Introduction

There exist many physical systems that are a combination of two separate states which interact with each other, thus modifying one state in a certain way will effect the other state. To clearly demonstrate the idea of coupled systems, it is easy to turn to coupled oscillators, since they undergo extremely visible changes in their oscillation patterns. One such coupled oscillator is the Wilberforce pendulum, which couples its longitudinal oscillation with its angular oscillation. The Wilberforce pendulum was named after its inventor Lionel Robert Wilberforce. Its basic construction is a long soft coiled spring (meaning it's longitudinal and angular spring coefficients are relatively small) with a mass hanging at the bottom that has a certain moment of inertia.

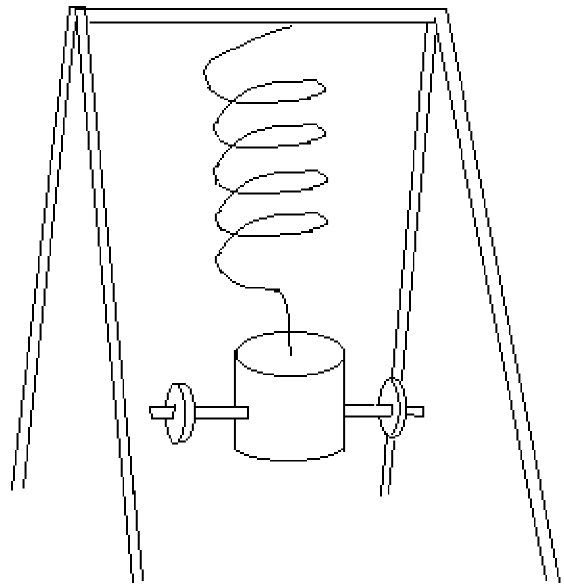

Fig.1 Diagram of a typical Wilberforce pendulum

Usually these masses have adjustable moments of inertia so as to experimentally verify the theory behind the pendulum. The coupling exists on the pendulum due to the torsional strain during the longitudinal compression and decompression of the spring and the axial strain during the twisting.

Theory

The kinetic energy of the pendulum is a combination of the longitudinal and angular kinetic energy. If k/2 is the longitudinal spring constant, δ/2 is the torsional spring constant, I is the moment of inertia, and m is the mass, the kinetic energy is therefore

$$T = 1/2mz'^2 + 1/2I\theta'^2 \ (1)$$

and the potential energy is

$$V = kz^2/2 + \delta\theta^2/2 + c\theta z \ (2)$$

where c is the coupling constant. The Lagrangian is

$$L = 1/2mz'^2 + 1/2I\theta'^2 - kz^2/2 - \delta\theta^2/2 - c\theta z \ (3)$$

The Lagrangian equations of motion become

$$mz'' + kz + c\theta = 0 \ (4)$$

$$I\theta'' + \delta\theta + cz = 0 \ (5)$$

At this point we assume oscillatory motion in the system. This allows us to state

$$z(t) = Ae^{i\omega t} \ (6)$$

is a solution to equation (4). It's a safe assumption to state this since the Wilberforce pendulum does indeed have oscillatory motion, and eventually we shall see that ω is an average of the angular frequencies of the eigenstates. Substituting equation (6) into equation (4) we find a solution for θ(t).

$$\theta(t) = (m\omega^2/c - k/c)Ae^{i\omega t} \ (7)$$

This equation makes the assumption that ω is approximately equal for both z and θ. While this may seem like a bad assumption to make, experimentally it holds within reason for this pendulum, and the calculations become extremely unlikely outside of numerical analysis if it is not made. Later in the paper the true meaning of ω will become apparent. Substituting (7) back in to (5), we arrive at an equation for ω:

$$-m\omega^4 Ae^{i\omega t}/c + k\omega^2 Ae^{i\omega t}/c + \delta m\omega^2 Ae^{i\omega t}/(lc) + cAe^{i\omega t}/l - \delta kAe^{i\omega t}/(lc) = 0 \quad (8)$$

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Cleaning this equation up and substituting in $k/m = \omega_z^2$, $\delta/l = \omega_\theta^2$, we have

$$\omega^4 - (\omega_z^2 + \omega_\theta^2)\omega^2 + \omega_z^2\,\omega_\theta^2 - c^2/(ml) = 0 \quad (9)$$

Note that the above substitutions are the relations that give the longitudinal and angular frequency for the given spring constants. At this point I would like to demonstrate the existence of normal modes in the pendulum. Normal modes exist as eigenstates to the actual system, and all transitional states are combinations of these modes. If the pendulum starts oscillating along a normal mode, the frequencies ω_z^2 and ω_θ^2 are equal, and the pendulum oscillates in simple harmonic motion. The calculations therefore become much simpler. Using the quadratic formula on equation (9) to discover a relation for ω^2 we discover:

$$\omega^2 = \omega_z^2 + \omega_\theta^2 \pm\sqrt{((\omega_z + \omega_\theta)(\omega_z - \omega_\theta) + 4c^2/(ml))} \quad (10)$$

However, as stated before, $\omega_z^2 = \omega_\theta^2$. Equation (10) therefore simplifies to

$$\omega^2 = \omega_z^2 + \omega_\theta^2 \pm\sqrt{(4c^2/(ml))} \quad (11)$$

To find the normal modes, we can make the substitution of $\omega_z^2 + \omega_\theta^2 = \omega_1^2 + \omega_2^2$, where ω_1^2 and ω_2^2 are the angular frequencies of the normal modes. and we can therefore state the normal modes as eigenfunctions to the original system:

$$\omega_1^2 = \omega^2 - \sqrt{(4c^2/(ml))} \quad (12)$$

$$\omega_2^2 = \omega^2 + \sqrt{(4c^2/(ml))} \quad (13)$$

We know that there can be only two normal modes, hence two frequencies to these modes, since there are only two degrees of freedom for the Wilberforce pendulum; however, there are an infinite amount of initial conditions that arrive to these two modes. Subtracting equation (12) from (13), we come across an interesting result:

$$\omega_1^2 - \omega_2^2 = -2\sqrt{(4c^2/(ml))} \quad (14)$$

Noting that $\omega_1^2 - \omega_2^2$ factors to $(\omega_1 - \omega_2)(\omega_1 + \omega_2)$, where $(\omega_1 - \omega_2) = \omega_b$, the beat frequency of the transfer of energy between the two modes, and making the approximation $\omega = (\omega_1 + \omega_2)$ we can finally solve for the coupling constant.

$$c = 2\omega_b\omega\sqrt{(ml)}/4 \quad (15)$$

Now that we have the coupling constant, we can solve the differential equations (4) and (5) listed above. After performing all the necessary substitutions, the end result for θ and z is

$$\Theta(t) = \frac{(2\omega(\omega_1 - \omega_2))\sqrt{(ml\, z_o(\cos(\omega_1 t)) - \cos(\omega_2 t))}}{4l(\omega_1^2 - \omega_2^2)} \quad (16)$$

$$+ \frac{\theta_0((\omega_1^2 - \omega^2)\cos(\omega_2 t) - (\omega_2^2 - \omega^2)\cos(\omega_1 t)}{\omega_1^2 - \omega_2^2}$$

$$z(t) = \frac{z_o((\omega_1^2 - \omega^2)\cos(\omega_1 t) - (\omega_2^2 - \omega^2)\cos(\omega_2 t))}{\omega_1^2 - \omega_2^2} - \frac{8I\theta_0 m(\omega_2^2 - \omega^2)\,((\omega_1^2 - \omega^2)\,(\cos(\omega_1 t) - \cos(\omega_2 t))}{\sqrt{Im(\omega(\omega_1 - \omega_2))(\,\omega_1^2 - \omega_2^2)}} \qquad (17)$$

Plotting z versus time gives us the motion below

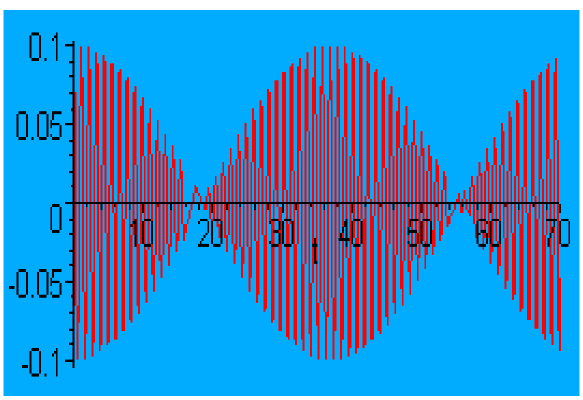

Fig. 2: z(t) vs. t

And plotting θ versus time gives us the plot in fig. 3. The values for the constants used in these plots are as follows: ω_1 = 5.80, ω_2 = 5.63, ω= 5.71, m = 0.266, I = 5.05 x 10^{-5}, θ_o = 0, z_o = 0.1, which are approximate experimental values for these constants.

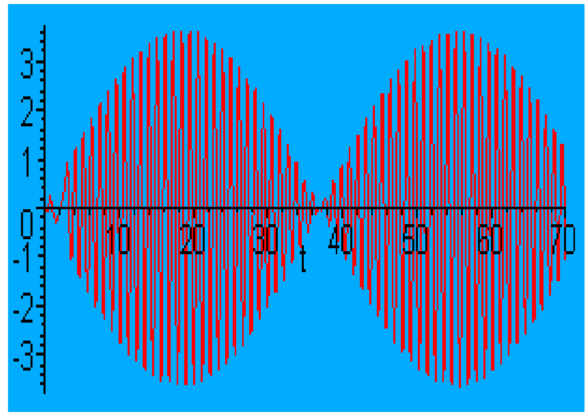

Fig 3: θ(t) vs. t

These plots display the coupled oscillation, where one can see the points of complete energy transfer from one type of oscillation to another. These points are approximately 18, 37 and 55 just by glancing at the plots. Finally, to find the normal modes, we must find where z(t) and θ(t) are zero. Substituting in we find that the normal modes are

$$z_o = 4\sqrt{(I/m)}\theta_o$$

$$z_o = -4\sqrt{(I/m)}\theta_o$$

A plot of one of the normal modes shows the harmonic oscillation:

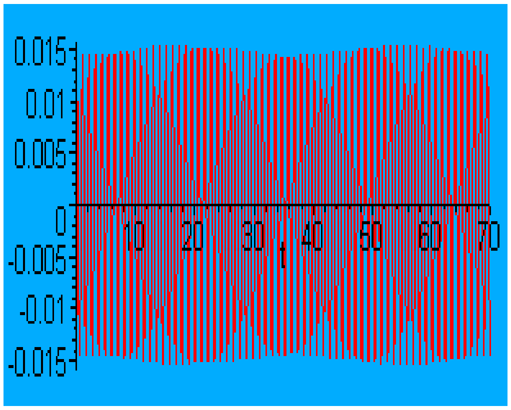

fig 4: Normal mode plot

Thus by using Langrange's equations of motion the Wilberforce pendulum's coupled oscillation is described, and it's eigenstates are found.

References:

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Quantitative Analysis of the Wilberforce Pendulum

Through Lagrangian Mechanics

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Phys349

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