Brief review of some necessary math

1 Vectors

Scalars: mathematical objects defined by a magnitude (one number). Examples are volume, mass, charge etc. Usual notation are letters: V, m, Q etc. Mathematical operations involving scalars are addition, subtraction, multiplication and division by another scalar. It is advisable to not divide by 0.

Vectors: mathematical objects defined by a magnitude and a direction. Examples are position $\vec{r}$, speed $\vec{v}$, momentum $\vec{p}$, force $\vec{F}$ etc. The little arrow is very important, since it distinguishes a vector from a scalar.

A vector can be expressed in terms of its components in a coordinate system. We use a right-handed, three-dimensional Cartesian system, and write (see Fig. 1):

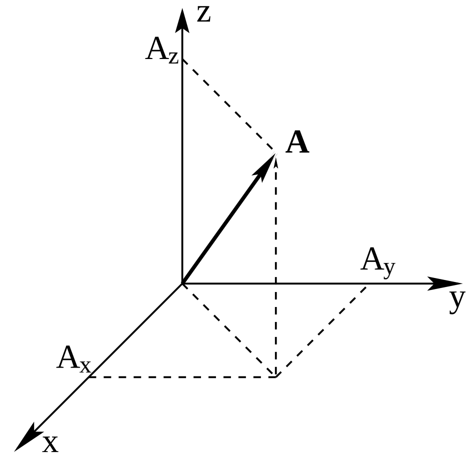

Fig 1. Right-handed Cartesian coordinate system.

$$\vec{A} = (A_x, A_y, A_z)$$

If all vectors are decomposed with respect to the same coordinate system, we can define:
- addition of vectors: $\vec{A} + \vec{B} = (A_x + B_x, A_y + B_y, A_z + B_z) = \vec{B} + \vec{A}$;
- multiplication by a constant c: $c\vec{A} = c(A_x, A_y, A_z) = (cA_x, cA_y, cA_z)$;
- dot product $\vec{A} \cdot \vec{B} = A_xB_x + A_yB_y + A_zB_z = \vec{B} \cdot \vec{A}$;
- cross product $\vec{A} \times \vec{B} = (A_yB_z - B_yA_z, A_zB_x - B_zA_x, A_xB_y - B_xA_y) = -\vec{B} \times \vec{A}$;

We introduce unit vectors (versors) for the three axis:

$$\vec{e}_x = (1, 0, 0); \qquad \vec{e}_y = (0, 1, 0); \qquad \vec{e}_z = (0, 0, 1);$$

Then, we can also write

$$\vec{A} = (A_x, A_y, A_z) = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z$$

The magnitude of a vector $\vec{A}$ is denoted by $|\vec{A}|$ or simply $A = \sqrt{A_x^2 + A_y^2 + A_z^2}$. Clearly, $|\vec{A}|^2 = \vec{A} \cdot \vec{A}$.

The angle θ between two vectors $\vec{A}$ and $\vec{B}$ is given by

$$\cos\theta = \frac{\vec{A} \cdot \vec{B}}{|\vec{A}||\vec{B}|}; \qquad \sin\theta = \frac{|\vec{A} \times \vec{B}|}{|\vec{A}||\vec{B}|}$$

The cross-product can be rewritten in the equivalent form

$$\vec{A} \times \vec{B} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ A_x & A_y & A_z \\ B_x & B_y & B_z \end{vmatrix} = (A_yB_z - B_yA_z)\vec{e}_x + (A_zB_x - B_zA_x)\vec{e}_y + (A_xB_y - B_xA_y)\vec{e}_z$$

Identities we will use later on:

$$\vec{A} \cdot (\vec{B} \times \vec{C}) = \begin{vmatrix} A_x & A_y & A_z \\ B_x & B_y & B_z \\ C_x & C_y & C_z \end{vmatrix} = \vec{B} \cdot (\vec{C} \times \vec{A}) = \vec{C} \cdot (\vec{A} \times \vec{B})$$

$$\vec{A} \times \left(\vec{B} \times \vec{C}\right) = \vec{B}(\vec{A} \cdot \vec{C}) - \vec{C}(\vec{A} \cdot \vec{B})$$

2 Cartesian, Cylindrical and Spherical Coordinates

Let $\vec{A} = A_x\vec{e}_x + A_y\vec{e}_y + A_z\vec{e}_z$ be the decomposition of a vector in Cartesian coordinates (Fig. 1). There are two other coordinate systems that may be more convenient, depending on the symmetry of the problem considered:

a) cylindrical coordinates (Fig. 2). We keep the z-axis projection, but in-plane we choose two new unit vectors: $\vec{e}_\rho$ – which is pointing in the radial direction (the direction in which ρ alone increases, while ϕ, z are unchanged) and $\vec{e}_\rho$ – which is pointing in the tangential direction (the direction to move to increase ϕ, while keeping ρ, z unchanged). Therefore:

$$\vec{A} = A_\rho\vec{e}_\rho + A_z\vec{e}_z$$

where

$$\begin{cases} \vec{e}_\rho = \cos\phi\vec{e}_x + \sin\phi\vec{e}_y \\ \vec{e}_\phi = -\sin\phi\vec{e}_x + \cos\phi\vec{e}_y \\ \vec{e}_z = \vec{e}_z \end{cases}$$

and

$$|\vec{A}|^2 = A_\rho^2 + A_z^2$$

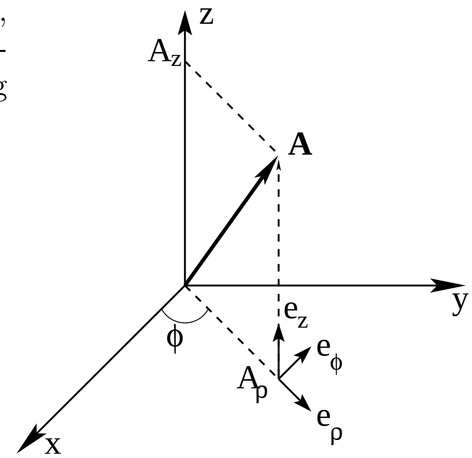

Fig 2. Cylindrical coordinate system.

b) spherical coordinates (Fig. 3). Here we have 3 new unit vectors: $\vec{e}_r$ – points in the direction in which r, the distance from origin, increases while keeping ϕ, θ are unchanged; and similarly, $\vec{e}_\phi, \vec{e}_\theta$ which point in the direction which would increase only ϕ, respectively only θ, while keeping the other coordinates fixed. Then:

$$\vec{A} = A_r\vec{e}_r$$

where

$$\begin{cases} \vec{e}_r = \sin\theta\cos\phi\vec{e}_x + \sin\theta\sin\phi\vec{e}_y + \cos\theta\vec{e}_z \\ \vec{e}_\theta = \cos\theta\cos\phi\vec{e}_x + \cos\theta\sin\phi\vec{e}_y - \sin\theta\vec{e}_z \\ \vec{e}_\phi = -\sin\phi\vec{e}_x + \cos\phi\vec{e}_y \end{cases}$$

and

$$|\vec{A}| = A_r$$

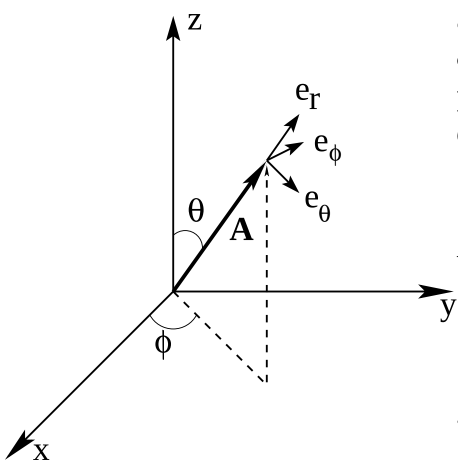

Fig 3. Spherical coordinate system.

3 Speed and acceleration vectors in various coordinate systems

Notation:

$$\frac{df(t)}{dt} = \dot{f}(t) \qquad \frac{d^2f(t)}{dt^2} = \ddot{f}(t)$$

(i) Cartesian coordinates: the triad $(\vec{e}_x, \vec{e}_y, \vec{e}_z)$ is fixed, therefore:

$$\vec{r}(t) = x(t)\vec{e}_x + y(t)\vec{e}_y + z(t)\vec{e}_z \tag{1}$$

$$\vec{v}(t) = \frac{d\vec{r}(t)}{dt} = \dot{x}(t)\vec{e}_x + \dot{y}(t)\vec{e}_y + \dot{z}(t)\vec{e}_z; \qquad |\vec{v}|^2 = (\dot{x})^2 + (\dot{y})^2 + (\dot{z})^2 \tag{2}$$

$$\vec{a}(t) = \frac{d\vec{v}(t)}{dt} = \frac{d^2\vec{r}(t)}{dt^2} = \ddot{x}(t)\vec{e}_x + \ddot{y}(t)\vec{e}_y + \ddot{z}(t)\vec{e}_z \tag{3}$$

(ii) Cylindrical coordinates: $\vec{e}_\rho$ and $\vec{e}_\phi$ change in time, because the angle $\phi(t)$ changes as the vector $\vec{r}(t)$ varies. We have:

$$\begin{cases} \dot{\vec{e}}_\rho = -\sin\phi\dot{\phi}\vec{e}_x + \cos\phi\dot{\phi}\vec{e}_y = \dot{\phi}\vec{e}_\phi \\ \dot{\vec{e}}_\phi = -\cos\phi\dot{\phi}\vec{e}_x - \sin\phi\dot{\phi}\vec{e}_y = -\dot{\phi}\vec{e}_\rho \end{cases}$$

Then

$$\vec{r}(t) = \rho(t)\vec{e}_\rho + z(t)\vec{e}_z \tag{4}$$

$$\vec{v}(t) = \dot{\vec{r}}(t) = \dot{\rho}\vec{e}_\rho + \rho\dot{\phi}\vec{e}_\phi + \dot{z}\vec{e}_z; \qquad |\vec{v}|^2 = \dot{\rho}^2 + \rho^2\dot{\phi}^2 + \dot{z}^2 \tag{5}$$

$$\vec{a}(t) = \ddot{\vec{r}}(t) = \left(\ddot{\rho} - \rho\dot{\phi}^2\right)\vec{e}_\rho + \left(2\dot{\rho}\dot{\phi} + \rho\ddot{\phi}\right)\vec{e}_\phi + \ddot{z}(t)\vec{e}_z \tag{6}$$

(iii) Spherical coordinates: all three unit vectors change in time as $\vec{r}(t)x$ varies. We have:

$$\begin{cases} \dot{\vec{e}}_r = \dot{\theta}\vec{e}_\theta + \sin\theta\dot{\phi}\vec{e}_\phi \\ \dot{\vec{e}}_\theta = -\dot{\theta}\vec{e}_r + \cos\theta\dot{\phi}\vec{e}_\phi \\ \dot{\vec{e}}_\phi = -\dot{\phi}\left(\sin\theta\vec{e}_r + \cos\theta\vec{e}_\theta\right) \end{cases}$$

Then

$$\vec{r}(t) = r(t)\vec{e}_r \tag{7}$$

$$\vec{v}(t) = \dot{r}\vec{e}_r + r\dot{\theta}\vec{e}_\theta + r\sin\theta\dot{\phi}e_\phi; \qquad |\vec{v}|^2 = \dot{r}^2 + r^2\dot{\theta}^2 + r^2\sin^2\theta\dot{\phi}^2 \tag{8}$$

$$\vec{a}(t) = \left[\ddot{r} - r\dot{\theta}^2 - r\dot{\phi}^2\sin^2\theta\right]\vec{e}_r + \left[r\ddot{\theta} + 2\dot{r}\dot{\theta} - r\dot{\phi}^2\sin\theta\cos\theta\right]\vec{e}_\theta + \left[r\ddot{\phi}\sin\theta + 2\dot{r}\dot{\phi}\sin\theta + 2r\dot{\phi}\dot{\theta}\cos\theta\right]\vec{e}_\phi \tag{9}$$

Luckily, we will not need to use the accelerations in this course, only the speeds.

4 Derivatives

a) function of one variable: Let $f(x)$ be some function of x. Then, we define:

$$\frac{df(x)}{dx} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x) - f(x)}{\Delta x}$$

Some examples:

$$\frac{dx^n}{dx} = nx^{n-1}; \frac{d\cos(x)}{dx} = -\sin x; \frac{d\sin x}{dx} = \cos x$$

$$\text{chain rule: } \frac{df(g(x))}{dx} = \frac{df}{dg}\frac{dg}{dx}$$

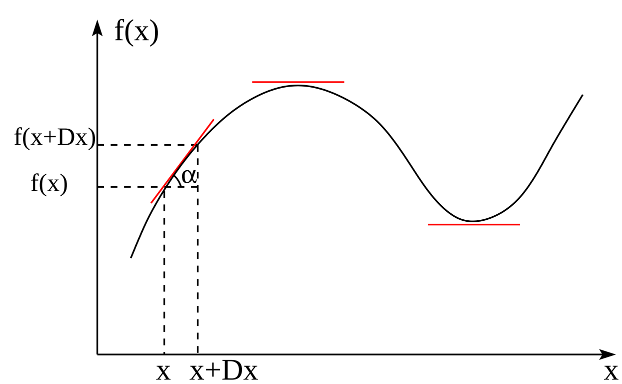

Fig 4. Derivative of a function, and extremum conditions.

Functions we encounter in physical problems are usually well-behaved (continuous etc), and derivatives are well defined. The geometric meaning of the derivative is that it equals the slope $\tan\alpha$ of the function, at the point of interest (see Fig. 4). It follows that a **local extremum** (maximum or minimum) condition is $\alpha = 0$, i.e.

$$\frac{df(x)}{dx} = 0$$

Whether this local extremum is a maximum or minimum depends on the sign of the second derivative: if $\frac{d^2f(x)}{dx^2} < 0 \rightarrow$ local maximum, if $\frac{d^2f(x)}{dx^2} > 0 \rightarrow$ local minimum.

b) function of several variables: assume a function of several variables $f(q_1, ..., q_N)$. We can define two types of derivatives, partial and total.

The **partial derivative** ∂ with respect to one variable q_i is given by the variation in the function when only that particular variable is allowed to change by a small amount:

$$\frac{\partial f}{\partial q_i} = \lim_{\Delta q_i \to 0} \frac{f(q_1, ..., q_i + \Delta q_i, ..., q_N) - f(q_1, ..., q_i, ..., q_N)}{\Delta q_i}$$

In other words, it is similar to taking a normal derivative when we assume that all other variables are simple constants.

Example: let

$$f(x, y, z, t) = 3tx^2 + 4xy + 5yz + 2(y^2 + z^2) + (x + y + 2z)\sin t$$

then

$$\frac{\partial f}{\partial x} = 6tx + 4y + \sin t; \quad \frac{\partial f}{\partial y} = 4x + 4y + \sin t; \quad \frac{\partial f}{\partial z} = 5y + 4z + 2\sin t; \quad \frac{\partial f}{\partial t} = 3x^2 + (x + y + 2z)\cos t$$

However, some of the arguments of a function may be themselves dependent on some other variables. For instance, let us consider a function $g(q_1, ...q_N, t)$ where $q_1, ...q_N$ are themselves functions of the variable t. Then, we can define the **total** derivative with respect to t as being given by the total variation of the function when we allow t to vary by a small amount:

$$\frac{dg}{dt} = \lim_{\Delta t \to 0} \frac{g(q_1(t + \Delta t), ..., q_N(t + \Delta t), t + \Delta t) - g(q_1(t), ..., q_N(t), t)}{\Delta t}$$

This is very different from the partial derivative with respect to t, when we keep $q_1, ..., q_N$ unchanged:

$$\frac{\partial g}{\partial t} = \lim_{\Delta t \to 0} \frac{g(q_1(t), ..., q_N(t), t + \Delta t) - g(q_1(t), ..., q_N(t), t)}{\Delta t}$$

From the chain rule, it follows that

$$\frac{dg}{dt} = \sum_{i=1}^{N} \frac{\partial g}{\partial q_i} \frac{dq_i}{dt} + \frac{\partial g}{\partial t}$$

Example: consider

$$g(x, y) = 3x + 2xy$$

Then

$$\frac{\partial g}{\partial x} = 3 + 2y; \qquad \frac{\partial g}{\partial y} = 2x; \qquad \frac{\partial g}{\partial t} = 0$$

and as a result:

$$\frac{dg}{dt} = (3 + 2y)\frac{dx}{dt} + 2x\frac{dy}{dt}$$

E.g., if $x(t) = 3t + 2$ and $y(t) = t^2$, it follows that

$$\frac{dg}{dt} = (3 + 2y)3 + 2x \cdot 2t = 9 + 8t + 18t^2 = \frac{d}{dt}\left(3[3t + 2] + 2[3t + 2]t^2\right)$$

If a function has no **explicit** dependence on a variable (i.e., no term in the function contains that variable), then the partial derivative of the function with respect to that variable is zero. For instance, in the last example $\partial g/\partial t = 0$. However, since both x and y depend on t, we say that the function g depends **implicitly** on t, and its total derivative with respect to t need not be zero.