Kepler's problem – gravitational attraction

1 Summary of formulas derived for two-body motion

Let the two masses be m_1 and m_2. The total mass is $M = m_1 + m_2$, the reduced mass is $\mu = m_1m_2/(m_1 + m_2)$.

The gravitational potential is

$$U(r) = -\frac{Gm_1m_2}{r} = -\frac{\alpha}{r}, \quad \text{where } \alpha = Gm_1m_2 > 0$$

The effective potential is then:

$$U_{eff}(r) = \frac{l^2}{2\mu r^2} + U(r) = \frac{l^2}{2\mu r^2} - \frac{\alpha}{r}$$

The conserved angular momentum $l = \mu r^2\dot{\phi}^2$ is equal to its initial value $l = [\vec{r}_0 \times \mu\vec{v}_0] \cdot \vec{e}_z$, where by definition $\vec{r}_0 = \vec{r}_1(0) - \vec{r}_2(0)$ and $\vec{v}_0 = \vec{v}_1(0) - \vec{v}_2(0)$. In general $l \neq 0$. (The case $l = 0$ is rather trivial: $l = 0 \to \dot{\phi} = 0$ i.e. $\phi = const$, and therefore the two bodies move either towards one another, or away from one another, along the line uniting them. This happens if $\vec{r}_0$ and $\vec{v}_0$ are parallel. I'll probably give this to you for homework, or we'll discuss it in the tutorial). For what follows, I assume $l \neq 0$.

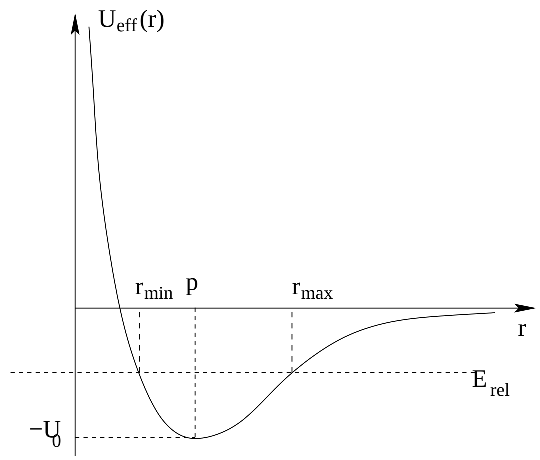

Fig 1. Effective potential for $l \neq 0$, and turning points for a value $E_{rel} < 0$.

The effective potential reaches its minimum value at $r = p$ given by

$$\frac{dU_{eff}}{dr}|_{r=p} = 0 \to p = \frac{l^2}{\mu\alpha} \tag{1}$$

where it takes the minimum value (see Fig. 1)

$$-U_0 = U_{eff}(p) \to U_0 = \frac{\mu\alpha^2}{2l^2} = \frac{\alpha}{2p} \tag{2}$$

To find out the allowed range for the relative distance r, we need to find the relative energy $E_{rel} = \mu\vec{v}_0^2/2 + U(r_0)$ from the initial conditions. We know that at any time

$$E_{rel} = \frac{\mu\dot{r}^2}{2} + U_{eff}(r) \geq U_{eff}(r) \tag{3}$$

since $\mu\dot{r}^2/2 \geq 0$. This means that motion is only possible if $E_{rel} > -U_0$, since that's the minimum value of U_{eff}.

It is very convenient to introduce the **eccentricity** e defined by

$$e = \sqrt{1 + \frac{E_{rel}}{U_0}} \rightarrow E_{rel} = (e^2 - 1)U_0 \tag{4}$$

It follows that if $0 \leq e < 1 \rightarrow -U_0 \leq E_{rel} < 0$; if $e = 1 \rightarrow E_{rel} = 0$; and if $e > 1 \rightarrow E_{rel} > 0$. We are now ready to find the trajectory of the relative coordinate.

2 The trajectory

The equation for the trajectory (obtained from Eq. 3 + the fact that $l = \mu r^2 \dot{\phi}$) is

$$\phi = \int \frac{l dr}{r^2 \sqrt{2\mu(E_{rel} - U_{eff}(r))}}$$

For simplicity, here I assume that the constant of integration is zero (that can always be arranged by orienting the system of coordinates properly).

We perform the integral in the following way: we introduce the new variable

$$u = \frac{p}{r} \rightarrow du = -\frac{pdr}{r^2} \rightarrow \frac{dr}{r^2} = -\frac{du}{p}$$

Also, following the definitions

$$E_{rel} - U_{eff}(r) = (e^2 - 1)U_0 - \frac{l^2}{2\mu r} + \frac{\alpha}{r}$$

We now replace $r = p/u = l^2/(\mu\alpha u)$ and $U_0 = \frac{\mu\alpha^2}{2l^2}$, and after some struggle we find that

$$E_{rel} - U_{eff}(r) = \frac{\mu\alpha^2}{2l^2}\left[e^2 - (u-1)^2\right]$$

We now substitute everything in the integral to find:

$$\phi = -l \int \frac{du}{p} \frac{1}{\sqrt{2\mu \frac{\mu\alpha^2}{2l^2}\left[e^2 - (u-1)^2\right]}} = -\int \frac{du}{\sqrt{e^2 - (u-1)^2}}$$

if one uses the fact that $p = l^2/(\mu\alpha)$. To do the last integral we request that $(u-1)^2 = e^2\cos^2 x$, such that the condition $0 \leq (u-1)^2 \leq e^2$ is always obeyed (this is necessary to insure that the quantity under square root is positive). Then,

$$u = 1 + e\cos x \rightarrow du = -e\sin x dx$$

and

$$\sqrt{e^2 - (u-1)^2} = e\sin x$$

In terms of x, the integral is trivial:

$$\phi = -\int \frac{-e\sin x dx}{e\sin x} = x$$

Since $u = p/r$ and $u = 1 + e\cos x$, the equation of the trajectory is:

$$\frac{p}{r} = 1 + e\cos\phi \tag{5}$$

Let us analyze this equation for various values of the eccentricity.

2.1 $e = 0$: the circle

If $e = 0$, i.e. $E_{rel} = -U_0$, the trajectory is $r = p$, i.e. a circle of radius p. This makes perfect sense if one looks at Fig. 1, and remembers that at all times we must have $E_{rel} \geq U_{eff}(r)$.

2.2 $0 < e < 1$: the ellipse

In this case, $-U_0 < E_{rel} < 0$, and we expect the trajectory to be finite (see Fig. 1). Let us go back to $x = r\cos\phi$ and $y = r\sin\phi$ (see Fig. 2) to understand what trajectory is described by this equation:

$$p = r + er\cos\phi = r + ex \rightarrow r = p - ex \rightarrow r^2 = x^2 + y^2 = p^2 - 2pex + e^2x^2$$

We group terms together to get:

$$y^2 + (1 - e^2)\left(x + \frac{ep}{1 - e^2}\right)^2 = \frac{p^2}{1 - e^2}$$

(expand the square and show that the two expressions are equivalent).

Let us define:

$$a = \frac{p}{1 - e^2}; b = \frac{p}{\sqrt{1 - e^2}}; x_0 = ea \tag{6}$$

(note: we can only do this for $e^2 < 1$!!) It follows that the equation of the trajectory is:

$$\frac{(x + x_0)^2}{a^2} + \frac{y^2}{b^2} = 1$$

which is an ellipse of large semi-axis a, small semi-axis b, and shifted by x_0 from its center (see Fig. 2). Note: if $e = 0 \rightarrow a = b = p, x_0 = 0$, i.e. the ellipse becomes a circle, as it should.

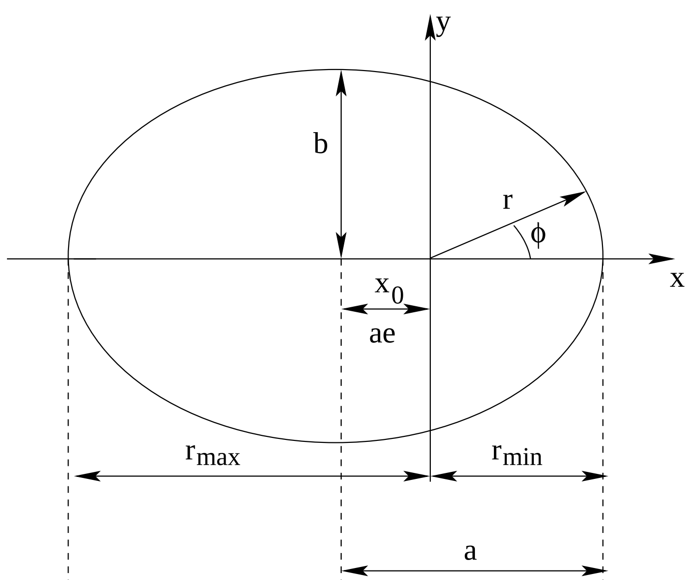

Fig 2. Elliptic trajectory for the relative coordinate $\vec{r}$.

Some interesting (and useful) relations about these quantities:

$$a = \frac{p}{1 - e^2} = -\frac{pU_0}{E_{rel}} \rightarrow E_{rel} = -\frac{\alpha}{2a}$$

(I used Eq. 4 and 2). So the large semiaxis is straightforward to find from E_{rel}. Another way to find the large semiaxis is the following: from Fig. 2 we see that the minimum/maximum distance between objects is

$$r_{min} = a - ae = a(1-e), \qquad r_{max} = a + ae = a(1+e) \tag{7}$$

and therefore $a = (r_{min} + r_{max})/2$. The values r_{min} and r_{max} are the return points, so they are given by the solutions of the equation $E_{rel} = U_{eff}(r)$. Another way to find them is from the trajectory equation: since $-1 \leq cos\phi \leq 1 \rightarrow 1 - e \leq \frac{p}{r} \leq 1 + e \rightarrow p/(1+e) \leq r \leq p/(1-e) \rightarrow r_{min} = p/(1+e); r_{max} = p/(1-e)$. These distances have special names, namely **perihelion** (r_{min}) and **ahelion** (r_{max}).

Now it would be interesting to know what each of the two objects is actually doing, when the relative trajectory is an ellipse. Let us assume that the CM is at rest (we can fix this by choosing an inertial reference system in which the CM speed is zero). Remember how $\vec{r}$ and $\vec{R}$ where linked to $\vec{r}_1$ and $\vec{r}_2$ (see Fig. 3): $\vec{r}_1 = \vec{R} + \frac{m_2}{M}\vec{r}$, $\vec{r}_2 = \vec{R} - \frac{m_1}{M}\vec{r}$.

If the CM is at rest, we can take $\vec{R} = 0$. We know that $\vec{r}$ rotates in time, describing an ellipse. So puting the two together, the motion of the bodies must be like in Fig. 4, with each object describing an ellipse, proportional to $\frac{m_2}{M}\vec{r}$, respectively $-\frac{m_1}{M}\vec{r}$. The CM is in the focus of both ellipses, and the objects are always opposite to one another.

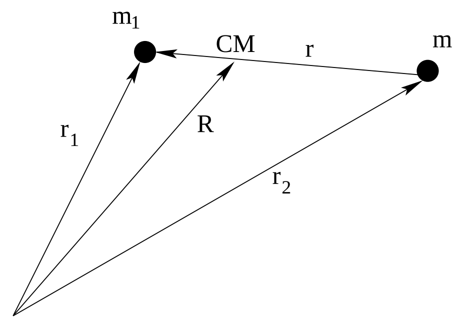

Fig 3. Relation between various vectors.

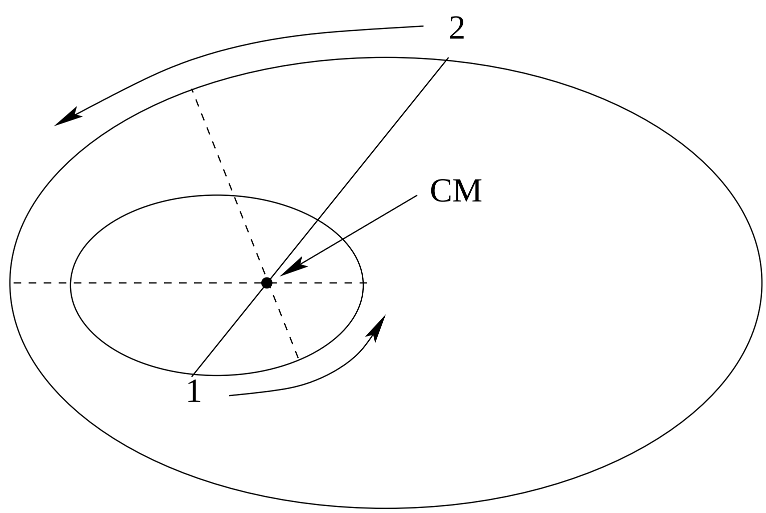

Fig 4. The motion of the two objects, when the CM (the focus of both ellipses) is at rest. If the CM is not at rest, just imagine moving the focus uniformly in some direction, as the masses rotate on ellipses about it. That's a bit too difficult for me to draw.

Finally, you can see what happens if one object is much much heavier than the other one (for instance, we have a Sun (1) and a planet (2)). The ellipse described by the planet is given by $M_S/(M_S + m_p)\vec{r} \approx \vec{r}$, while the Sun describes an ellipse given by $-m_p/(M_S + m_p)\vec{r} \approx 0$. In other words, the CM is basically in the Sun, which is so very much heavier. Therefore, the Sun stays in the focus, and the planet orbits around it at the relative distance $\vec{r}$. This is the confirmation for:

Kepler's 1^{st} law: All planetary orbits are ellipses, with the Sun at the focus.

We have already confirmed:

Kepler's 2^{nd} law: The area swept per unit time by the line joining the planet to its sun, is constant.

This is just equivalent with the conservation of the angular momentum, as discussed for general motion in a central field, where we showed that

$$\frac{d\mathcal{A}}{dt} = \frac{r^2\dot{\phi}}{2} = \frac{l}{2\mu} = const.$$

This allows us to find the period of the orbital motion right away, since this constant must be the total area of the ellipse πab divided by the total period T. Therefore:

$$\frac{\pi ab}{T} = \frac{l}{2\mu} = \frac{\sqrt{p\mu\alpha}}{2\mu} \rightarrow T = 2\pi\frac{ab\sqrt{\mu}}{\sqrt{p\alpha}} \tag{8}$$

(see Eq. 1). Now let's put all lengths in terms of a. We have (Eq. 6)

$$b = a\sqrt{1-e^2}; p = a(1-e^2) \rightarrow \frac{ab}{\sqrt{p}} = a^{\frac{3}{2}}$$

So we get the beautiful formula

$$T = 2\pi\sqrt{\frac{\mu a^3}{\alpha}} = 2\pi\sqrt{\frac{a^3}{GM}}$$

In other words,

Kepler's 3^{rd} law: The square of the period is proportional to the cube of the large semiaxis.

If you remember, we showed that this must be the right relationship using scaling laws (mechanical similarity). This should convince you that those scaling laws are very useful (although they don't tell us what the proportionality constant is!)

Let's now look at the other possible eccentricities:

2.3 $e = 1, e > 1$: the parabola and the hyperbola

We proceed in the same way as for the ellipse, and find that for $e = 1$ we have

$$p = r + x \rightarrow r^2 = (p-x)^2 \rightarrow y^2 = p^2 - 2px \rightarrow x = \frac{1}{2p}(p^2 - y^2)$$

As you know, this is the equation of a rotated parabola (see Fig. 5). The minimum distance is $p/2$ (when $y = 0$). Indeed, looking at Fig. 1 we see that this is the turning point when $E_{rel} = 0$. Finally, the case $e > 1$ is treated similarly, and the resulting curve is called a hyperbola. The shortest distance there (the turning point) is $r_{min} = p/(1+e)$, since $\cos\phi \leq 1$. The shape is somewhat like a distorted parabola, so I won't draw another figure. Again, if you want to think about the motion of the two objects, just put the CM in the focus (origin), draw two trajectories scaled in rations of m_2/M and $-m_1/M$ etc. The right-side picture in Fig. 5 should help visualize this when $m_1 \approx m_2$.

I will post on the website a link to a PhET simulation that will allow you to look at the trajectories of both objects, to better visualize how this really works.

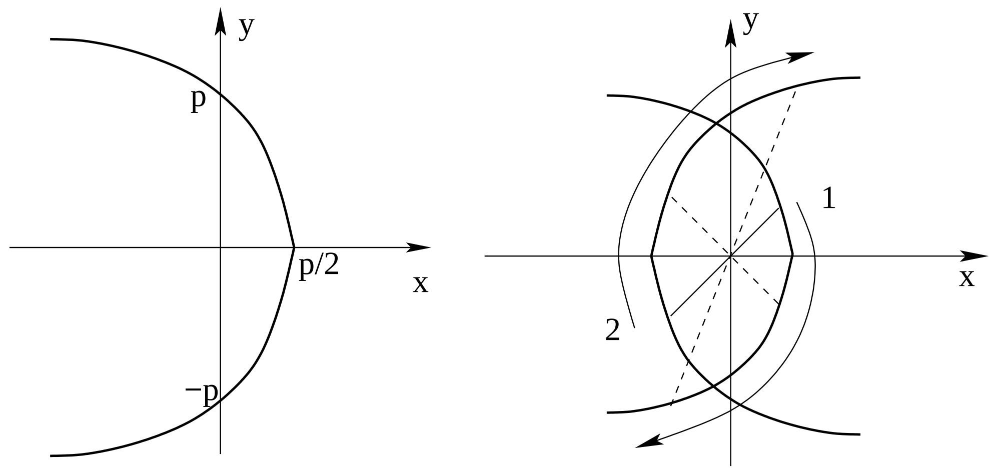

Fig 5. The parabola for $e = 1$. The hyperbola for $e > 1$ looks somewhat similar, except that the shortest distance is not $p/2$, but $p/(1+e)$.

3 Time dependence of the motion on the ellipse

To find $r(t)$, we need to integrate the equation:

$$t = \int \frac{dr}{\sqrt{\frac{2}{\mu}[E_{rel} - U_{eff}(r)]}}$$

Here it is convenient to express everything in terms of a; remember that $E_{rel} = -\alpha/2a$. The term proportional to l^2/μ in the effective potential can be written in terms of p and therefore in terms of $a = p/(1-e^2)$. After some struggle, the expression becomes:

$$t = \sqrt{\frac{\mu a}{\alpha}} \int \frac{r dr}{\sqrt{e^2a^2 - (r-a)^2}}$$

Surprise surprise ... we will ask that

$$(r-a)^2 = e^2a^2\cos^2\xi \to r - a = -ea\cos\xi \to r = a(1 - e\cos\xi)$$

(the minus sign and the ξ notation are just conventions for this particular problem). Then

$$dr = ae\sin\xi d\xi, \sqrt{e^2a^2 - (r-a)^2} = ea\sin\xi$$

and the integral becomes quite trivial:

$$t = \sqrt{\frac{\mu a}{\alpha}} \int a(1 - e\cos\xi)d\xi \to t = \sqrt{\frac{\mu a^3}{\alpha}}\,(\xi - e\sin\xi)$$

This is the parametric version: for any time t we can find the corresponding ξ and from that find the value of $r = a(1 - e\cos\xi)$, so we have $r(t)$. The period can be easily found now: a full revolution is described by $\xi = 0 \to 2\pi$, since in this case $r = r_{min} \to r_{max} \to r_{min}$. But if ξ varies by 2π, you can see that this corresponds to an increase in time:

$$T = 2\pi\sqrt{\frac{\mu a^3}{\alpha}}$$

The problem can be solved similarly for the parabola and hyperbola -type of trajectories. The textbook gives you thefinal results. You should try deriving them, it's good math practice.