September 2015 Physics & Astronomy Qualifying Exam
for Advancement to Candidacy
Day 1: September 3, 2015

Do not write your name on the exam. Instead, write your student number on each exam booklet. This will allow us to grade the exams anonymously. We'll match your name with your student number after we finish grading. If you use extra exam booklets, write your student number on the extra exam books as well. Write all answers in the blank exam booklet(s), not on this printout!

Today's portion of the exam has 8 questions. Answer *any five* of the eight. Do not submit answers to more than 5 questions—if you do, only the first 5 of the questions you attempt will be graded. If you attempt a question and then decide you don't want to it count, clearly cross it out and write "don't grade".

You have 4 hours to complete 5 questions.

You are allowed to use one $8.5'' \times 11''$ formula sheet (both sides), and a handheld, non-graphing calculator.

Here is a possibly useful table of physical constants and formulas:

absolute zero	0 K	-273°C
air pressure at sea level	1 atm	10^5 N/m^2
atomic mass unit	1 amu	1.66×10^{-27} kg
Avogadro's constant	N_A	6.02×10^{23}
Boltzmann's constant	k_B	1.38×10^{-23} J/K
charge of an electron	e	1.6×10^{-19} C
distance from earth to sun	1 AU	1.5×10^{11} m
Laplacian in spherical coordinates	$\nabla^2 f =$	$\frac{1}{r}\frac{\partial^2}{\partial r^2}(rf) + \frac{1}{r^2 \sin\theta}\frac{\partial}{\partial\theta}\left(\sin\theta\frac{\partial f}{\partial\theta}\right) + \frac{1}{r^2\sin^2\theta}\frac{\partial^2 f}{\partial\phi^2}$
mass of an electron	m_e	0.511 MeV/c^2
mass of a proton	m_p	938 MeV/c^2
mass of a neutron	m_n	940 MeV/c^2
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
nuclear magneton	μ_N	5×10^{-27} J/T
permittivity of free space	ϵ_0	8.9×10^{-12} C^2 N^{-1}/m^2
permeability of free space	μ_0	$4\pi \times 10^{-7}$ N/A^2
Planck's constant	h	6.6×10^{-34} J·s
radius of the Earth	R_{earth}	6.4×10^6 m
speed of light	c	3.0×10^8 m/s
Stefan-Boltzmann constant	σ	5.67×10^{-8} W m^{-2} K^{-4}
Stirling's approximation	$N!$	$e^{-N}N^N\sqrt{2\pi N}$

1. Order of magnitude calculation: Imagine that a tea cup could be made impervious to neutrons (spin = 1/2). How many neutrons could the cup hold without running over, if the cup and its neutrons were cooled as close to absolute zero as possible? Assume that the cup is in vacuum in a constant gravitational field $g = 9.8$ m/s^2. Don't forget to consider the difference in gravitational potential energy between the top and bottom of the cup.

2. A solid superconducting sphere of radius a is placed in a uniform external magnetic field $\vec{B} = B_0\hat{z}$. Because it is superconducting, the sphere expels magnetic fields, and $\vec{B} = 0$ everywhere inside the sphere. The superconductor achieves this by acquiring a uniform magnetization (magnetic moment density) $\vec{M} = M\hat{z}$ inside its volume. It can be shown that the magnetic field due to a uniformly magnetized sphere is a perfect dipole field in the region outside of the sphere.

A. Calculate the *total* magnetic field everywhere outside of this superconducting sphere. Hint: the $\vec{B}$ field never penetrates the sphere's surface.

B. Calculate the surface current density on the sphere as a function of the polar angle θ, measured from the $+z$ axis.

3. Consider a beam of neutrons of mass m_N, momentum p and magnetic moment $\vec{\mu} = \gamma \vec{s}$. The beam's polarization P is defined by

$$P = \frac{n_\uparrow - n_\downarrow}{n_\uparrow + n_\downarrow}$$

where $n_\uparrow$ and $n_\downarrow$ are the numbers of neutrons with spin up and spin down in the z direction, respectively. Note that $\vec{s}$ is the spin angular momentum.

An unpolarized beam is incident from $x = -\infty$ and it encounters a constant magnetic field which ramps up in magnitude from $B = 0$ when $x < 0$ to field strength $B > 0$ when $x > 0$. Assume that the magnetic field profile can be approximated by a step function, that is,

$$\vec{B}(\vec{x}) = B\hat{z}\theta(x)$$

A. For what range of momentum will the transmitted beam be 100% polarized?

B. Calculate the polarization of the reflected beam as a function of p.

4. Consider two electrons (spin-$\frac{1}{2}$ fermions) placed in a three-dimensional, isotropic harmonic potential given by $U(r) = (1/2)m\omega^2 r^2$.

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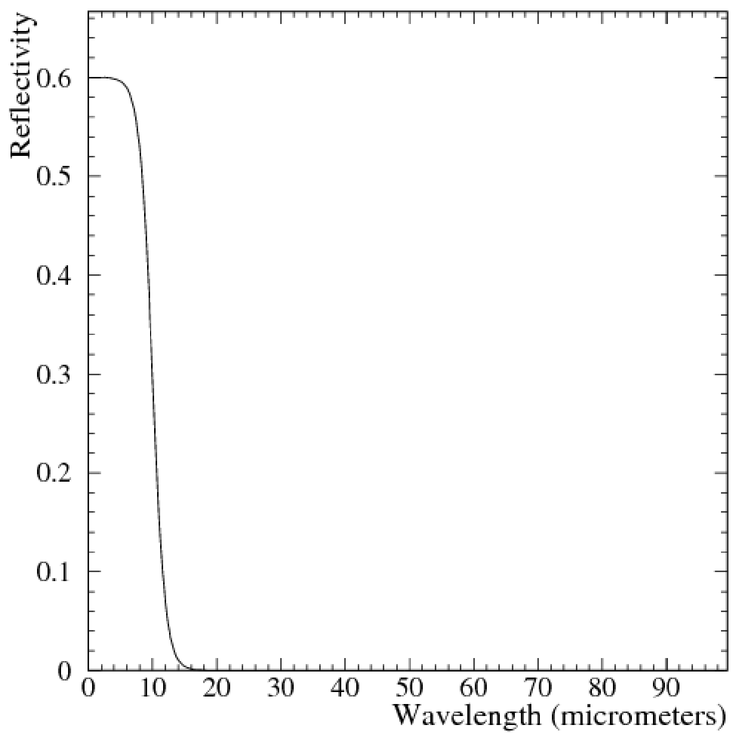
A. What are the energies and degeneracies of the two lowest energy levels of the two-particle system?

B. Suppose that the electrons have a magnetic moment and interact via a weak spin-spin interaction of the form $\hat{V} = \alpha \hat{\vec{s}}_1 \cdot \hat{\vec{s}}_2$. How are the energies and degeneracies of the states in Part A changed by this spin-spin interaction?

5. Assuming that the Sun is a perfect black body radiator and that all of Pluto's thermal energy comes from the Sun, estimate the expected average temperature of the surface of Pluto given that the surface of the Sun is at 5778 K.

Required constants: the Pluto-to-Sun distance is about 5.4 light-hours and the Sun's radius is about 2.3 light seconds. The surface reflectivity of Pluto is measured to be:



6. A white dwarf is a stellar remnant supported by electron degeneracy pressure. Electrons become degenerate when they are packed closely enough that the Pauli exclusion principle produces an additional form of pressure to keep them apart. The electron degeneracy pressure is a consequence of the Heisenberg uncertainty principle. If you call N_e the number density of electrons, what is the uncertainty Δx on the electron location in a white dwarf? For non-relativistic electrons, what is the uncertainty in velocity Δv? Explain why it makes sense to write the degeneracy pressure as $P_{degen} \sim N_e m_e (\Delta v)^2$, where m_e is the electron mass. At the center of a white dwarf of mass M and radius R, the gravitational pressure is $P_{grav} \sim GM^2/R^4$. Derive the relation between R and M for a stable white dwarf (i.e. when $P_{degen} \sim P_{grav}$). How does R scale with mass M?

7. The *Voyager 1* space probe has a 3.6 m diameter parabolic antenna that beams data back to Earth at a wavelength of 3.6 cm. It is presently 125 AU from the Sun. The transmitter power is 20 W.

A. What is the flux (W/m^2) of *Voyager*'s signal at Earth, assuming perfect optics and no absorption?

B. Communications theory says that the maximum bit rate of a data transmission is given by:

$$\text{Bit Rate} = B \log_2 \left(1 + \frac{S}{N} \right)$$

where B is a constant, S is the power of the received signal, and N is the power of the noise. When *Voyager* was at Jupiter (distance from Sun = 5.2 AU), the bit rate from Voyager was 115 kilobits/second. At Saturn (9.6 AU) the bit rate was 60 kilobits/second. Assuming that the same receiver continued to be used, estimate the present-day bit rate from Voyager.

8. The speed of ocean waves on the surface is found to depend on the density ρ of the fluid, the gravitational acceleration g, and the wavelength λ according to $v = K\rho^a g^b \lambda^c$, provided that the water is very deep. Here K is a dimensionless constant.

A. Determine the exponents a, b, and c.

B. Consider two sinusoidal ocean waves with frequencies ω and $\omega + \Delta\omega$, where $\Delta\omega \ll \omega$. If superimposed these will form a beat pattern. If the velocity at frequency ω is v, calculate the effective group velocity of the beat pattern.

September 2015 Physics Qualifying Exam
for Advancement to Candidacy
Day 2: September 4, 2015

If you are in the Ph.D. in astronomy program, don't write this exam! Ask a proctor for the astronomy version of today's exam!

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electron volt	1 eV	1.6×10^{-19} J
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Larmor formula		$P_{rad} = \mu_0(\ddot{\vec{m}})^2/6\pi c^3$
mass of an electron	m_e	0.511 MeV/c^2
mass of a proton	m_p	938 MeV/c^2
Newton's gravitational constant	G	6.7×10^{-11} N m^2 kg^{-2}
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9. A supercapacitor is an energy storage device with a very high capacitance C in series with an internal resistance R_i. It can be charged to a maximum voltage V_{max}. You desire to use one to power a load that requires a constant power input P. The load automatically varies its resistance R_L in order to keep the delivered power in R_L constant.

A. How long could the capacitor provide constant power P, if it is initially charged to V_{max}?

B. What fraction of the capacitor's total stored energy is still in the capacitor at the point when it can no longer deliver the requested power?

10. Do an order of magnitude calculation of the atomic number Z at which atomic electrons begin to move at relativistic speeds (defined here as $pc > mc^2/2$, where p is the electron's momentum and m is its rest mass).

11. Consider two entangled identical spin-1/2 particles, with magnetic moments related to their spin by $\vec{m} = \mu\vec{s}$, separated by a large distance. The four Bell states are given as,

$$|\Phi_\pm^{(AB)}\rangle = \frac{1}{\sqrt{2}}\left[|z_+, z_+\rangle \pm |z_-, z_-\rangle\right]$$

$$|\Psi_\pm^{(AB)}\rangle = \frac{1}{\sqrt{2}}\left[|z_+, z_-\rangle \pm |z_-, z_+\rangle\right]$$

where $|z_\pm\rangle$ are the two spin states along or against the z-axis, the first $z_\pm$ refers to particle A which is nearby, and the second $z_\pm$ refers to particle B which is far away. Consider two operations, both of which entail applying a magnetic field only at location A. In the first operation, the magnetic field in in the x-direction. In the second operation, the magnetic field is in the z- direction. In both cases, the fields act for a time t.

This problem can be solved by writing down the Hamiltonians H_z and H_x for the two operations above, along with the time evolution operators, $U_z(t, 0)$ and $U_x(t, 0)$.
Hint: $U(t, 0)$ obeys the equations $i\hbar\frac{d}{dt}U(t, 0) = HU(t, 0)$, $U(t, 0)^\dagger = U(t, 0)^{-1}$, $U(0, 0) = \mathcal{I}$.

A. Find a time duration, t_A, which results in $U_z(t_A, 0)|z_+\rangle = -i|z_+\rangle$.

B. Using time, t_A, find $U_z(t_A, 0)|z_-\rangle$, and $U_x(t_A, 0)|z_\pm\rangle$ (Hint: relate the $|z_\pm\rangle$ basis, which are eigenstates of the z-components of spin, to the $|x_\pm\rangle$ basis which are eigenstates of the x-components of spin.)

C. Using the two operations above show that any one of the Bell states can be transformed into any other, operating only at location A.

Hint: the Pauli spin matrices are:

$$\sigma_1 = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad \sigma_2 = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad \sigma_3 = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}$$

12. A point mass with mass m hangs from a massless string with length a. The top end of the string is tied to the ceiling. The mass is set in motion so that it orbits smoothly in a horizontal circle with the string making an angle of 45° to vertical.

A. Calculate the velocity of the mass.

B. Suppose now that the mass is started in a horizontal orbit at a string angle of $45^\circ + \epsilon$, with the same initial velocity as in Part A. Calculate the frequency of oscillation of the string's angle about 45°.

13. On a dry, clear and calm day, the sea level temperature is measured to be 20°C. A group of hikers plan to hike to a summit at 3000m elevation. Estimate the temperature at the summit that day. Show details of your calculations.

14. An electron with non-relativistic velocity v is approaching a proton at rest at an impact parameter b. Calculate the distance of closest approach between the electron and proton. (Do not treat the proton as an immobile object—it will recoil!)

15. A charged object is at distance $L = 1$ cm away from the centre of a 1 m $\times$ 1 m surface of a metal. When released, it takes about 2 seconds to reach the surface. Calculate how long it takes when released from a point 10 cm away from the center.

16. The solar radiation reaching the top of the Earth's atmosphere is 1400 W/m^2. For stars of the mass of the Sun and less, the proton-proton chain (PP chain) is the main source of energy. The most important fusion branch of the PP chain is the one that converts 4 protons ($m_p = 1.6473 \times 10^{-27}$ kg) into one helium nucleus ($m_{He} = 6.6446 \times 10^{-27}$ kg). Calculate the lifetime of the Sun assuming that the emitted energy is coming from the PP chain. The lifetime of the Sun is attained when 70% of the protons in the Sun's core have been converted into helium nuclei. The Sun's core represents only 20% of the Sun mass and, at its birth, the primordial composition of the Sun's mass was approximately 75% of hydrogen and 25% of helium. The distance of the Earth from the Sun is 1.5×10^{11} m, and a year has 365 days.