

Motion of Stars about a Galactic Center

The Henon-Heiles Hamiltonian describes the motion of stars around a galactic center, assuming the motion is restricted to the xy plane.

$$H = \frac{P_x^2}{2m} + \frac{P_y^2}{2m} + \frac{1}{2}k(x^2 + y^2) + \lambda(x^2 y - \frac{1}{3}y^3)$$

The Equations of Motion

In order to simplify the Hamiltonian assume:

$$P_x = m\dot{x} \qquad \text{and} \qquad P_y = m\dot{y}$$

And rewrite the Hamiltonian in a dimensionless, normalized form with lambda equal to one:

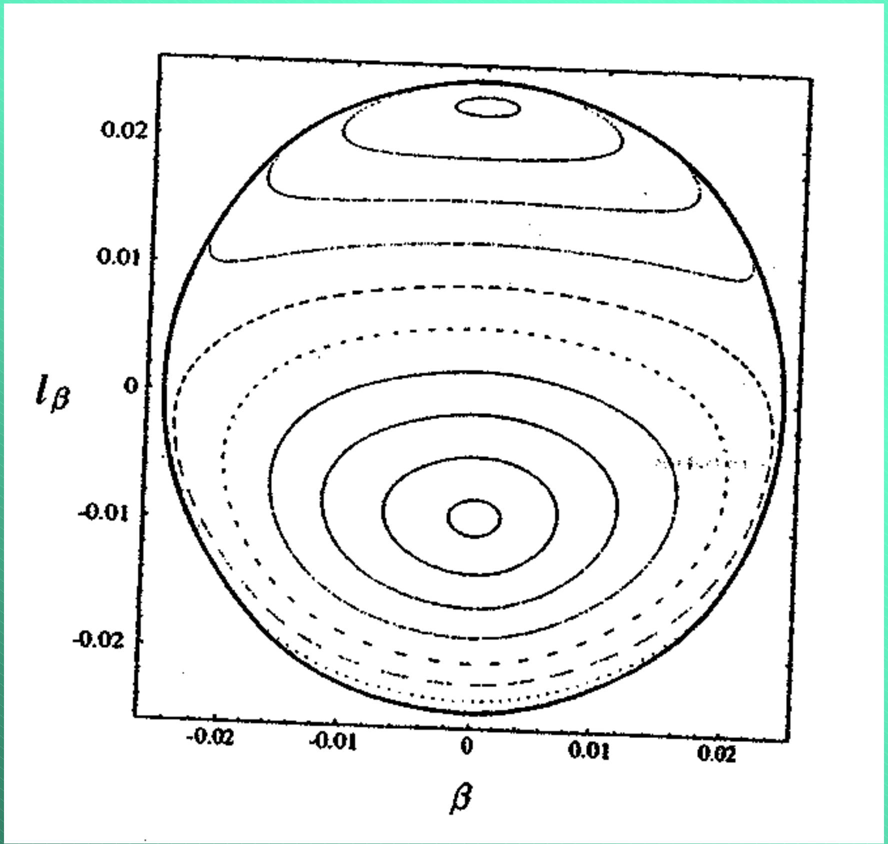
$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3}y^3$$

This yields the following equations of motion:

$$\ddot{x} = -x - 2xy \qquad \text{and} \qquad \ddot{y} = -y - x^2 + y^2$$

Poincare Maps

A Poincare map is a two dimensional slice of a systems four dimensional phase space



Poincare map for the double pendulum

Beta is the angle between the first and second pendulums

Step 1:
Determine Appropriate Initial Conditions

Appropriate initial conditions are determined by the total energy of the system

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2}(x^2 + y^2) + x^2 y - \frac{1}{3} y^3$$

We will be examining Poincare sections in the $y\dot{y}$ plane, so initially we may set x=0

$$E = \frac{1}{2}(\dot{x}^2 + \dot{y}^2) + \frac{1}{2} y^2 - \frac{1}{3} y^3$$

The bounding curve for the available phase space can be determined by setting x=0

$$E = \frac{1}{2}\dot{y}^2 + \frac{1}{2}y^2 - \frac{1}{3}y^3$$

If we choose values of y and $\dot{y}$ that lie within this bounding Curve, we may use the equation

$$\dot{x} = (2E - \dot{y}^2 - y^2 + \frac{2}{3}y^3)^{\frac{1}{2}}$$

This gives us our appropriate set of initial conditions

Step 2:

Integrate the Equations of Motion

$$\ddot{x} = -x - 2xy \qquad \ddot{y} = -y - x^2 + y^2$$

In order to solve these equations we will first cast them into canonical, first order form

$$y_1 = x \qquad y_1' = y_3$$

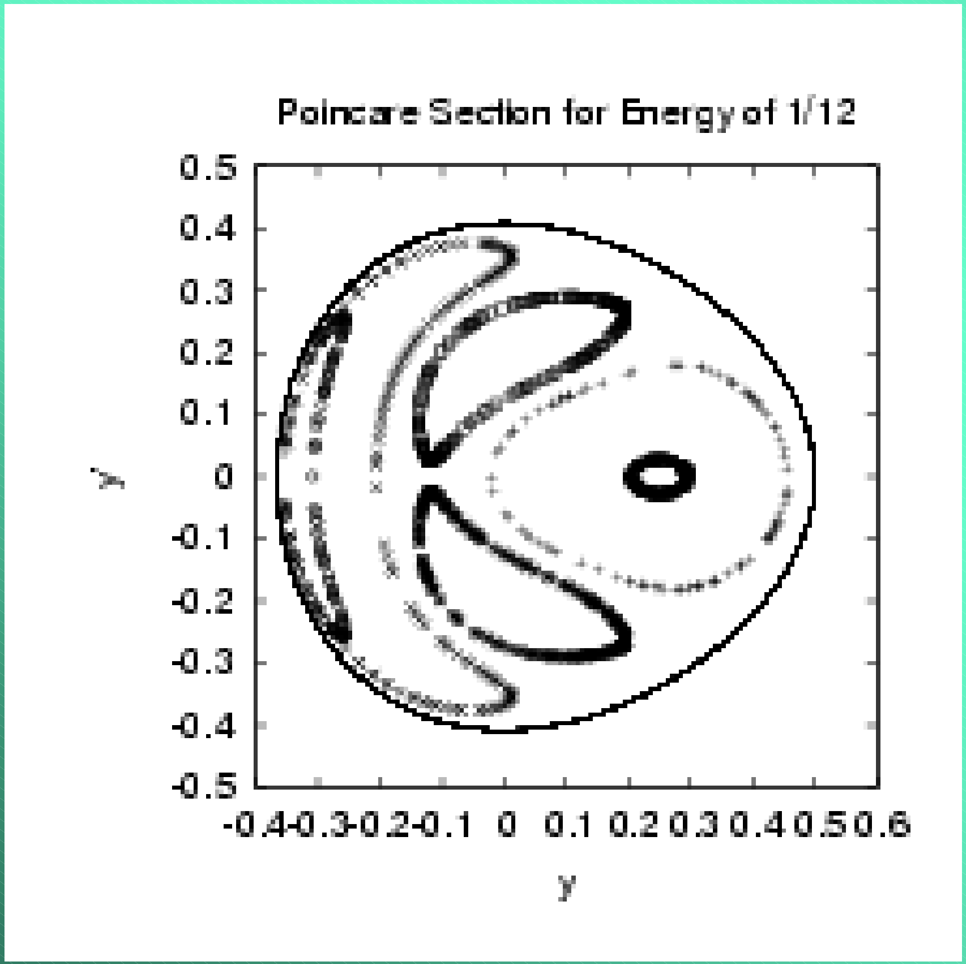
$$y_2 = y \qquad y_2' = y_4$$

$$y_3 = \dot{x} \qquad y_3' = -y_1 - 2y_1y_2$$

$$y_4 = \dot{y} \qquad y_4' = -y_2 - y_1^2 + y_2^2$$

These four first order equations may now be integrated using the Fortran routine LSODA

Results



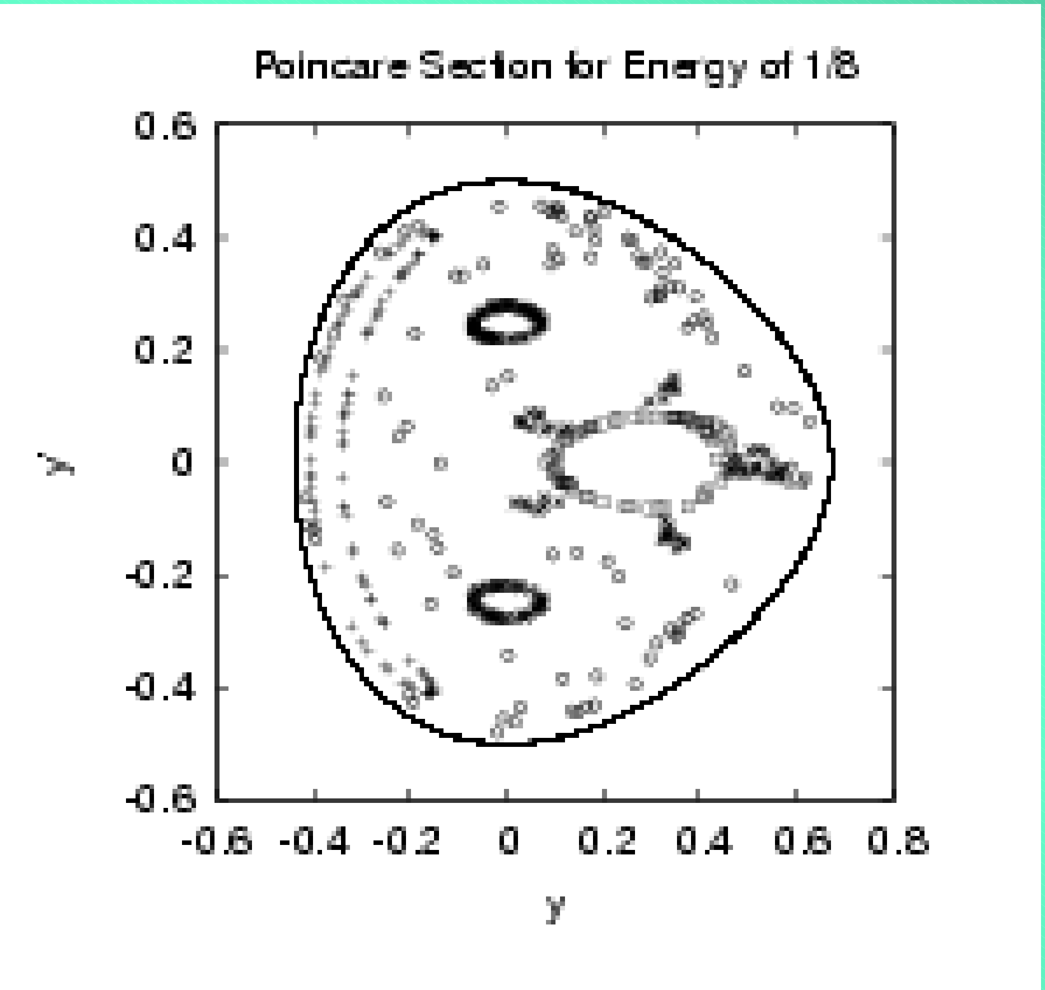
Poincare section for E=1/12

Four regions with elliptical orbits

At the center of each region is an elliptical fixed point

Where the boundary of each region meets is a hyperbolic point

If we increase the energy to E=1/8 we start to see chaotic behavior

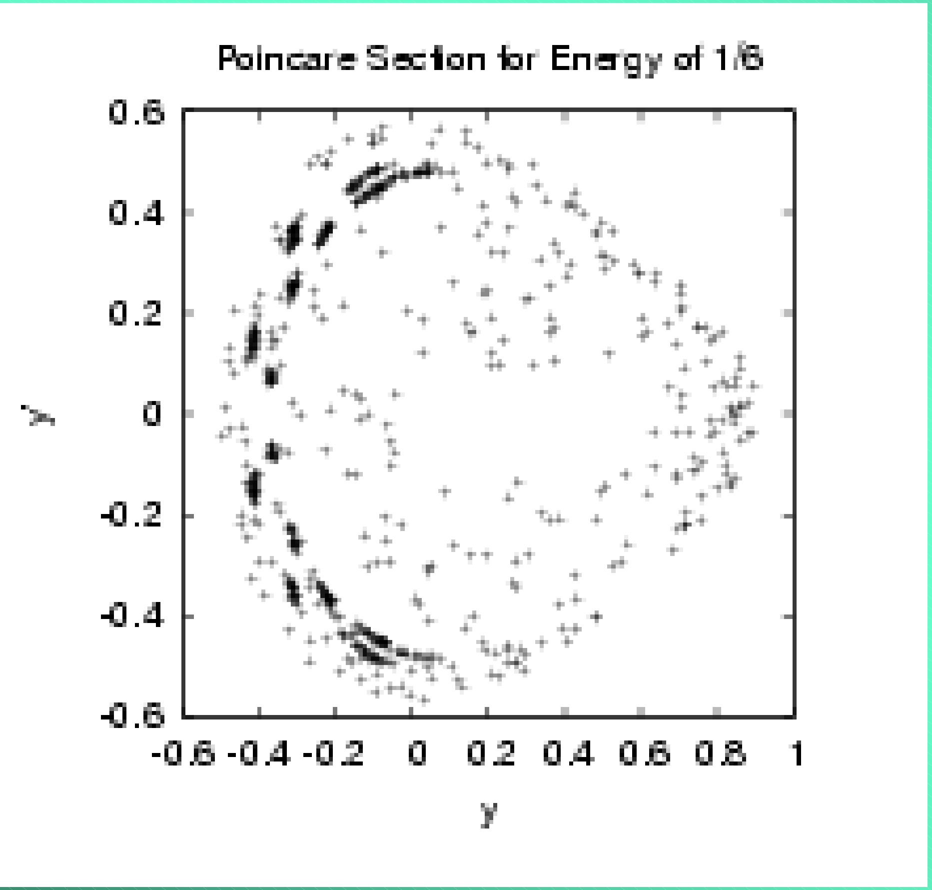


There are still “islands” of non-chaotic behavior with elliptic fixed points At their centers

However, the regions between these islands are now filled with a completely random set of points generated from a single set of initial conditions

These bounded areas of chaos are cross sections of a strange attractor

If the energy is increased to E=1/6 the strange attractor pretty much fills up all of the available phase space



This Poincare map was created with a single set of initial conditions