The effects of the Coriolis force on projectile trajectories

For an object in motion with respect to the Earth, the largest non-inertial force is the Coriolis force $2m\vec{v}\times\vec{\Omega}$, where Ω is the angular velocity of the Earth around its North Pole - South Pole axis (of magnitude $2\pi/24h$). If we neglect the other 3 smaller, non-inertial forces, the equation of motion of a projectile with respect to a non-inertial frame tied to the Earth, is:

$$\vec{a} = \vec{g} + 2\vec{v}\times\vec{\Omega}$$

since the gravitational force is $m\vec{g}$. Let's pick this NIRS as shown in the Figure below, with x-axis pointing towards South, z-axis point radially outward from the center of the Earth, and therefore y-axis is pointing towards East. In this frame $\vec{g} = -g\vec{e}_z$.

We use the following strategy (= perturbation theory) to solve the equation of motion:

(1) find "zero"-order solution, i.e. the solution if $\Omega = 0$. Let's call this solution $\vec{r}_g(t)$ and $\vec{v}_g(t)$, which describes the motion of the projectile only under the influence of gravity. This is simple to solve, since in this case:

$$\frac{d^2x_g}{dt^2} = 0; \qquad \frac{d^2y_g}{dt^2} = 0; \qquad \frac{d^2z_g}{dt^2} = -g \rightarrow$$

$$x_g(t) = x(0) + v_{0,x}t; \qquad y_g(t) = y(0) + v_{0,y}t; \qquad z_g(t) = z(0) + v_{0,z}t - g\frac{t^2}{2} \rightarrow$$

$$\vec{r}_g(t) = \vec{r}_0 + \vec{v}_0 t - \frac{gt^2}{2}\vec{e}_z; \qquad \vec{v}_g(t) = \vec{v}_0 - gt\vec{e}_z$$

where $\vec{r}_0$ and $\vec{v}_0$ describe the initial position and velocity of the projectile. In particular, if we launch the projectile from the point where we centered the NIRS, we have $\vec{r}_0 = 0$. The projectile will then hit the ground again at the time $T = 2v_{0,z}/g$ when its heigth is again $z_g(T) = 0$.

(2) find the "first-order" corrections (i.e., terms proportional to Ω) to this solution. Let's call these corrections $\vec{r}_c(t)$ and $\vec{v}_c(t)$, since they are due to the Coriolis force. Both these quantities should be proportional to Ω (if $\Omega = 0$, there is no correction; on the other hand, we will ignore terms of order Ω^2 or higher powers, which are much smaller since Ω is so small. Moreover, we know we've already neglected the centrifugal force, which is of order Ω^2, so it would be wrong to keep other such terms).

The total solution in this approximation will be $\vec{r}(t) = \vec{r}_g(t) + \vec{r}_c(t)$, $\vec{v}(t) = \vec{v}_g(t) + \vec{v}_c(t)$. If we substitute this into the equation of motion, we find that

$$\frac{d^2\vec{r}_g}{dt^2} + \frac{d^2\vec{r}_c}{dt^2} = \vec{g} + 2\left[\vec{v}_g(t) + \vec{v}_c(t)\right]\times\vec{\Omega}$$

We already know that $\frac{d^2\vec{r}_g}{dt^2} = \vec{g}$, because that's how we constructed that solution. So this means that we must have:

$$\frac{d^2\vec{r}_c}{dt^2} = 2\vec{v}_g(t)\times\vec{\Omega}$$

The reason we neglected the term $\vec{v}_c(t)\times\Omega$ is because we know that $\vec{v}_c$ is proportional to Ω, so this product would be proportional to Ω^2 (too small). But (see Figure):

$$\vec{v}_g(t)\times\vec{\Omega} = \begin{vmatrix} \vec{e}_x & \vec{e}_y & \vec{e}_z \\ v_{0,x} & v_{0,y} & v_{0,z} - gt \\ -\Omega\cos\lambda & 0 & \Omega\sin\lambda \end{vmatrix} = \vec{e}_x v_{0,y}\Omega\sin\lambda - \vec{e}_y\left[v_{0,x}\Omega\sin\lambda + (v_{0,z} - gt)\Omega\cos\lambda\right] + \vec{e}_z v_{0,y}\Omega\cos\lambda$$

We can now directly integrate these equations, since we know that $\vec{r}_c(o) = \vec{v}_c(0) = 0$:

$$\frac{d^2 x_c}{dt^2} = 2v_{0,y}\Omega \sin\lambda \rightarrow x_c(t) = v_{0,y}\Omega \sin\lambda t^2$$

$$\frac{d^2 y_c}{dt^2} = -2\left[v_{0,x}\Omega \sin\lambda + (v_{0,z} - gt)\Omega \cos\lambda\right] \rightarrow y_c(t) = -\left[v_{0,x}\Omega \sin\lambda + v_{0,z}\Omega \cos\lambda\right] t^2 + g\Omega \cos\lambda \frac{t^3}{3}$$

$$\frac{d^2 z_c}{dt^2} = 2v_{0,y}\Omega \cos\lambda \rightarrow z_c(t) = v_{0,y}\Omega \cos\lambda t^2$$

So a knowledge of the initial speed, as well as the time of flight T, allows us to find the deflections $x_c(T)$ and $y_c(T)$ of the projectile from where it would have hit the ground, if $\Omega = 0$. You might think that because $z_c(T)$ is generally not zero, one would also have to correct the time of flight by some quantity that would be proportional to Ω. However, such corrections to T add only higher order powers of Ω to x_c and y_c, so they can be ignored.