The Restricted Three-Body Problem : Earth,Jupitar,Sun

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1 Abstract

How Jupitar's gravitational pull affects the Earth's motion around the sun is known as the Three-Body Problem - a problem which can't be solved analyticaly. By numerically integrating the motion of the Earth under Jupitar's influence, this paper explores the chaotic motion of the Earth, and as a consequence, the solar system.

2 Introduction

The motion of the Earth under the influence of the sun is a fairly simple problem to solve. You just relate the gravitaional force to the centripical force and, ta-dah, you have an analytic solution for all time. But what happens if you add another celestial body, say Jupitar, to the problem. Now the problem is not so simple. Jupitar, while small in relation to the Sun, adds a tiny perturbation to the Earth's motion. In fact, in Pointcare's famous "discovery" of chaos, he proved that this Three-Body Problem cannot be solved analyticaly. But how chaotic is it? The Earth obviously continues to circle the sun in an elliptical fashion, and has done so for at least thousands of years. This paper demonstrates some of the aspects of chaos displayed by the Three-Body Problem: a circular Poincare map and divergent solutions, as-well-as explores how increasing Jupitar's mass affects the Earth's motion.

3 The Hamiltonian

The promblem contains two planetary bodies (Earth and Jupitar), and a central sun. Given that the sun's mass is much greater than that of Jupitar or the Earth, we assume that the center of the sun is the center of mass for the system and thus the sun doesn't move. We also assume that the Earths gravitaional pull on Jupitar is negligable and does not effect the circular motion of the larger planet. For simplicity's sake, Earth and Jupitar can be considered to circle the sun on the same plane of motion. The problem, then, reduces to a system with only two degrees of freedom: the x and y coordinates of the Earth.

Jupitar is considered to follow a circular orbit around the sun with a radius,r_j, of 5.203 AU form the sun and period 11.86 Earth years. Its coordinates, x_j and y_j, are given in terms of r_j and $w_j = \frac{2\pi}{11.86}$, its angular frequency around the sun.

$$x_j = r_j \cos(w_j t) \tag{1}$$

$$y_j = r_j \sin(w_j t) \tag{2}$$

Where t is given in Earth years. The Hamiltonian of the earth becomes:

$$H = \frac{p_x^2}{2m} + \frac{p_y^2}{2m} - \frac{GM_S m}{\sqrt{x^2 + y^2}} - \frac{GM_J m}{\sqrt{(x - x_j)^2 + (y - y_j)^2}} \tag{3}$$

Where $\sqrt{x^2+y^2}$ is the distance from the Earth to the Sun, and $\sqrt{(x-x_j)^2+(y-y_j)^2}$ is the distance from the Earth to Jupitar. Hamilton's equations give:

$$\dot{x} = \frac{\partial H}{\partial p_x} = \frac{p_x}{m} \tag{4}$$

$$\dot{y} = \frac{\partial H}{\partial p_y} = \frac{p_y}{m} \tag{5}$$

$$\dot{p_x} = -\frac{\partial H}{\partial x} = -Gm(\frac{xM_S}{(x^2+y^2)^{3/2}} + \frac{M_J}{((x-x_j)^2+(y-y_j)^2)^{3/2}}(x-x_j)) \tag{6}$$

$$\dot{p_y} = -\frac{\partial H}{\partial y} = -Gm(\frac{yM_S}{(x^2+y^2)^{3/2}} + \frac{M_J}{((x-x_j)^2+(y-y_j)^2)^{3/2}}(y-y_j)) \tag{7}$$

Since the Earth circles the sun in one Earth year with a mean radius,r_{mean}, of 1AU, w_E is 2π. Relating the centripital and gravitational forces acting on the Earth(ignoring Jupitar):

$$F = \frac{GM_S m}{r_{mean}^2} = m w_E^2 r_{mean} \tag{8}$$

$$GM_S = w_E^2 r_{mean}^3 \tag{9}$$

$$GM_S = 4\pi^2 \tag{10}$$

Comparing Equations 4 & 6 and 5 & 7 with equation 10, we find the equations of motion of the Earth.

$$\ddot{x} = 4\pi^2(\frac{x}{(x^2+y^2)^{3/2}} + \xi\frac{x-x_j}{((x-x_j)^2+(y-y_j)^2)^{3/2}}) \tag{11}$$

$$\ddot{y} = 4\pi^2(\frac{y}{(x^2+y^2)^{3/2}} + \xi\frac{y-y_j}{((x-x_j)^2+(y-y_j)^2)^{3/2}}) \tag{12}$$

Or

$$\ddot{\vec{r}} = 4\pi^2(\frac{\vec{r}}{|\vec{r}|^3} + \xi\frac{\vec{r}-\vec{r_j}}{|\vec{r}-\vec{r_j}|^3}) \tag{13}$$

Where ξ is the ratio of Jupitars mass to the Sun's mass. As proved by Poincare in the 1830's, the above equations of motion cannot be solved analytically. In fact, as the next section will show, the 3-Body problem is actually weakly chaotic.

4 Integrating the Equations of Motion

The equations of motion given by Equation 13 can be written as a main term plus a small perturbation due to the presence of Jupitar.

$$\ddot{\vec{r}} = Z_0 + \xi Z' \tag{14}$$

What is the effect of this perturbation? To find out, the equations of motion were numerically evaluated using the dverk dsolve method available on maple.[1] Initial conditions were chosen so that Earth and Jupitar both started in the same line from the sun (ie along the x axis). Figure 1 shows the position of the Earth over a period of one Jupitarian year. We see that the Earth's motion is not much perturbed. We cannot distinguish the 12 different trajectories the Earth took around the sun. It still seems to still follow a circular path about the sun. If the system is chaotic, then, it most be weakly chaotic.

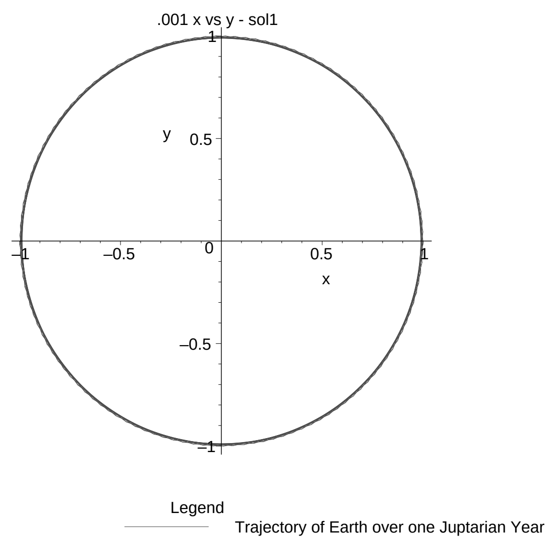
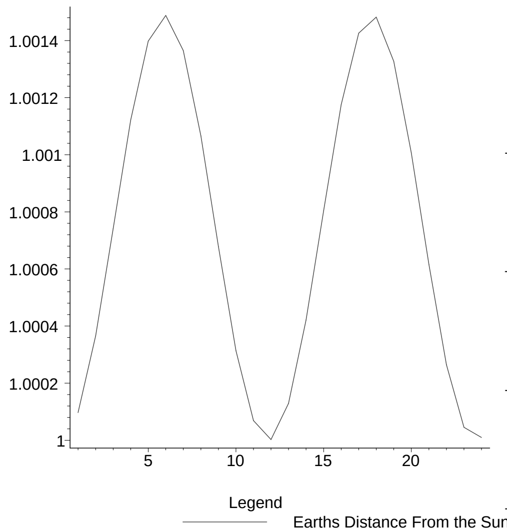


Figure 1: Motion of Earth around Sun with Jupitarian Perturbation

In figure 2 the distance of the Earth from the Sun is plotted as a function of time. We can see a periodic behaviour of approximately 12 Earth years.

[1]When ξ was set to zero, the method gave back the original conditions to within 6 decimal places after 600 earth years.



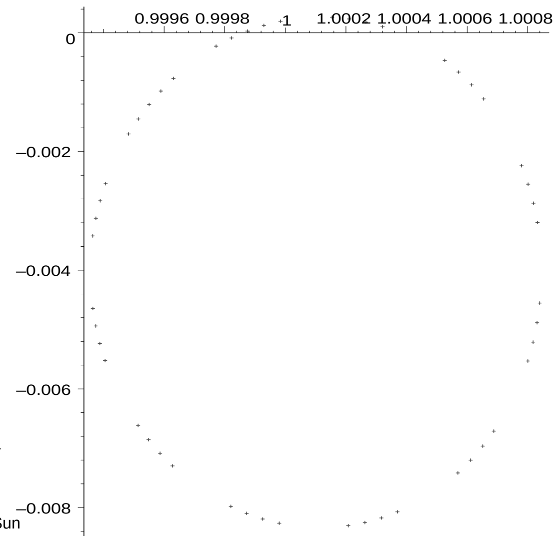


Figure 2: Distance of the Earth from the sun with actual Jupitarian Perturbation- Here we can see a period of slightly less than 12 Earth years

Figure 3: Beginning of a Pointcare map of Earths motion around the sun under the influence of actual Jupitarian Perturbation. We can clearly see the winding number of slightly less than 12

This corresponds with the Jupitarian period of almost 12 Earth years. As expected, the system almost returns to its initail conditions after 12 years. This 12 year periodic motion is confirmed be the first steps of a Pointcare map as shown in Figure 3. Here we can see 12 distinct steps in the position/momentum map.

The Poincare map points out another important fact: the system does not return to its exact initial conditions. After each iteration, the next Pointcare point is slightly displaced from the one 12 years earlier. As we follow the motion of the Earth over longer time periods, the points in figure 3 join to form a sold line (see Figure 6 in the next section).

Examining the system in a different way, the initial conditions were changed slightly by adding a phase of 1° to Jupitar's motion. We can then compare the two solutions (one with no phase change; one with a slight phase change). Figure 4 shows a plot of the difference in distances from the sun of the two solutions. We see that the distances between the two solutions diverge..and then get closer together, in a somewhat periodic fashion. To understand this, imagine marking two strands sitting next to each other on a ball of twine. When the twine is wound up the distance between the marks is small. But, if you unwind the twine, the distances between the two marks is much bigger. Similarly, for the distances between the two solutions in our problem to decrease, one solution most be avancing at a faster rate than the other. Thus, the solutions are actually diverging! The system is weakly chaotic!

5 Increasing Jupitars Mass

What happens when the perturbation factor in equation 14 is increased? One would expect Jupitar's mass to have more of an effect upon the Earth, thus increasing its chaotic behaviour. Figure 5 shows the trajectory of the Earth over 12 years (one Jupitarian year), for various values of ξ. As Jupitars mass increases, the radius of the Earth from the sun takes on more and more possible values. It gets less and less likely that the Earth will ever return to its initial conditions. The Poincare maps of figures 6 and 7 show how the Earth's motion diverges from a fixed point as ξ increases. By the time Jupitar's mass gets to be able half of that of the Sun's ($\xi = .4$), and Jupitar's potential accounts for about 10 percent of the total Potential acting on the Earth, the Poincare map begins to break up into sections-Extreme Chaos.

Figure 8 shows the log of the difference in two initailly close solutions (see last section), for increasing ξ values. We can see that the higher the mass the father the two solutions are from each other and the shorter (and more random), the periodic motion is - a sign of rapidly diverging solutions.

6 Conclusion

In his famous paper, Poincare used the Three-Body Problem to show that the equations of motion of the solar system could not be solved analyitically, and that the system was chaotic. This paper has explored the degree of chaotic motion that the Earth-Jupitar-Sun system displays. The small perturbation Jupitar adds to the gravitational force acting on Earth is tiny - much less than .1 percent of the Sun's. But, it is enough to take the stable motion of the Earth into a weakly chaotic orbit. As We increase this perturbation, the chaotic motion is seen more readily: the Pointcare maps diverge from a fixed point; and the orbits over just one Jupitarian year cover large ranges of radii from the sun. The Three-Body Problem definately displays chaotic motion.

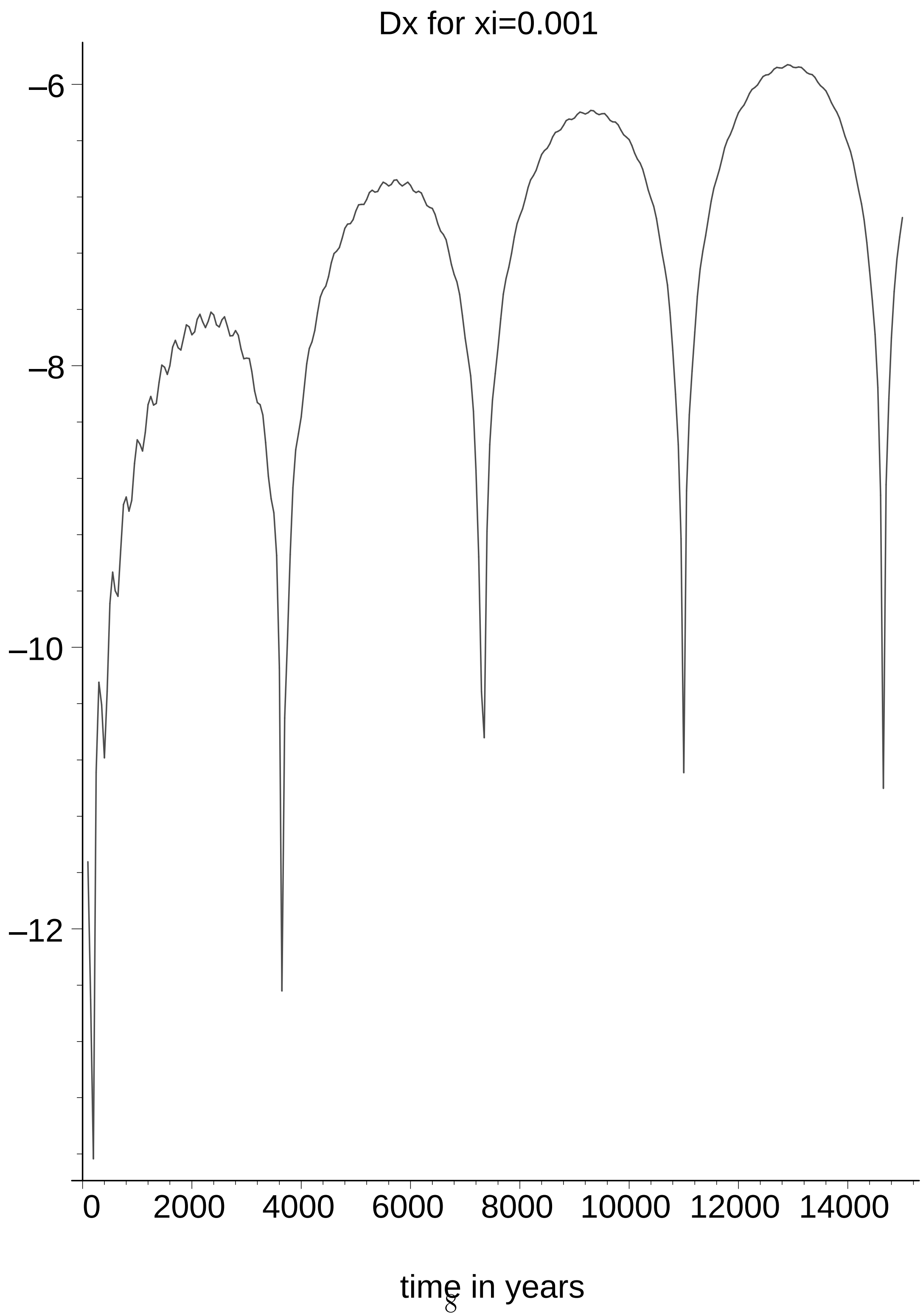


Figure 4: Log of difference in X-positions From the Sun of two slightly different initialconditions as a function of time under actual Jupitarian influence

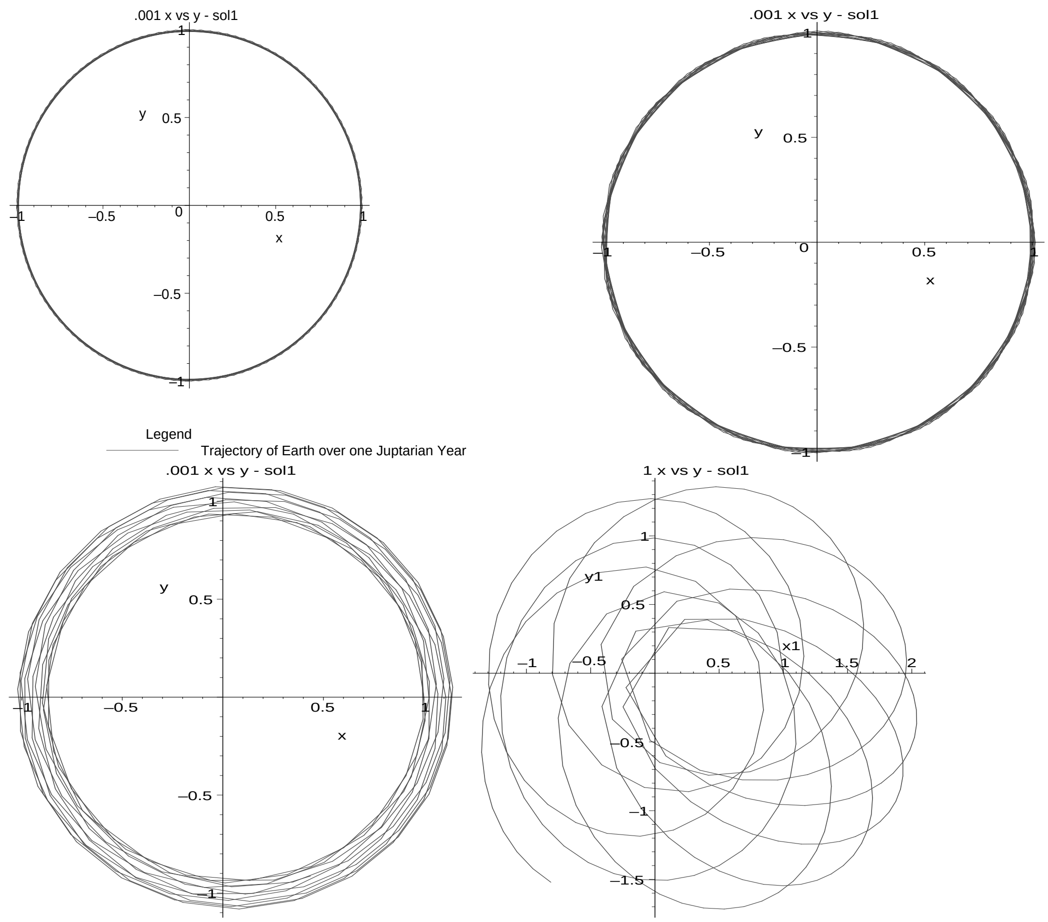


Figure 5: Motion of Earth over 12 years. a) $\xi = .001$ b)$\xi = 0.01$ c)$\xi = 0.1$, d)$\xi = 1$

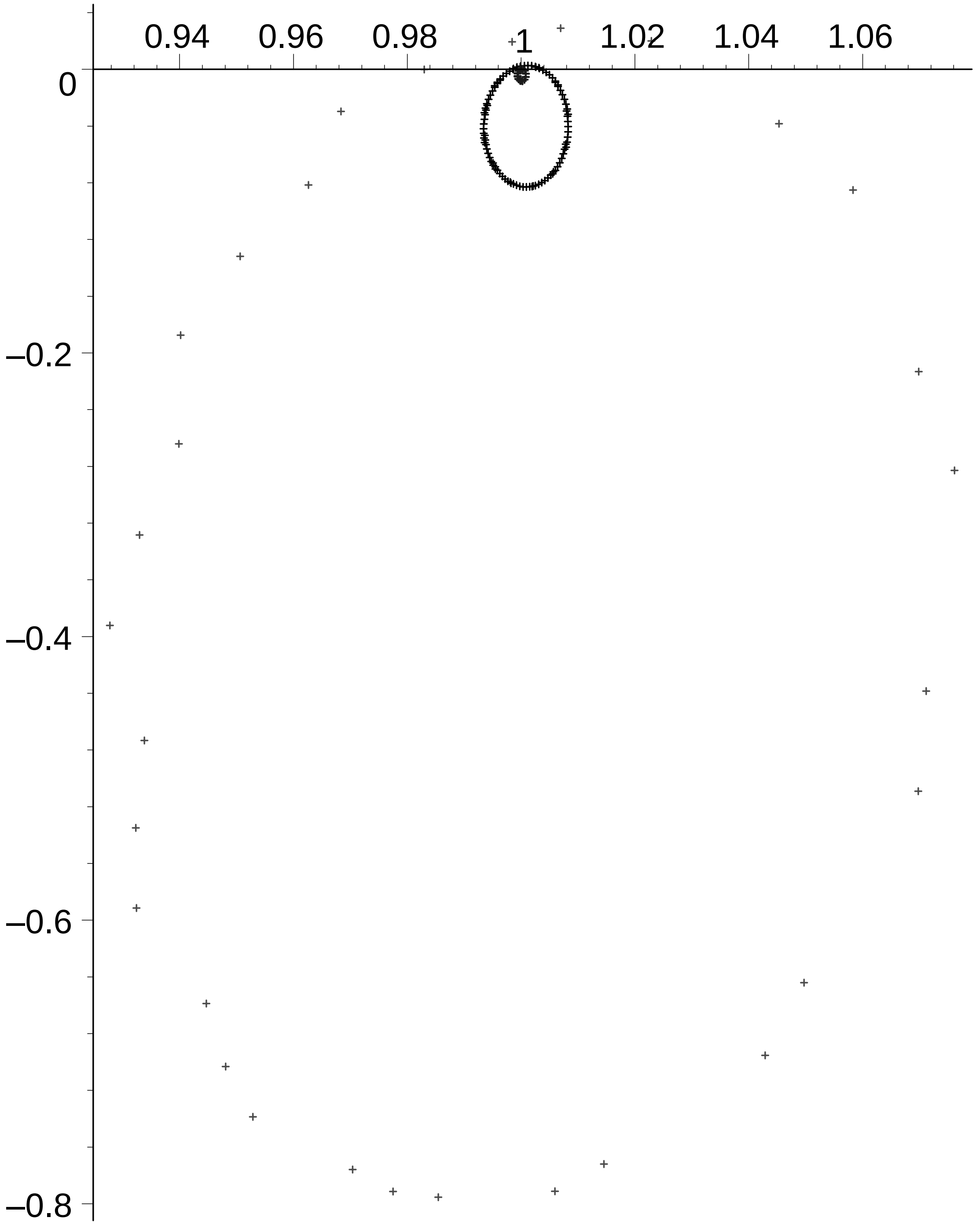


Figure 6: Poincare maps for (from the center outwards), $\xi = .001$, $\xi = .01$, and $\xi = .1$

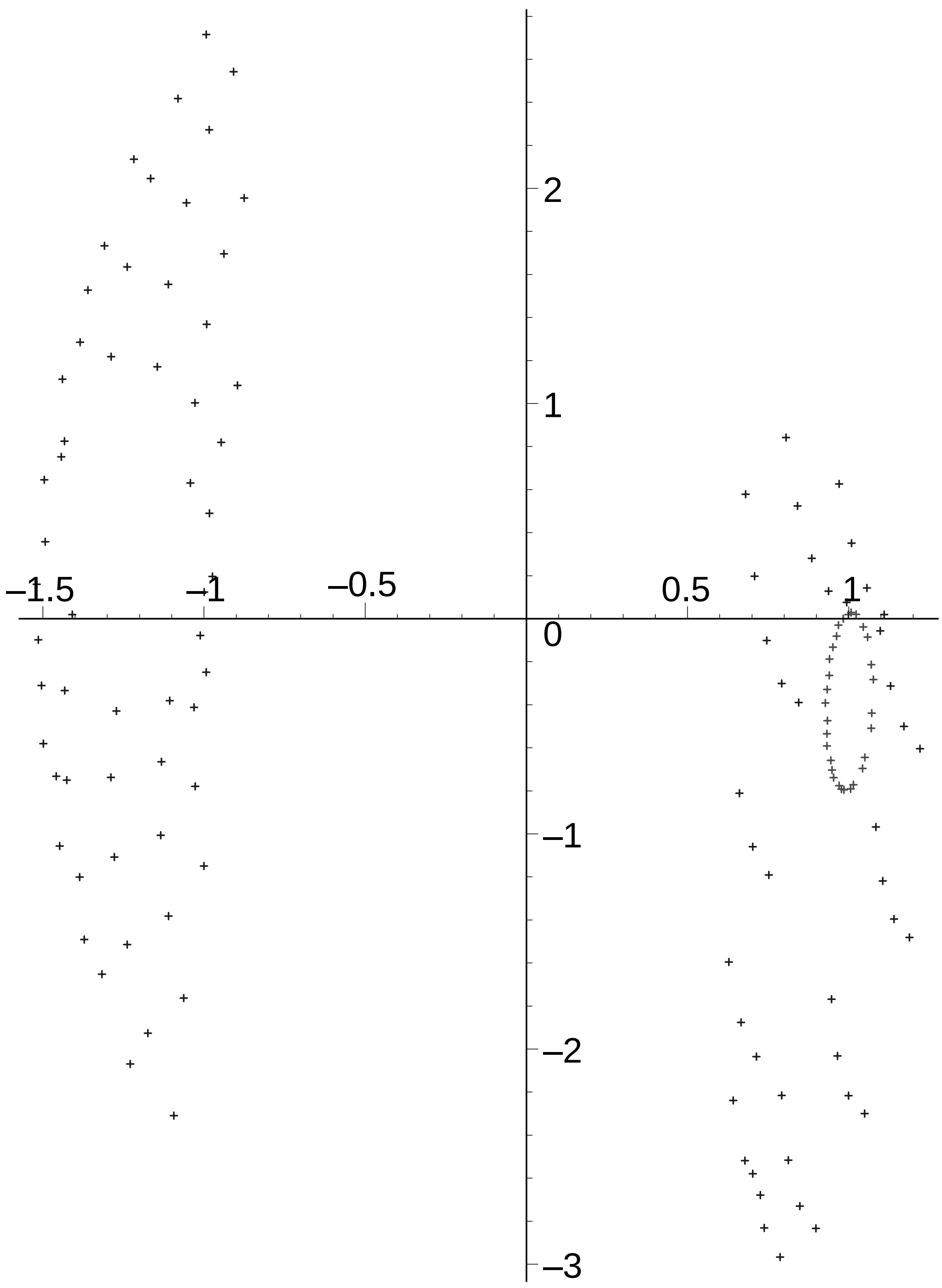


Figure 7: Poincare maps for $\xi = .1$ and $\xi = .4$

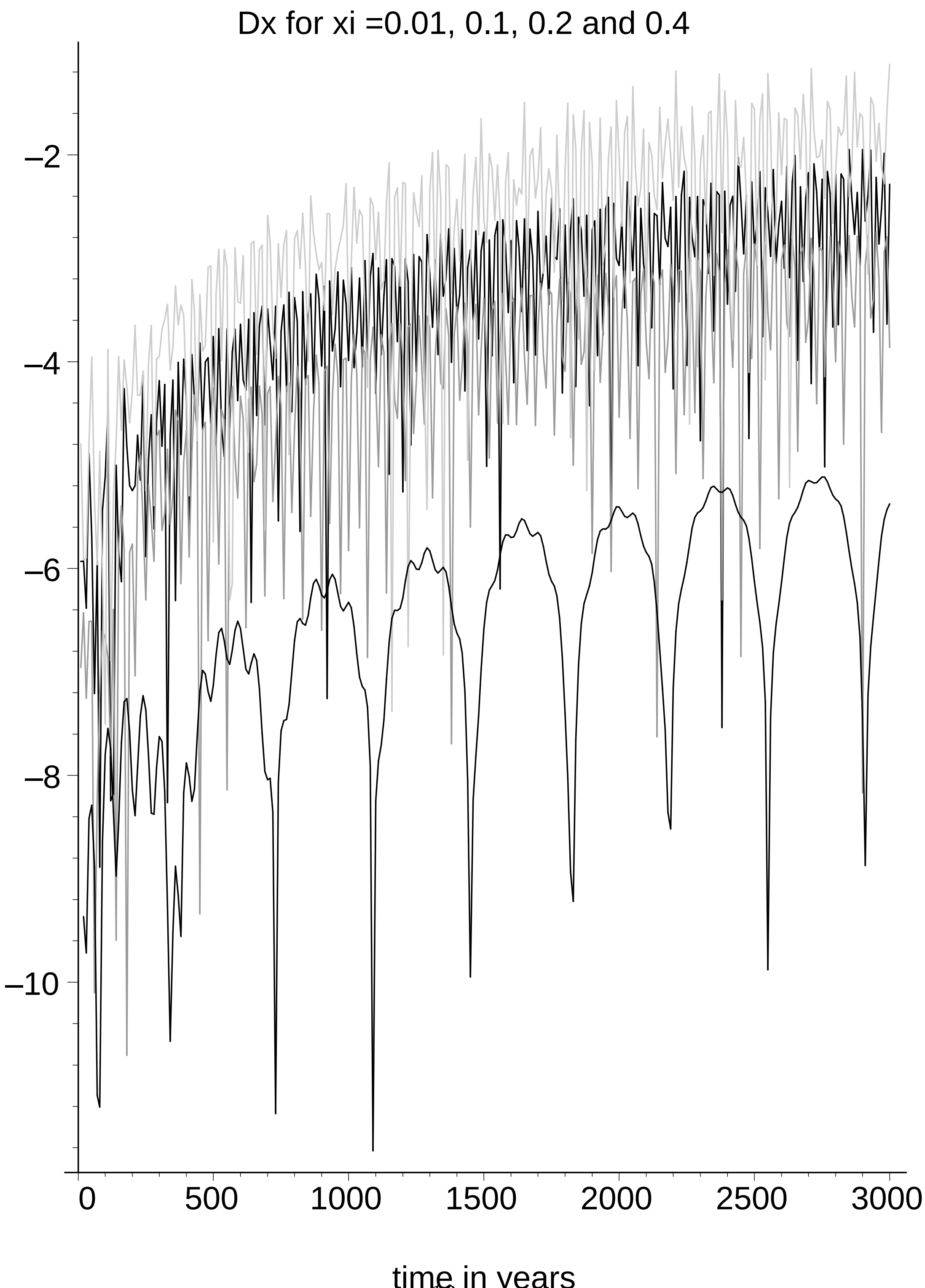


Figure 8: Log of difference in X-positions From the Sun of two slightly different initialconditions as a function of time under various ξ values. From the bottom up, $\xi = .01$, $\xi = .1$, $\xi = .2$, $\xi = .4$